Seminar on Condensed mathematics

August 18, 2023

1 Outline of the talks

In this seminar we aim to review the basics of the theory of condensed mathematics of Clausen and Scholze [Sch19, CS20, CS22]. In the first three talks we introduce the notion of condensed set and condensed abelian group, we compare the notions of condensed sets with the one of topological space, and study condensed vs sheaf cohomology for compact Hausdorff spaces. In talk 4 we introduce the notion of analytic ring; this is the base for the theory of analytic geometry obtained via condensed mathematics. The rest of the talks are dedicated to present the two most important analytic rings at the moment: the solid integers $\mathbb{Z}_\blacksquare$ and the liquid real numbers $\mathbb{R}_{<p}$. [1]

2 Distribution of the talks

Talk 1: Condensed sets

Give a general introduction to condensed sets: give the definition of condensed sets via profinite sets, extremally disconnected sets or Hausdorff spaces. Discuss the set theoretical subtleties and how to solve them. Prove Theorem 2.16 [Sch19] or Proposition 1.2 [CS20] about the characterization of quasi-separated condensed sets and properties of the "condensification functor". Follow the material of Lecture I and Appendix to Lecture II of [Sch19]; Lecture I [CS20]. For a completely formal reference see [Man22, §2.1].

Talk 2: Condensed abelian groups

Define condensed objects in a general category, see [Man22, Definition 2.1.1 and Remark 2.1.2] (for simplicity restrict to the setting of classical categories). Then consider the category of condensed abelian groups: it can be described as both commutative group objects in Cond(Set) or condensed objects in abelian groups. Explain the proof of Theorem 2.2 of [Sch19], in particular the key role of the compact projective generators. Mention the symmetric monoidal structure of Cond(Ab) as well as its internal $\underline{\mathrm{Hom}}$. Prove Proposition 2.1 of [CS20] about the explicit description of the free condensed abelian groups generated by profinite sets. Finally, discuss general properties of quasi-separated sets and the preservation of the abelian structure. Follow the material of Lecture II of [Sch19]; Lecture II and Appendix to Lecture IV of [CS20].

[1] Actually, the ring $\mathbb{R}_{<p}$ will appear as a specialization of the more fundamental ring $\mathbb{Z}((T))_{>r}$ of r-liquid Laurent series.

Talk 3: Cohomology

In this talk we do the first computations in condensed cohomology. Explain the proofs of Theorems 3.2 and 3.3 of [Sch19], computing condensed cohomology with $\mathbb{Z}$-coefficients as sheaf/Čech cohomology, and vanishing of condensed cohomology with (condensed) $\mathbb{R}$-coefficients. If possible explain how the proof can be adapted when $\mathbb{R}$ is replaced by a Banach space. Follow the material of Lecture III of [Sch19].

Talk 4: Analytic rings

A main motivation to use condensed mathematics is the great framework it provides to do analytic geometry. In particular, one of the most important notions is that of analytic ring. In this talk we motivate the definition of analytic rings via its category of modules, cf. [CS20, Proposition 12.20]: an analytic ring $\mathcal{A}$ ought to give the data of a condensed (eg. topological) ring $\underline{\mathcal{A}}$, together with a "nice" category of "complete" modules $\mathrm{Mod}_{\mathcal{A}}$ (for simplicity restrict to the situation of static analytic rings, i.e. those of [Sch19, Definitions 7.1 and 7.4]). Define morphisms between analytic rings and show that they induce pullbacks between their categories of modules. Then, give the forthcoming examples arising from the solid and the liquid theories. Follow Lecture VII of [Sch19]; Lectures XI, XII and Appendix to Lecture XII of [CS20].

Talk 5: Locally compact abelian groups

In order to define the analytic ring of solid integers $\mathbb{Z}_\blacksquare$ we need some important cohomological computations of locally compact abelian groups seen as condensed abelian groups ([Sch19, Theorem 5.8]). Prove Theorem 4.3 of [Sch19] concerning Ext's groups between locally compact abelian groups. If possible, also explain the construction of the EM-Breen-Deligne resolution (this might take a while, so we can either skip it or take it as an additional talk). Follow Lecture IV of [Sch19] and its appendix.

Talks 6-7: Solid abelian groups

Discuss the construction of the analytic ring $\mathbb{Z}_\blacksquare$, equivalently, the construction of the category of solid abelian groups. Describe the tensor product of compact projective generators, give examples of solid abelian groups and some additional tensor products (power series rings, p-adic numbers, $\mathbb{Q}_p$-Banach spaces, etc). Can also explain the simpler proof in the case of solid $\mathbb{F}_p$-vector spaces of [CS20, Lecture II]. Follow Lectures V and VI of [Sch19]; Lecture II of [CS20]. For tensor products of $\mathbb{Q}_p$-vector spaces see [Bos21, Appendix A].

Talks 8-9: Condensed and $\mathcal{M}$-complete $\mathbb{R}$-vector spaces

We now switch towards the archimedian theory. In these talks we discuss the general theory of condensed $\mathbb{R}$-vector spaces. In particular, how locally convex vector spaces embed into condensed $\mathbb{R}$-vector spaces (cf. Propositions 3.2 and 3.4 of [CS20]). Define $\mathcal{M}$-complete vector spaces, prove the characterization of Proposition 4.6 and the duality between Smith and Banach spaces of Theorem 4.7. Finally, explain how the projective and injective tensor products in classical functional analysis are related with $\mathcal{M}$-completions of condensed tensor products (Propositions 4.10 and 4.12). Follow Lectures III and IV of [CS20].

Talks 10-11: Towards liquid vector spaces

Give the example of an extension of Banach spaces that is not Banach (Corollary 5.4 [CS20]). Introduce the notion of p-locally convex vector space, the analytic ring of p-liquid $\mathbb{R}$-vector spaces and the category of p-liquid $\mathbb{R}$-vector spaces (Theorem 6.4 [CS20]). State the explicit characterization for a quasi-separated vector space to be liquid (Theorem 2.14. of [CS22]), and prove that the compact projective generators $\mathcal{M}_{<p}(S)$ are flat for the liquid tensor product (Theorem 3.14 [CS22]). Give as many examples as possible, even if they come without a proof (eg. examples of power series rings). Take material from Lectures V-VI of [CS20]; Lectures II-IV of [CS22].

References

[Bos21] Guido Bosco. On the p-adic pro-étale cohomology of Drinfeld symmetric spaces, 2021.

[CS20] Dustin Clausen and Peter Scholze. Lectures on Analytic Geometry. https://www.math.uni-bonn.de/people/scholze/Analytic.pdf, 2020.

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[Man22] Lucas Mann. A p-adic 6-Functor Formalism in Rigid-Analytic Geometry. https://arxiv.org/abs/2206.02022, 2022.

[Sch19] Peter Scholze. Lectures on condensed mathematics. 2019.