THE SOLID INTEGERS $\mathbb{Z}_\blacksquare$

In this talk we will prove one of the foundational theorems in condensed mathematics. Let $\mathbb{Z}_\blacksquare$ be the datum $(\underline{\mathbb{Z}}, \mathbb{Z}_\blacksquare[-])$ consisting on the condensed ring defined by the integers, and the functor from Extdis (or Prof) to condensed abelian groups mapping $S = \varprojlim_i S_i$ to $\mathbb{Z}_\blacksquare[S] := \varprojlim_i \mathbb{Z}[S_i]$.

Theorem 0.1 ([Sch19, Theorem 5.8]). *The object $\mathbb{Z}_\blacksquare$ is an analytic ring. Moreover, the following hold:*

(1) *Let* Solid $\subset$ CondAb *be the full subcategory consisting on condensed abelian groups M such that for all profinite set S the natural map*

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_\blacksquare[S], M) \to \mathrm{Map}(S, M)$$

is a bijection. Then Solid *is an abelian category stable under all small limits, colimits and extensions. The objects $\prod_I \mathbb{Z} \in \mathrm{Solid}$, for any index set I, form a family of compact projective generators. The inclusion* Solid $\subset$ CondAb *has a left adjoint given by the solidification*

$$M \mapsto \mathbb{Z}_\blacksquare \otimes_{\mathbb{Z}} M = M^\blacksquare : \mathrm{CondAb} \to \mathrm{Solid}.$$

(2) *The category* Solid *has a unique symmetric monoidal structure $\otimes_\blacksquare$ making the functor $\mathbb{Z}_\blacksquare \otimes_{\mathbb{Z}} -$ symmetric monoidal.*

(3) *Let $\mathscr{C} = \mathscr{D}(\mathbb{Z}_\blacksquare) \subset \mathscr{D}(\underline{\mathbb{Z}})$ be the full subcategory defined by the analytic ring $\mathbb{Z}_\blacksquare$. Then $C \in \mathscr{D}(\underline{\mathbb{Z}})$ belongs to $\mathscr{C}$ if and only if one of the following equivalent conditions hold:*

 (i) *For all profinite set S the natural map*

$$R\underline{\mathrm{Hom}}(\mathbb{Z}_\blacksquare[S], C) \to R\underline{\mathrm{Hom}}(\mathbb{Z}[S], C)$$

 is an equivalence.

 (ii) *For all profinite set S the natural map*

$$R\mathrm{Hom}(\mathbb{Z}_\blacksquare[S], C) \to R\mathrm{Hom}(\mathbb{Z}[S], C)$$

 is an equivalence.

 (iii) $H^i(C) \in \mathrm{Solid}$ *for all $i \in \mathbb{Z}$.*

 In particular, the inclusion $\mathscr{C} \subset \mathscr{D}(\underline{\mathbb{Z}})$ is stable under all limits and colimits, and has a left adjoint $\mathbb{Z}_\blacksquare \otimes^L_{\mathbb{Z}} -$ given by the left derived functor of $\mathbb{Z}_\blacksquare \otimes_{\mathbb{Z}} -$.

(4) *The category $\mathscr{C}$ has a unique symmetric monoidal structure $\otimes^L_\blacksquare$ making $\mathbb{Z}_\blacksquare \otimes^L_{\mathbb{Z}} -$ symmetric monoidal. The functor $\otimes^L_\blacksquare$ is the left derived functor of $\otimes_\blacksquare$. Moreover, for any index sets I and J, we have that*

$$\prod_I \mathbb{Z} \otimes^L_\blacksquare \prod_J \mathbb{Z} = \prod_{I \times J} \mathbb{Z}.$$

The proof of the previous theorem will required several reductions to the key $R\underline{\mathrm{Hom}}$ computations between locally compact abelian groups of the previous talk. Before that we need some preparations.

1. Pseudo-coherent objects

Let $\mathscr{C}$ be an abelian category with all colimits admitting compact projective generators, let $\mathscr{D}(\mathscr{C})$ be its (∞-)derived category and $\mathscr{D}_{\geq 0}(\mathscr{C}) \subset \mathscr{D}(\mathscr{C})$ the full subcategory consisting on connective complexes (equivalently, the animation of $\mathscr{C}$).

Definition 1.1. Let $n \geq 1$ be an integer. An object $A \in \mathscr{C}$ is n-pseudo-coherent if $\mathrm{Ext}^i(A, -)$ commutes with filtered colimits for all $0 \leq i \leq n-1$. We say that A is pseudo-coherent if it is n-pseudo-coherent for all $n \geq 1$.

Let $\mathscr{P} = \{P_i\}_I$ be a fixed family of compact projective generators of $\mathscr{C}$.

Proposition 1.2. *An object $A \in \mathscr{C}$ is n-pseudo-coherent if and only if it has a partial resolution*

$$C_n \to \cdots \to C_1 \to C_0 \to A \to 0 \tag{1.1}$$

where each C_i is a finite direct sum of compact projective objects in $\mathscr{P}$

Proof. We just give the argument for $n = 1$, we leave the general case to the reader. Note that A is 1-pseudo-coherent means that $\mathrm{Hom}_{\mathscr{C}}(A, -)$ commutes with filtered colimits, i.e. that A is a compact object of $\mathscr{C}$. We find a partial resolution

$$\bigoplus_{j\in J} P_j \to \bigoplus_{i \in I} P_i \to A \to 0.$$

Then, we can write A as a filtered colimit of the cokernels X_k of maps

$$\bigoplus_{J'_k} P_j \to \bigoplus_{I'_k} P_i$$

where $J'_k \subset J$ and $I'_k \subset I$ are finite subsets. Then, since X is compact and $X = \varinjlim_k X_k$, it is a retract of some X_k, and one has the desired partial resolution of X after a suitable modification. Conversely, if X has a resolution as in (1.1), we have an exact sequence

$$0 \to \mathrm{Hom}(A, -) \to \mathrm{Hom}(C_0, -) \to \mathrm{Hom}(C_1, -),$$

then, since each $\mathrm{Hom}(C_i, -)$ commute with filtered colimits in $\mathscr{C}$, then so does A as wanted. ☐

Lemma 1.3. *Let $A \in \mathscr{C}$ and suppose we have a resolution*

$$\cdots \to M_1 \to M_0 \to A \to 0$$

with each M_i pseudo-coherent. Then A is pseudo-coherent.

Proof. The stupid truncation of the previous complex gives rise an hypercohomology spectral sequence with E_1-page

$$E_1^{p,q} = \mathrm{Ext}^q(M_p, -) \Rightarrow \mathrm{Ext}^{p+q}(A, -),$$

since each M_q is pseudo-coherent, the functors $\mathrm{Ext}^p(M_q, -)$ preserve filtered colimits for all p, q, and so does $\mathrm{Ext}^n(A, -)$, proving that A is pseudo-coherent. ☐

When specialized to condensed abelian groups we have the following properties:

Proposition 1.4. *Let X be a compact Hausdorff space, then $\mathbb{Z}[X]$ is a pseudo-coherent condensed abelian group. Moreover, let A be a compact abelian group, then $A \otimes_{\mathbb{Z}} \mathbb{Z}[X]$ is a pseudo-coherent condensed abelian group.*

Proof. Let $S_\bullet \to X$ be an hypercover by extremally disconnected sets, then we have a projective resolution

$$\cdots \to \mathbb{Z}[S_1] \to \mathbb{Z}[S_0] \to \mathbb{Z}[X] \to 0,$$

proving that $\mathbb{Z}[X]$ is pseudo-coherent. Now let A be a compact abelian group, by the Breen-Deligne theorem we have a resolution

$$\cdots \to \mathbb{Z}[A^2] \to \mathbb{Z}[A] \to A \to 0$$

where each term is a finite direct sum of terms of the form $\mathbb{Z}[A^r]$ for $r \in \mathbb{N}$. Tensoring with $\mathbb{Z}[X]$ we get a resolution with terms given by finite sums of free condensed abelian groups generated by compact Hausdorff spaces (note that the free condensed abelian groups are flat being the sheafification of a flat pre-sheaf). The proposition follows by the first statement and Lemma 1.3. ☐

2. Some preparations before the proof

Let $S = \varprojlim_i S_i$ be a profinite set, recall that we have defined the *free solid abelian group generated by S* as

$$\mathbb{Z}_\blacksquare[S] = \varprojlim_i \mathbb{Z}[S_i],$$

equivalently it can be constructed as the space of *$\mathbb{Z}$-valued measures*

$$\mathbb{Z}_\blacksquare[S] = \underline{\mathrm{Hom}}_{\mathbb{Z}}(C(S, \mathbb{Z}), \mathbb{Z}).$$

The following theorem gives a clear description of this object as condensed abelian group.

Theorem 2.1 (Nöbelin-Specker, [Sch19, Theorem 5.4]). *For any profinite set S, the abelian group $C(S,\mathbb{Z})$ of continuous maps from S to $\mathbb{Z}$ is a free abelian group. In particular, if $C(S,\mathbb{Z}) \cong \bigoplus_I \mathbb{Z}$ is a fixed basis, we have an isomorphism of condensed abelian groups*

$$\mathbb{Z}_\blacksquare[S] \cong \prod_I \mathbb{Z}.$$

Proof. Let us take an injection $S \hookrightarrow \prod_I \{0,1\}$ for some set I. Choose a well ordering on I, so I is some ordinal λ and elements in I identify with ordinals $\mu < \lambda$. For each $\mu < \lambda$ we get $e_\mu \in C(S,\mathbb{Z})$ the idempotent object mapping to the μ-component of $\prod_I \{0,1\}$. Order the products $e_{\mu_1} \cdots e_{\mu_r}$ with $\mu_1 > \cdots > \mu_r$ with the lexicographic order (including the empty product corresponding to $r = 0$). Let E be the set of such products that cannot be written as linear combinations of smaller such products. We claim that E is a basis of $C(S,\mathbb{Z})$.

We argue by induction of λ, the case $\lambda = 0$ being trivial. For any $\mu < \lambda$ let S_μ be the image of S in $\prod_{\mu' < \mu} \{0,1\}$. If λ is a limit ordinal, then the result follows formally from the result of the S_μ by taking colimits of the sets E_μ (this holds since $S_{\mu'} \subset S_\mu$ for $\mu' \subset \mu$). Then, we can assume that $\lambda = \rho + 1$ and let us write $\overline{S} = S_\rho$. We have a closed immersion $S \hookrightarrow \overline{S} \times \{0,1\}$. Let $S_i = S \cap \overline{S} \times \{i\}$ for $i = 0, 1$; these are open and closed subspaces covering S, and so closed subspaces of $\overline{S}$ covering $\overline{S}$. Let $\overline{S}'$ be the intersection of S_1 and S_2 in $\overline{S}$. Then we have a short exact sequence

$$0 \to C(\overline{S},\mathbb{Z}) \to C(S,\mathbb{Z}) \to C(\overline{S}',\mathbb{Z}) \to 0$$

where the second map sends f to the difference of the two restrictions in $\overline{S}'$.

By induction, the part of the basis vectors of E that do not start with e_ρ form a basis of $C(\overline{S},\mathbb{Z})$. On the other hand, the basis vectors of E that start with e_ρ project to a basis of $C(\overline{S}',\mathbb{Z})$ (by applying the induction hypothesis to $\overline{S}'$ with its closed immersion into $\prod_{\mu<\rho}\{0,1\}$; note that the image of $e_\rho e_{\mu_1} \cdots e_{\mu_k} \in C(S,\mathbb{Z})$ in $C(\overline{S}',\mathbb{Z})$ is just $e_{\mu_1} \cdots e_{\mu_k}$). Thus, E defines a basis of $C(S,\mathbb{Z})$ as desired. □

Remark 2.2. If S is metrizable then we can embed $S \hookrightarrow \prod_{\mathbb{N}}\{0,1\}$. Thus, the previous proof simplifies in this situation as the images of S in $\prod_{i=1}^n \{0,1\}$ is finite and a simple induction does the work.

A consequence of the previous theorem is that hypercovers still produce resolutions for the free solid abelian groups.

Proposition 2.3. *Let $S_\bullet \to S$ be an hypercover of a profinite sets S by profinite sets. Then the corresponding complex*

$$\cdots \mathbb{Z}_\blacksquare[S_1] \to \mathbb{Z}_\blacksquare[S_0] \to \mathbb{Z}_\blacksquare[S] \to 0$$

is exact.

Proof. Since a profinite set has no higher condensed cohomology for discrete coefficients, we have an exact sequence

$$0 \to C(S,\mathbb{Z}) \to C(S_0,\mathbb{Z}) \to C(S_1,\mathbb{Z}) \to \cdots.$$

Taking duals and using Theorem 2.1 we get the proposition. □

As a first approximation to Theorem 0.1 we can show that the free solid abelian groups are indeed solid:

Proposition 2.4 ([Sch19, Proposition 5.7]). *For any profinite sets S and S', the natural map*

$$R\underline{\mathrm{Hom}}(\mathbb{Z}_\blacksquare[S'], \mathbb{Z}_\blacksquare[S]) \to R\underline{\mathrm{Hom}}(\mathbb{Z}[S'], \mathbb{Z}_\blacksquare[S])$$

is an equivalence.

Proof. By Theorem 0.1 we have isomorphisms $\mathbb{Z}_\blacksquare[S'] \cong \prod_J \mathbb{Z}$ and $\mathbb{Z}_\blacksquare[S] \cong \prod_I \mathbb{Z}$ depending on a basis of the spaces of $\mathbb{Z}$-valued continuous functions of S' and S respectively. Thus, we can assume without loss of generality that $S = *$ and $\mathbb{Z}_\blacksquare[S] = \mathbb{Z}$. Consider the short exact sequence

$$0 \to \prod_J \mathbb{Z} \to \prod_J \mathbb{R} \to \prod_J \mathbb{T} \to 0.$$

Then the proposition follows from the fact that

$$R\underline{\mathrm{Hom}}(\prod_J \mathbb{T}, \mathbb{Z}) = \bigoplus_J \mathbb{Z}[-1],$$

$$R\underline{\mathrm{Hom}}(\prod_J \mathbb{R}, \mathbb{Z}) = R\underline{\mathrm{Hom}}_{\mathbb{R}}(\prod_J \mathbb{R}, R\underline{\mathrm{Hom}}(\mathbb{R}, \mathbb{Z})) = 0,$$

and

$$R\underline{\mathrm{Hom}}(\mathbb{Z}[S'], \mathbb{Z}) = C(S', \mathbb{Z}) = \bigoplus_J \mathbb{Z}.$$

□

3. Proof of Theorem 0.1

We want to show the following theorem (which is precisely part of the definition of being an analytic ring).

Theorem 3.1. *Let S be a profinite set and let $C_\bullet$ be a connective complex with terms C_i isomorphic to direct sums of terms of the form $\prod_I \mathbb{Z}$ for I varying index sets. Then the natural map*

$$R\underline{\mathrm{Hom}}(\mathbb{Z}_\blacksquare[S], C_\bullet) \to R\underline{\mathrm{Hom}}(\mathbb{Z}[S], C_\bullet)$$

is an equivalence.

Proof. Recall that

$$\mathbb{Z}_\blacksquare[S] = \mathcal{M}(S, \mathbb{Z}) = \underline{\mathrm{Hom}}(C(S, \mathbb{Z}), \mathbb{Z}).$$

Define

$$\mathcal{M}(S, \mathbb{R}) = \underline{\mathrm{Hom}}(C(S, \mathbb{Z}), \mathbb{R}) \text{ and } \mathcal{M}(S, \mathbb{T}) = \underline{\mathrm{Hom}}(C(S, \mathbb{Z}), \mathbb{T}),$$

we have a short exact sequence of condensed abelian groups

$$0 \to \mathcal{M}(S, \mathbb{Z}) \to \mathcal{M}(S, \mathbb{R}) \to \mathcal{M}(S, \mathbb{T}) \to 0.$$

Then, it suffices to prove the following claim:

Claim. For all $C_\bullet$ as in the statement of the theorem, and all profinite set S, we have

$$R\underline{\mathrm{Hom}}(\mathcal{M}(S, \mathbb{T}), C_\bullet) \cong R\underline{\mathrm{Hom}}(\mathbb{Z}[S], C_\bullet)[-1] = R\underline{\Gamma}(S, C_\bullet).$$

Indeed, suppose the claim holds, by taking $S = *$ we get that

$$R\underline{\mathrm{Hom}}(\mathbb{T}, C_\bullet) = C_\bullet[-1] = R\underline{\mathrm{Hom}}(\mathbb{Z}[1], C_\bullet).$$

Then $R\underline{\mathrm{Hom}}(\mathbb{R}, C_\bullet) = 0$. Therefore, for a general S we get that

$$R\underline{\mathrm{Hom}}(\mathcal{M}(S, \mathbb{R}), C_\bullet) = R\underline{\mathrm{Hom}}_{\mathbb{R}}(\mathcal{M}(S, \mathbb{R}), R\underline{\mathrm{Hom}}(\mathbb{R}, C_\bullet)) = 0,$$

and we get a natural equivalence

$$R\underline{\mathrm{Hom}}(\mathcal{M}(S, \mathbb{Z}), C_\bullet) = R\underline{\mathrm{Hom}}(\mathcal{M}(S, \mathbb{T}), C_\bullet)[1] = R\underline{\mathrm{Hom}}(\mathbb{Z}[S], C_\bullet).$$

Let us now prove the claim, we do it in three main steps:

Step 1. We first assume that $C_\bullet = C_0[0]$ is concentrated in degree 0. By Proposition 1.4 we know that $\mathcal{M}(S, \mathbb{T}) \otimes \mathbb{Z}[S']$ is a pseudo-coherent condensed abelian group for any profinite set S', then $\mathcal{M}(S, \mathbb{T})$ is internally pseudo-coherent (i.e. the internal $\underline{\mathrm{Ext}}^i(\mathcal{M}(S, \mathbb{T}), -)$ functors commute with filtered colimits), and we can formally reduced to the case $C_0 = \prod_I \mathbb{Z}$ which follows from Proposition 2.4.

Step 2 Now suppose that $C_\bullet$ has only finitely many non-zero terms, the claim hods from Step 1 by taking the stupid filtration of $C_\bullet$.

Step 3. Now suppose that $C_\bullet$ is arbitrary. It suffices to show that there is some $k \geq 0$, independent of $C_\bullet$ and S, such that both $R\underline{\mathrm{Hom}}(\mathcal{M}(S, \mathbb{T}), C_\bullet)$ and $R\underline{\mathrm{Hom}}(\mathbb{Z}[S], C_\bullet)$ are in homological degrees $\geq -k$ (eq. cohomological degrees $\leq k$); we will see that we can actually take $k = 1$. Indeed, this property will imply that both functors above coincide for $C_\bullet$ and its stupid truncation $\sigma_{\leq k+n+1} C_\bullet$, for homology groups in degrees $i \leq n$, reducing the Claim to complexes supported in finitely many degrees, which follows by Step 2.

We now focus in Step 3 above. Let us write $C_n = \bigoplus_{i\in J_n}\prod_{K_{i,n}}\mathbb{Z}$. We define a new complex $C_{\bullet,\mathbb{R}}$ whose terms are $C_{n,\mathbb{R}} = C_n = \bigoplus_{i\in J_n}\prod_{K_{i,n}}\mathbb{R}$, obtained by naturally extending the differentials of $C_\bullet$. For this, it suffices to show that the natural map

$$\mathrm{Hom}(C_{n+1,\mathbb{R}}, C_{n,\mathbb{R}}) \to \mathrm{Hom}(C_{n+1}, C_{n,\mathbb{R}})$$

is an isomorphism. We can formally assume that $C_{n+1} = \prod_I \mathbb{Z}$, and it is enough to see that

$$\mathrm{RHom}(\prod_I \mathbb{T}, C_{n,\mathbb{R}}) = 0.$$

Since $\prod_I \mathbb{T}$ is pseudo-coherent, we can commute with direct sums, and then we can formally commute with products. We then are reduced to the statement

$$R\mathrm{Hom}(\prod_I \mathbb{T}, \mathbb{R}) = 0$$

which was proved in the previous lecture.

Therefore, we have a complex of condensed $\mathbb{R}$-vector spaces $C_{\bullet,\mathbb{R}}$. Define $C_{\bullet,\mathbb{T}}$ to fit in a short exact sequence of complexes

$$0 \to C_\bullet \to C_{\bullet,\mathbb{R}} \to C_{\bullet,\mathbb{T}} \to 0.$$

Thus, it is enough to prove that

$$R\underline{\mathrm{Hom}}(\mathcal{M}(S,\mathbb{T}), C_{\bullet,\mathbb{R}}),\ ; R\underline{\mathrm{Hom}}(\mathcal{M}(S,\mathbb{T}), C_{\bullet,\mathbb{T}}),\ ; R\underline{\mathrm{Hom}}(\mathbb{Z}[S], C_{\bullet,\mathbb{R}}) \text{ and } R\underline{\mathrm{Hom}}(\mathbb{Z}[S], C_{\bullet,\mathbb{T}})$$

are concentrated in cohomological degrees ≤ 1. By writing $C_{\bullet,\mathbb{R}}$ and $C_{\bullet,\mathbb{T}}$ as limits of their canonical truncations $\tau_{\leq i}C_{\bullet,\mathbb{R}}$ and $\tau_{\leq i}C_{\bullet,\mathbb{T}}$, it suffices to prove the statement for the truncations. These are finite complexes, sot it suffices to prove the statement for each term. We are then reduce to prove the cohomological bound for the condensed abelian groups given by the kernels of maps $d_{i,\mathbb{R}} : C_{i,\mathbb{R}} \to C_{i-1,\mathbb{R}}$ and $d_{i,\mathbb{T}} : C_{i,\mathbb{T}} \to C_{i-1,\mathbb{T}}$. Then, by taking filtered colimits, and since $\mathcal{M}(S,\mathbb{T})$ and $\mathbb{Z}[S]$ are pseudo-coherent, we can further assume that $C_i \cong \prod_I \mathbb{Z}$ and $C_{i-1} \cong \prod_J \mathbb{Z}$.

Case $\ker d_{i,\mathbb{T}}$. the group K is the kernel of a map of the form $\prod \mathbb{T} \to \prod \mathbb{T}$. Then K is a compact abelian group, and its Pontrjagin dual $\mathbb{D}(K)$ is discrete and admits a resolution

$$0 \to F_1 \to F_2 \to \mathbb{D}(K) \to 0$$

with F_i free abelian groups. This shows that K fits in a short exact sequence

$$0 \to K \to \prod_I \mathbb{T} \to \prod_J \mathbb{T} \to 0,$$

and we reduce to prove the bound for products of tori, then formally for just $\mathbb{T}$, and finally for $\mathbb{Z}$ or $\mathbb{T}$ which follows by previous cohomological computations.

Case $\ker d_{i,\mathbb{R}}$. the group $K = \ker d_{i,\mathbb{T}}$ is the kernel of a map $\prod_I \mathbb{R} \to \prod_I \mathbb{R}$ that arises from the $\mathbb{R}$-linear extension of a map $\prod_I \mathbb{Z} \to \prod_J \mathbb{Z}$. This second map is the dual of a map $\bigoplus_J \mathbb{Z} \to \bigoplus_I \mathbb{Z}$, and so $d_{i,\mathbb{R}}$ is the dual of the $\mathbb{R}$-linear extension

$$\bigoplus_J \mathbb{R} \to \bigoplus_I \mathbb{R}.$$

But any map between real vector spaces factors by a split surjection followed by a split injection, and any real vector space is free. This implies that K is the dual of a real vector space of the form $\bigoplus \mathbb{R}$, so isomorphic to $\prod \mathbb{R}$. We then can reduce to $K = \mathbb{R}$ and the statement follows from previous cohomological computations.

$\square$

Most of the Theorem 0.1 is deduced from the abstract definition of analytic ring (see [Sch19, Lemma 5.9]), the only non-trivial statement is point (4).

Proposition 3.2 ([Sch19, Proposition 6.3])**.** *For any two index sets I and J we have*

$$\prod_I \mathbb{Z} \otimes^L_\blacksquare \prod_J \mathbb{Z} = \prod_{I\times J} \mathbb{Z},$$

Moreover, $\otimes^L_\blacksquare$ is the left derived functor of $\otimes_\blacksquare$.

Proof. Since the solidification functor is a left adjoint of the inclusion $\mathscr{D}(\mathbb{Z}_\blacksquare) \subset \mathscr{D}(\underline{\mathbb{Z}})$, we have that for any extremally disconnected sets S and S'

$$\mathbb{Z}_\blacksquare[S] \otimes^{\mathbb{L}}_\blacksquare \mathbb{Z}_\blacksquare[S'] = \mathbb{Z}_\blacksquare \otimes^L_{\underline{\mathbb{Z}}} (\mathbb{Z}[S \times S']).$$

Now let $T_\bullet \to S \times S'$ be an hypercover by extremally disconnected sets, we get that $\mathbb{Z}_\blacksquare \otimes^L_{\underline{\mathbb{Z}}} (\mathbb{Z}[S \times S'])$ is equivalent to the complex

$$\cdots \to \mathbb{Z}_\blacksquare[T_1] \to \mathbb{Z}_\blacksquare[T_0] \to 0,$$

but Proposition 2.3 implies that it is equivalent to $\mathbb{Z}_\blacksquare[S \times S']$, proving that

$$\mathbb{Z}_\blacksquare[S] \otimes^{\mathbb{L}}_\blacksquare \mathbb{Z}_\blacksquare[S'] = \mathbb{Z}_\blacksquare[S \times S'].$$

This shows that $\otimes^L_\blacksquare$ is the left derived functor of $\otimes_\blacksquare$. Now, by fixing basis of $C(S, \mathbb{Z})$ and $C(S, \mathbb{Z})$ indexed by I and J respectively, the space $C(S \times S', \mathbb{Z})$ has a basis indexed by $I \times J$. This shows that, by identifying $\mathbb{Z}_\blacksquare[S] \cong \prod_I \mathbb{Z}$ and $\mathbb{Z}_\blacksquare[S'] \cong \prod_J \mathbb{Z}$, we get

$$\prod_I \mathbb{Z} \otimes^L_\blacksquare \prod_J \mathbb{Z} = \prod_{I\times J} \mathbb{Z}.$$

For general index sets I and J we can find extremally disconnected sets S and S' such that $\prod_I \mathbb{Z}$ and $\prod_J \mathbb{Z}$ are direct summands of $\mathbb{Z}_\blacksquare[S]$ and $\mathbb{Z}_\blacksquare[S']$, the corollary follows. ☐

4. Examples of solid abelian groups

After this important theorem we show some examples of solid abelian groups that appear in the nature.

Proposition 4.1 ([Sch19, Example 6.5])**.** *Let X be a CW complex, and let $H_\bullet(X)$ denote the singular homology complex. Then there is a canonical isomorphism*

$$\mathbb{Z}_\blacksquare[X] \cong H_\bullet(X).$$

Proof. The functor $\mathbb{Z}_\blacksquare[-] = \mathbb{Z}_\blacksquare \otimes^L_{\underline{\mathbb{Z}}} (\mathbb{Z}[-])$ commutes with filtered colimits, so does the homology complex. Then, we can assume that X is a finite CW complex, in particular a compact Hausdorff space. Then, by Proposition 1.4 $\mathbb{Z}[X]$ is a pseudo-coherent condensed abelian group, in particular it admits a projective resolution with terms given by $\mathbb{Z}[S]$ with S extremally disconnected sets. This shows that $\mathbb{Z}_\blacksquare[X]$ is equivalent to a complex $C_\bullet$ with $C_n \cong \prod_{I_n} \mathbb{Z}$, implying that $\mathbb{Z}_\blacksquare[X]$ is reflexive. Then, it suffices to find a natural identification

$$\mathbb{Z}_\blacksquare[X]^\vee = R\underline{\mathrm{Hom}}(\mathbb{Z}_\blacksquare[X], \mathbb{Z}) = R\underline{\mathrm{Hom}}(\mathbb{Z}[X], \mathbb{Z}) \cong H^\bullet(X)$$

where $H^\bullet(X)$ is the singular cohomology complex. But the LHS is also sheaf cohomology, producing the desired equivalence

$$\mathbb{Z}_\blacksquare[X]^\vee \cong H^\bullet(X)$$

since X is a CW complex. ☐

Some first computations of solid tensor products are recollected in the following proposition.

Proposition 4.2 ([Sch19, Example 6.4])**.** *The following hold:*

(1) *We have* $\mathbb{Z}[[T_1]] \otimes^L_\blacksquare \mathbb{Z}[[T_2]] = \mathbb{Z}[[T_1, T_2]]$.
(2) $\mathbb{Z}_p \otimes^L_\blacksquare \mathbb{Z}[[T]] = \mathbb{Z}_p[[T]]$. *More generally,* $\mathbb{Z}_p \otimes^L_\blacksquare \prod_I \mathbb{Z} = \prod_I \mathbb{Z}_p$.
(3) *The tensor* $\mathbb{Z}_p \otimes^L_\blacksquare \mathbb{Z}_\ell$ *is* 0 *if* $p \neq \ell$ *and* $\mathbb{Z}_p$ *if* $p = \ell$.
(4) *We have* $\mathbb{Z}_\blacksquare \otimes^L_{\underline{\mathbb{Z}}} \mathbb{R} = 0$.

Proof. We can write $\mathbb{Z}[[T_i]] \cong \prod_{n\in\mathbb{N}} \mathbb{Z}T_i^n$, then part (1) follows from Proposition 3.2. For part (2), we have a short exact sequence of condensed abelian groups

$$0 \to \mathbb{Z}[[T]] \xrightarrow{T-p} \mathbb{Z}[[T]] \to \mathbb{Z}_p \to 0,$$

by Proposition 3.2 we see that $\mathbb{Z}_p \otimes^L_\blacksquare \prod_I \mathbb{Z}$ is represented by the complex

$$\prod_I \mathbb{Z}[[T]] \xrightarrow{T-p} \prod_I \mathbb{Z}[[T]]$$

which is equivalent to $\prod_I \mathbb{Z}_p$, proving what we wanted.

For (3), $\mathbb{Z}_p \otimes^L_\blacksquare \mathbb{Z}_\ell$ is represented by the complex

$$\mathbb{Z}_p[[T]] \xrightarrow{T-\ell} \mathbb{Z}_p[[T]].$$

If $\ell = p$ this is equivalent to $\mathbb{Z}_p$, if $\ell \neq p$ then $T-\ell$ is invertible and the complex is 0.

Finally, for (4), by the proof of Theorem 3.1 we know that for all conective complex $C_\bullet$ with terms given by direct sums of products of $\mathbb{Z}$, we have that

$$R\underline{\mathrm{Hom}}(\mathbb{R}, C_\bullet) = 0,$$

but $\mathscr{D}_{\geq 0}(\mathbb{Z}_\blacksquare)$ is generated under sifted colimits of objects of the form $\prod_I \mathbb{Z}$, and so they are represented by such a kind of complex $C_\bullet$. This implies that for all $N \in \mathscr{D}_{\geq}(\mathbb{Z}_\blacksquare)$ we have

$$R\underline{\mathrm{Hom}}(\mathbb{R}, M) = 0,$$

implying that $\mathbb{Z}_\blacksquare \otimes^L_{\mathbb{Z}} \mathbb{R} = 0$. □

In the next lecture we will give more examples of solid tensor products, we will also construct more solid analytic rings which are relevant for algebraic and rigid geometry.

References

[Sch19] Peter Scholze. Lectures on Condensed Mathematics. `https://www.math.uni-bonn.de/people/scholze/Condensed.pdf`, 2019.