MORE ON SOLID ANALYTIC RINGS

We have constructed the ring of solid integers $\mathbb{Z}_{\blacksquare}$ whose category of complete modules Solid $\subset$ CondAb satisfies the following features:

(1) It is an abelian category stable under small limits, colimits and extensions in CondAb.
(2) It has by compact projective generators the condensed abelian groups of the form $\prod_I \mathbb{Z}$ for arbitrary index sets I.
(3) The inclusion Solid $\subset$ CondAb has by left adjoint the solidification functor

$$(-)^{\blacksquare} = \mathbb{Z}_{\blacksquare} \otimes_{\mathbb{Z}} - : \mathrm{CondAb} \to \mathrm{Solid}$$

which is the unique colimit preserving functor mapping $\mathbb{Z}[S]$ to $\mathbb{Z}_{\blacksquare}[S]$, for S extremally disconnected.
(4) There is a unique symmetric monoidal tensor product $\otimes_{\blacksquare}$ on Solid making $(-)^{\blacksquare}$ symmetric monoidal.
(5) The derived category of Solid is a full subcategory of $\mathscr{D}(\underline{\mathbb{Z}})$, stable under small limits and colimits. Furthermore, the solidification $(-)^{\blacksquare}$ has a left derived functor $(-)^{L\blacksquare}$, and $\mathscr{D}(\mathrm{Solid})$ has a unique symmetric monoidal structure $\otimes^L_{\blacksquare}$ such that $(-)^{L\blacksquare}$ is symmetric monoidal. Moreover, $\otimes^L_{\blacksquare}$ is the left derived functor of $\otimes_{\blacksquare}$.
(6) The solid tensor product satisfies

$$\prod_I \mathbb{Z} \otimes^L_{\blacksquare} \prod_J \mathbb{Z} = \prod_{I\times J} \mathbb{Z}.$$

In this lecture we will give more examples of solid abelian groups that appear in nature, as well as some important solid tensor products that recover classical complete tensor products of Banach and Frechét spaces.

1. Examples of solid abelian groups

The category of solid abelian groups consists on condensed abelian groups that are constructed from products of $\mathbb{Z}$ under limits, colimits and extensions. In particular, the following objects are solid abelian groups:

Example 1.1. (1) Any discrete abelian group M is a solid abelian group. Indeed, it is a quotient of direct sums of $\mathbb{Z}$:

$$\bigoplus_I \mathbb{Z} \to \bigoplus_J \mathbb{Z} \to M \to 0.$$

(2) Let A be a commutative ring, and let $I^{(n)}$ be an ideal filtration of A, namely, $I^{(0)} = A$ and $I^{(n)}I^{(m)} \subset I^{n+m}$. Then the completion $\widehat{A} = \varprojlim_n A/I^{(n)}$ is a solid abelian group being the limit of discrete abelian groups.
(3) In the previous example, we let A_{top} be the ring A endowed with topology defined by the filtration, considered as a condensed abelian group. Then it is not true in general that A_{top} is a solid abelian group. Indeed, if $(a_n)_{\mathbb{N}}$ is a sequence in A converging to 0, we have a natural map

$$\mathbb{Z}[\mathbb{N} \cup \{\infty\}]/(\infty) \to A$$

that (in general) cannot be extended to $\mathbb{Z}_{\blacksquare}[\mathbb{N}\cup\{\infty\}]/(\infty)$ (the solid abelian group of null-sequences) since the sum $\sum_n a_n$ does not necessarily converge in A_{top}.
(4) Let $\mathbb{Z}_{p-top}$ be the ring of integers $\mathbb{Z}$ considered with the p-adic topology. It is not clear whether $(\mathbb{Z}_{p-top})^{L\blacksquare} = \mathbb{Z}_p$ (one can show however that $\mathbb{Z}_p$ is a retract of the solidification of $\mathbb{Z}_{p-top}$). The reason why this apparently simple computation is not so easy is because $\mathbb{Z}_{p-top}$ has not a simple resolution by profinite sets: the p-adic topology gives rise power series on p with *arbitrary* coefficients on $\mathbb{Z}$.

(5) Let $\mathbb{Z}'_{p-top}$ be the condensed abelian group given by the quotient

$$0 \to \mathbb{Z}[\mathbb{N}\cup\{\infty\}]/(\infty) \xrightarrow{T-p} \mathbb{Z}[\mathbb{N}\cup\{\infty\}]/(\infty) \to \mathbb{Z}'_{p-top} \to 0,$$

(exercise, describe explicitly the S-points of this condensed abelian for $S \in \mathrm{Extdis}$). Then, by construction, we have that

$$(\mathbb{Z}'_{p-top})^{L\blacksquare} = \mathbb{Z}_p.$$

The difference between $\mathbb{Z}_{p-top}$ and $\mathbb{Z}'_{p-top}$ is that, in the solidifications, the first allows power series expansions on p with *arbitrary* coefficients on $\mathbb{Z}$, while $\mathbb{Z}'_{p-top}$ only allows power series in p with *uniformly bounded coefficients*. Note that any element $x \in \mathbb{Z}_p$ has a unique power series development

$$x = \sum_n a_n p^n$$

with $a_n \in \{0, \ldots, p-1\}$.

A reality check for the solid tensor product to capture non-archimedean analysis is that it coincides with I-complete tensor products for finitely generated ideals. For instance, we want that the solid tensor product behaves as expected for Tate algebras:

$$\mathbb{Q}_p\langle T_1\rangle \otimes^L_{\blacksquare} \mathbb{Q}_p\langle T_2\rangle = \mathbb{Q}_p\langle T_1, T_2\rangle.$$

Surprisingly, this is not obvious and to show this property we need to do some extra analysis. The difficulty appears since we need to understand completed direct sums as colimits of spaces of the form $\prod_I \mathbb{Z}$, namely, we must write the space $\mathbb{Q}_p\langle T\rangle$ as the colimit of converging sequences with respect to any possible rate of convergence, which is a presentation as an uncountable indexed colimit.

Before proving this fact we need to have access to some additional solid rings:

Definition 1.2. Let A be a condensed animated ring whose underlying animated condensed abelian group is solid. We define the *induced* analytic ring $A_{\mathbb{Z}_\blacksquare/} = (A, \mathbb{Z})_\blacksquare$ to have underlying condensed ring A, and *functor of measures*

$$(A, \mathbb{Z})_\blacksquare[S] = A \otimes^L_\blacksquare \mathbb{Z}_\blacksquare[S]$$

for $S \in \mathrm{Extdis}$.

Lemma 1.3. *The object $(A, \mathbb{Z})_\blacksquare$ is an analytic ring. We denote by $\mathscr{D}((A, \mathbb{Z})_\blacksquare)$ its derived category of complete $(A, \mathbb{Z}_\blacksquare)$-modules, and let $\otimes^L_{A,\blacksquare}$ be its symmetric monoidal structure. A condensed A-module M is $(A, \mathbb{Z})_\blacksquare$-complete if and only if it is $\mathbb{Z}_\blacksquare$-complete.*

Proof. We let $\mathscr{C} \subset \mathscr{D}_{\geq 0}(\underline{A})$ be the full subcategory of condensed animated A-modules consisting on objects M whose underlying condensed animated abelian group is $\mathbb{Z}_\blacksquare$-complete. We want to show that $(A, \mathscr{C})$ is an analytic ring, and that the left adjoint F of the inclusion map satisfies

$$F(\underline{A}[S]) = A \otimes^L_\blacksquare \mathbb{Z}_\blacksquare[S] \tag{1.1}$$

for $S \in \mathrm{Extdis}$.

It is clear that $\mathscr{C}$ is stable under all limits, colimits, and the formation of $M \mapsto R\underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathbb{Z}[S], M)$ for $S \in \mathrm{Extdis}$. It is left to show that the left adjoint F exists and that it is given by (1.1). For this, note that since, the solidification functor $(-)^{L\blacksquare}$ is symmetric monoidal, it sends condensed A-modules to $\mathbb{Z}_\blacksquare$-complete A-modules. Then, by writing $\mathscr{D}_{\geq 0}(\underline{A})$ as the sifted Ind-category of the objects $\underline{A}[S]$ (the free condensed A-module on S), it suffices to check that for $M \in \mathscr{C}$ and $S \in \mathrm{Extdis}$ the following map is an equivalence

$$R\underline{\mathrm{Hom}}_A(A \otimes^L_\blacksquare \mathbb{Z}_\blacksquare[S], M) \xrightarrow{\sim} R\underline{\mathrm{Hom}}_A(\underline{A}[S], M).$$

The previous follows from the computation

$$\begin{aligned} R\underline{\mathrm{Hom}}_A(A \otimes^L_\blacksquare \mathbb{Z}_\blacksquare[S], M) &= R\underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathbb{Z}_\blacksquare[S], M) \\ &= R\underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathbb{Z}[S], M) \\ &= R\underline{\mathrm{Hom}}_A(\underline{A}[S], M). \end{aligned}$$

$\square$

Example 1.4. (1) Consider $\mathbb{Z}[[X]]_{\blacksquare} = (\mathbb{Z}[[X]], \mathbb{Z})_{\blacksquare}$, then we have

$$\mathbb{Z}[[X]]_{\blacksquare} = \mathbb{Z}[[X]] \otimes^L_{\blacksquare} \mathbb{Z}_{\blacksquare}[S] = \varprojlim_i \mathbb{Z}[[X]][S_i]$$

where $S = \varprojlim_i S_i$ is written as a limit of finite sets. In particular, we have that

$$\prod_I \mathbb{Z}[[X]] \otimes^L_{\mathbb{Z}[[X]],\blacksquare} \prod_J \mathbb{Z}[[X]] = \prod_{I\times J} \mathbb{Z}[[X]].$$

(2) The ring $\mathbb{Z}[[X]]$ is an idempotent $(\mathbb{Z}[X], \mathbb{Z})_{\blacksquare}$-algebra. Indeed, we have the presentation

$$0 \to \mathbb{Z}[[T]] \otimes_{\mathbb{Z},\blacksquare} \mathbb{Z}[X] \xrightarrow{X-T} \mathbb{Z}[[T]] \otimes_{\mathbb{Z},\blacksquare} \mathbb{Z}[X] \to \mathbb{Z}[[X]] \to 0,$$

so that the tensor product $\mathbb{Z}[[X]] \otimes^L_{\mathbb{Z}[X]} \mathbb{Z}[[X]]$ is represented by the complex

$$\begin{aligned} &\big[\mathbb{Z}[[T]] \otimes_{\mathbb{Z},\blacksquare} \mathbb{Z}[[X]] \xrightarrow{X-T} \mathbb{Z}[[T]] \otimes_{\mathbb{Z},\blacksquare} \mathbb{Z}[[X]]\big] \\ &= \big[\mathbb{Z}[[T,X]] \xrightarrow{X-T} \mathbb{Z}[[T,X]]\big] \\ &\cong \mathbb{Z}[[X]][0], \end{aligned}$$

proving the idempotency. In particular, for solid $(\mathbb{Z}[[X]], \mathbb{Z})_{\blacksquare}$-modules N and M we have

$$N \otimes^L_{\mathbb{Z}[X],\blacksquare} M = N \otimes^L_{\mathbb{Z}[[X]],\blacksquare} M \text{ and } R\underline{\mathrm{Hom}}_{\mathbb{Z}[X]}(N, M) = R\underline{\mathrm{Hom}}_{\mathbb{Z}[[X]]}(N, M).$$

Proposition 1.5 ([Man22, Proposition 2.12.10])**.** *Let A be a discrete ring, $I \subset A$ a finitely generated ideal, and N and M I-adically complete A-modules considered as condensed abelian groups via the formulas*

$$M = \varprojlim_n M/I^n \text{ and } N = \varprojlim_n N/I^n.$$

Then $M \otimes_{A,\blacksquare} N = M \widehat{\otimes}_A N$.

Proof. For simplicity we prove the statement in the case where $A = \mathbb{Z}[X]$ and $N = M = \widehat{\bigoplus}_{\mathbb{N}} \mathbb{Z}[[X]]$ are the completed direct sums with respect to the X-adic topology, namely, we will show that

$$\widehat{\bigoplus}_{\mathbb{N}} \mathbb{Z}[[X]] \otimes^L_{\mathbb{Z}[X],\blacksquare} \widehat{\bigoplus}_{\mathbb{N}} \mathbb{Z}[[X]] = \widehat{\bigoplus}_{\mathbb{N}\times\mathbb{N}} \mathbb{Z}[[X]].$$

The proof of the general case can be reduced to this one by a devisage process, see [Man22, Lemma 2.12.9]. Since $\mathbb{Z}[[X]]$ is an idempotent $(\mathbb{Z}[X], \mathbb{Z})_{\blacksquare}$-algebra by Example 1.4 (2), we have that

$$M \otimes^L_{\mathbb{Z}[X],\blacksquare} N = M \otimes^L_{\mathbb{Z}[[X]],\blacksquare} N.$$

Let $\mathscr{S}$ be the poset of functions $f : \mathbb{N} \to \mathbb{N}$ converging to ∞ with order $f \leq g$ if and only if $f(n) \geq g(n)$ for all $n \in \mathbb{N}$. We can write

$$\widehat{\bigoplus}_{\mathbb{N}} \mathbb{Z}[[X]] = \varinjlim_{f\in\mathscr{S}} \prod_{\mathbb{N}} \mathbb{Z}[[X]] X^{f(n)}.$$

To see this, note that a map $S \to \widehat{\bigoplus}_{\mathbb{N}} \mathbb{Z}[[X]]$ is given by a null-sequence $(a_n)_{n\in\mathbb{N}}$ of continuous functions $a_n : S \to \mathbb{Z}[[X]]$, so we can find a function $f \in \mathscr{S}$ converging to ∞ such that $(a_n)_{n\in\mathbb{N}}$ is in $\prod_{\mathbb{N}} C(S, \mathbb{Z}[[X]]) X^{f(n)}$. Thus, we can compute

$$\widehat{\bigoplus}_{\mathbb{N}} \mathbb{Z}[[X]] \otimes^L_{\mathbb{Z}[[X]],\blacksquare} \widehat{\bigoplus}_{\mathbb{N}} \mathbb{Z}[[X]] = \varinjlim_{f,g\in\mathscr{S}} \prod_{\mathbb{N}\times\mathbb{N}} \mathbb{Z}[[X]] X^{f(n)+g(m)},$$

but the set of functions $f+g : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ with $f, g \in \mathscr{S}$ is cofinal in the poset $\mathscr{G}$ of functions $h : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ converging to ∞. Indeed, given $h \in \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ we can define $f(n) = \min_{i\geq n, j\in\mathbb{N}} \lfloor \frac{1}{2} h(i,j) \rfloor$ and $g(m) = \min_{i\in\mathbb{N}, j\geq m} \lfloor \frac{1}{2} h(i,j) \rfloor$, we have $f, g \in \mathscr{S}$ and $f(n) + g(m) \leq h(n,m)$. The previous shows that

$$\widehat{\bigoplus}_{\mathbb{N}} \mathbb{Z}[[X]] \otimes^L_{\mathbb{Z}[[X]],\blacksquare} \widehat{\bigoplus}_{\mathbb{N}} \mathbb{Z}[[X]] = \varinjlim_{h\in\mathscr{G}} \prod_{\mathbb{N}\times\mathbb{N}} \mathbb{Z}[[X]] X^{h(n,m)} = \widehat{\bigoplus}_{\mathbb{N}\times\mathbb{N}} \mathbb{Z}[[X]].$$

proving what we wanted. $\square$

Another important computation that appears in "solid analysis" is the incarnation of the projective tensor product of two Fréchet spaces, see Corollary 1.10 down below. To show this property we need a couple of additional computations.

Proposition 1.6. *We have*

$$\left(\prod_{\mathbb{N}} \mathbb{Z}((X))\right) \otimes^L_{\mathbb{Z}[[X]],\blacksquare} \left(\prod_{\mathbb{N}} \mathbb{Z}((X))\right) = \prod_{\mathbb{N}\times\mathbb{N}} \mathbb{Z}((X)).$$

Proof. By (AB6*) we can write

$$\prod_{\mathbb{N}} \mathbb{Z}((X)) = \varinjlim_{f:\mathbb{N}\to\mathbb{Z}} \prod_{n\in\mathbb{N}} \mathbb{Z}[[X]]X^{-f(n)}.$$

where $f \leq g$ if $f(n) \leq g(n)$ for all n. Then, we have

$$\left(\prod_{\mathbb{N}} \mathbb{Z}((X))\right) \otimes^L_{\mathbb{Z}[[X]],\blacksquare} \left(\prod_{\mathbb{N}} \mathbb{Z}((X))\right) = \varinjlim_{f,g:\mathbb{N}\to\mathbb{Z}} \prod_{(n,m)\in\mathbb{N}\times\mathbb{N}} \mathbb{Z}[[X]]X^{-f(n)-g(m)}.$$

The functions of the form $f(n)+g(m)$ are cofinal among the functions $h : \mathbb{N}\times\mathbb{N} \to \mathbb{Z}$, namely, given h we can define

$$f(n) = g(n) = \max_{i,j\leq n} h(i,j),$$

then $f(n) + g(m) \geq h(n,m)$. This shows that

$$\left(\prod_{\mathbb{N}} \mathbb{Z}((X))\right) \otimes^L_{\mathbb{Z}[[X]],\blacksquare} \left(\prod_{\mathbb{N}} \mathbb{Z}((X))\right) = \varinjlim_{h:\mathbb{N}\times\mathbb{N}\to\mathbb{Z}} \prod_{(n,m)\in\mathbb{N}\times\mathbb{N}} \mathbb{Z}[[X]]X^{h(n,m)} = \prod_{\mathbb{N}\times\mathbb{N}} \mathbb{Z}((X)).$$

□

Proposition 1.7. *Let $M = \widehat{\bigoplus}_{\mathbb{N}}\mathbb{Z}((X))$, then*

$$M \otimes^L_{\mathbb{Z}[[X]],\blacksquare} \prod_{\mathbb{N}} \mathbb{Z}((X)) = \prod_{\mathbb{N}} M. \tag{1.2}$$

Proof. Let us give explicit description of both terms as colimits of products of $\mathbb{Z}[[X]]$. Recall that we have

$$M = \varinjlim_{\substack{f:\mathbb{N}\to\mathbb{Z}\\ f(n)\to\infty}} \prod_{\mathbb{N}} \mathbb{Z}[[X]]X^{f(n)}$$

and

$$\prod_{\mathbb{N}} \mathbb{Z}((X)) = \varinjlim_{g:\mathbb{N}\to\mathbb{Z}} \prod_{\mathbb{N}} \mathbb{Z}[[X]]X^{-g(m)}.$$

Then, we get that

$$M \otimes^L_{\mathbb{Z}[[X]],\blacksquare} \prod_{\mathbb{N}} \mathbb{Z}((X)) = \varinjlim_{\substack{f:\mathbb{N}\to\mathbb{Z}\\ f(n)\to\infty\\ g:\mathbb{N}\to\mathbb{Z}}} \prod_{(n,m)\in\mathbb{N}\times\mathbb{N}} \mathbb{Z}[[X]]X^{f(n)-g(m)}.$$

On the other hand, the (AB6*) property implies that

$$\prod_{\mathbb{N}} M = \varinjlim_{\substack{h:\mathbb{N}\times\mathbb{N}\to\mathbb{Z}\\ \forall m,\ h(n,m)\to\infty}} \mathbb{Z}[[X]]X^{h(n,m)}$$

with order $h_1 \leq h_2$ if and only if $h_2(n,m) \leq h_1(n,m)$ for all $(n,m) \in \mathbb{N}^2$.

Therefore, in order to deduce (1.2), it suffices to show that the functions $h_{f,g}(n,m) = f(n) - g(m)$ with $f(n) \to \infty$ are cofinal among all the functions $h : \mathbb{N}\times\mathbb{N} \to \mathbb{Z}$ such that for fixed m we have $h(n,m) \to \infty$. Let h be one of such functions, for $m \in \mathbb{N}$ consider the function $h_m : \mathbb{N} \to \mathbb{Z}$ given by $h_m(n) = h(n)$. Then, we want to find a function $f : \mathbb{N} \to \mathbb{Z}$ converging to ∞ such that for all $m \in \mathbb{N}$ we have

$$\lim_{n\to\infty} h_m(n) - f(n) = \infty. \tag{1.3}$$

Indeed, suppose such a function exists, for $m \in \mathbb{N}$ let N_m be the minimum integer such that for all $k > N_m$ we have $h_m(n) \geq f(n)$, then define

$$g(m) = \max_{0 \leq n \leq N_m} \{f(n) - h_m(n)\},$$

it is immediate to see that $h(n,m) \geq f(n) - g(m)$ for all $n, m \in \mathbb{Z}$, so that $h \leq h_{f,g}$.

We now prove the existence of f. Given $s \geq 1$ let N_s be the minimum integer such that

$$h_j(i) \geq s$$

for all $i \geq N_s$ and all $1 \leq j \leq s$. We have that $N_s < N_{s+1}$, define the function $f(n)$ by $f(n) = \lfloor \frac{1}{2} s \rfloor$ for $N_s \leq n < N_{s+1}$. Then the function f satisfies (1.3), which ends the proof of the proposition. $\square$

Remark 1.8. The countability assumption in the product in Proposition 1.7 is crucial in order to make the diagonal argument.

Remark 1.9. The previous propositions represent very well how many computations in condensed mathematics are done: we know some basic computations, such as $\prod_I \mathbb{Z} \otimes^L_{\blacksquare} \prod_J \mathbb{Z} = \prod_{I \times J} \mathbb{Z}$. Next, we try to resolve new objects like $\mathbb{Z}[[X]]$, $\mathbb{Z}_p$, $\mathbb{Q}_p\langle T \rangle$ or $\widehat{\bigoplus}_I \mathbb{Z}((X))$, in terms of previously known objects as colimits or limits, then we use formal commutations of $\otimes$ or $\underline{\mathrm{Hom}}$ with (co)limits, and finally we try to describe the output in terms of more classical objects.

Corollary 1.10. *Let* $M = \widehat{\bigoplus}_{\mathbb{N}} \mathbb{Z}((X))$*, then*

$$\prod_{\mathbb{N}} M \otimes^L_{\mathbb{Z}[[X]],\blacksquare} \prod_{\mathbb{N}} M = \prod_{\mathbb{N} \times \mathbb{N}} (M \otimes_{\mathbb{Z}[[X]],\blacksquare} M).$$

Proof. By Proposition 1.7 we have that $\prod_{\mathbb{N}} M = \prod_{\mathbb{N}} \mathbb{Z}((X)) \otimes^L_{\mathbb{Z}[[X]],\blacksquare} M$. Then, using Propositions 1.5 and 1.6 we deduce that

$$\begin{aligned}
\prod_{\mathbb{N}} M \otimes^L_{\mathbb{Z}[X]],\blacksquare} \prod_{\mathbb{N}} M &= \left(\prod_{\mathbb{N}} \mathbb{Z}((X)) \otimes^L_{\mathbb{Z}[[X]],\blacksquare} M \right) \otimes^L_{\mathbb{Z}[[X]],\blacksquare} \left(\prod_{\mathbb{N}} \mathbb{Z}((X)) \otimes^L_{\mathbb{Z}[[X]],\blacksquare} M \right) \\
&= \left(\prod_{\mathbb{N}} \mathbb{Z}((X)) \otimes^L_{\mathbb{Z}[[X]],\blacksquare} \prod_{\mathbb{N}} \mathbb{Z}((X)) \right) \otimes^L_{\mathbb{Z}[[X]],\blacksquare} \left(M \otimes^L_{\mathbb{Z}[[X]],\blacksquare} M \right) \\
&= \prod_{\mathbb{N} \times \mathbb{N}} \mathbb{Z}((X)) \otimes^L_{\mathbb{Z}[[X]],\blacksquare} \left(M \otimes^L_{\mathbb{Z}[[X]],\blacksquare} M \right) \\
&= \prod_{\mathbb{N} \times \mathbb{N}} \left(M \otimes_{\mathbb{Z}[[X]],\blacksquare} M \right).
\end{aligned}$$

$\square$

References

[Man22] Lucas Mann. A p-adic 6-Functor Formalism in Rigid-Analytic Geometry. https://arxiv.org/abs/2206.02022, 2022.