On a result of Deligne

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In this note, we discuss canonical resolutions attached to abelian groups. The construction is most natually expressed as resolutions in certain abelian functor categories. Rather surprisingly, the existence of these canonical resolutons rely on results from stable homotopy theory. We have written these notes with the goal of de-mystifying the relationship.

0.1. **Framing the problem**

Given an abelian group A, we want to associate a natural free resolution $F_A^\bullet \longrightarrow A$. The claim is that there is a functorial resolution:

$$\ldots \to \oplus_{j=1}^{n_i}\mathbb{Z}[A^{r_{i,j}}] \to \ldots \longrightarrow \mathbb{Z}[A^3] \oplus \mathbb{Z}[A^2] \to \mathbb{Z}[A^2] \to \mathbb{Z}[A] \to A \to 0$$

We may construct the first few terms rather easily, in particular, the map $d_1 : \mathbb{Z}[A^2] \longrightarrow \mathbb{Z}[A]$ is associated to the set map

$$(a,b) \longmapsto [a+b] - [a] - [b].$$

One finds that constructing this resolution by hand captures some non-trivial group relations that are very hard to write down. For example, we observe that $[(a,b)] - [(b,a)]$ and $[(a,b+c)] + [(b,c)] - [(a+c,b)] - [(a,c)]$ are in $\mathrm{Ker}(d_1)$.

0.2. Let us begin by acknowledging that construction a functorial resolution implies we should be looking at functor categories. In particular, its not even clear what this resolution should look like on the full subcategory Latt of finite free $\mathbb{Z}$-modules, so let us start there.

Consider the abelian category $\mathcal{A} := \mathrm{Fun}_{\mathrm{Ab}}(\mathrm{Latt}, \mathrm{Ab})$;[1] it has all limits and colimits computed termwise, and moreover, it has compact projective generators. A set of such compact projective is given by the functors taking $P \in \mathrm{Latt}$ to $\mathbb{Z}[P^n]$ for some $n \geq 0$. Indeed, $P^n = \mathrm{Hom}_{\mathrm{Latt}}(\mathbb{Z}^n, P)$, so for any $F \in \mathrm{Fun}(\mathrm{Latt}, \mathrm{Ab})$, one has

$$\mathrm{Hom}_{\mathrm{Fun}_{\mathrm{Ab}}(\mathrm{Latt},\mathrm{Ab})}\left(\mathbb{Z}\left[\mathrm{Hom}_{\mathrm{Latt}}\left(\mathbb{Z}^n,-\right)\right],F\right) = \mathrm{Hom}_{\mathrm{Fun}(\mathrm{Latt},\mathrm{Sets})}\left(\mathrm{Hom}_{\mathrm{Latt}}\left(\mathbb{Z}^n,-\right),F\right) = F\left(\mathbb{Z}^n\right),$$

which commutes with all limits and colimits, and as F is determined by the values $F(\mathbb{Z}^n)$, one also sees that they form a generating family. (One may write F as a filtered colimit using its functor of elements construction). Our notation for this generating set will be $\left\{h^{\mathbb{Z}}_{\mathbb{Z}^n}\right\}_{n\geq 0}$, in analogy with Yoneda lemma.

The claim is that the functor $\psi : \mathrm{Latt} \longrightarrow \mathrm{Ab}$ taking $P \longmapsto P$ has a resolution by these compact projectives. If this is the case, one has a resolution in $\mathrm{Fun}_{\mathrm{Ab}}(\mathrm{Latt}, \mathrm{Ab})$ as:

$$C_*(-) := \ldots \to \bigoplus_{j=1}^{k_i} h^{\mathbb{Z}}_{\mathbb{Z}^{r_{i,j}}} \to \ldots \to \bigoplus_{j=1}^{k_0} h^{\mathbb{Z}}_{\mathbb{Z}^{r_{0,j}}} \to \psi \to 0.$$

[1] We use the subscript Ab to denote abelian functors i.e. normal functors whose induced map on Hom sets is additive.

Equivalently, we can ask for a resolution of the form:

$$C_*(P) := \ldots \to \bigoplus_{j=1}^{k_i} \mathbb{Z}\left[P^{r_{i,j}}\right] \overset{d_{r_{i,j}}}{\to} \ldots \to \bigoplus_{j=1}^{k_0} \mathbb{Z}\left[P^{r_{0,j}}\right] \to P \to 0,$$

$\forall P \in \mathrm{Latt}$.

0.3. *Extending this to arbitrary abelian groups*
One notes that the differentials d_{i,r_j} above are determined independently of P; indeed they are built from morphisms $h^{\mathbb{Z}}_{\mathbb{Z}^m} \longrightarrow h^{\mathbb{Z}}_{\mathbb{Z}^n}$, which by Yoneda are determined by formal $\mathbb{Z}$-linear sums of $M^{m\times n} : \mathbb{Z}^m \longrightarrow \mathbb{Z}^n$. We define $C_*(A)$ to be the complex formed by substituting A for P where A is an arbitrary abelian group. The main point here is that each chain map in $C_*(A)$ is uniquely defined via the universal morphisms discussed above.

We first want to see $C_*(A)$ is an actual complex. We know the result for finite free abelian groups A, and thus for free abelian groups of arbitrary rank by taking filtered colimits. Every abelian group A has a surection $P \longrightarrow A$ where P is free, which leads to a termwise surjection $C_*(P) \longrightarrow C_*(A)$, and consequently we conclude $C_*(A)$ is complex. We thus have a functor $C_*(-) : \mathrm{Ab} \longrightarrow \mathrm{Ch}_+(Ab)$, which can extend termwise to a functor $C_*(-) : \mathrm{Ch}_+(Ab) \longrightarrow \mathrm{Ch}_+(Ab)$, which by a spectral sequence argument will preserve quasi-isomorphisms. As such, we can then conclude $C_*(A)$ is a resolution of A by taking a projective resolution $P^\bullet \longrightarrow A$ and using the the result for each $P^j \in P^\bullet$.

0.4. *Can't we just construct the complex by hand?*
One candidate is the Bar construction. Let us recall its definition. First, we recall that the *Dold-Kan* equivalence gives an equivalence of model categories between

$$\begin{tikzcd} \mathrm{sAb} \arrow[r, bend right, "N"'] & \mathrm{Ch}^+ \arrow[l, bend right, "K"'] \end{tikzcd} .$$

We have Eilenberg-Maclean spaces $K(A,n) := K(A[n])$, and in particular, one has a Bar construction $B(-) : \mathrm{Ab} \longrightarrow \mathrm{sAb}, \quad A \longmapsto K(A[1])$. We can extend $B(-)$ to a functor sAb termwise and then take the underlying simplicial set of the bi-simplicial set. By a spectral sequence argument, one sees that $B(-)$ defined in this way sends quasi-isomorphisms to quasi-isomorphisms.

Associating $A \longmapsto BA$ gives us a simplicial resolution[2] of A, but this is only a projective resolution when A is projective/free. To remedy this, consider consider the adjunction

$$\begin{tikzcd} \mathrm{Ab} \arrow[r, bend right, "i"'] & \mathrm{Set} \arrow[l, bend right, "\mathbb{Z}(-)"'] \end{tikzcd} ,$$

[2]This is not strictly true; by construction $\pi_1(BA) = A$, and are 0 elsewhere. A suitable shift would make it a resolution of A, but for expository purposes, its easier to just think of it as a resolution.

and extend termwise to get a functor $\mathbb{Z}[-] : \mathrm{sSet} \longrightarrow \mathrm{sAb}$. This functor is not exact, the problem being that it is not additive on morphisms between simplicial abelian groups. Nevertheless, we get a complex $\mathbb{Z}[BA]$. What are the homotopy groups of this?

Recall that we have an equivalence of model categories

$$\begin{tikzcd} \mathrm{Top} \arrow[r, bend right, "S(-)"'] & \mathrm{sSet} \arrow[l, bend right, "|-|"'] \end{tikzcd} .$$

In particular, one has $|BA| \simeq K(A,1)$, and one sees $\mathbb{Z}[X]$ is precisely the singular chains of the simplicial space $|X|$, and hence $\pi_*(\mathbb{Z}[X]) = H_*(|X|;\mathbb{Z})$. Moreover, we have a justification for a passing comment made earlier that $\mathbb{Z}[-]$ does not preserve quasi-isomorphisms of associated chain complexes; computing the homotopy groups of $\mathbb{Z}[BA]$ is as hard as computing the homology groups of $K(A,1)$, which is hard.

Not all is lost. These homology groups are understood in the stable limit. In other words, every abelian group A associates a spectrum HA, and for a spectrum $X := \{X(n)\}_{n\geq 0}$ one associates the integral stable homology groups $H_i^{\mathrm{st}}(X) := \lim_n H_{n+i}(X(n);\mathbb{Z})$. It is easy to represent $K(A,n)$ as a simplicial set; one iteratively applies B to BA. The statement from homotopy theory is as follows:

Theorem:[1, 11.1] For a finite free abelian group A, the homology groups

$$H_i\left(\mathbb{Z}\left[B^n A\right]\right) = H_i(K(A,n),\mathbb{Z})$$

vanish for $i < n$ and are given by $M_{i-n} \otimes_{\mathbb{Z}} A$ for $n \leq i < 2n$, where $M_* := \pi_*(H\mathbb{Z} \otimes_{\mathbb{S}} H\mathbb{Z})$, the dual Steenrod algebra of $\mathbb{Z}$, and are finitely generated abelian groups.

As we will see, as n gets large, we get sequence $\mathbb{Z}\left[B^n P\right][-n] \longrightarrow P \longrightarrow 0$, for all $P \in \mathrm{Latt}$. While this sequence is not exact, its existence can be interpretted as a complex of compact projectives for our original functor ψ. Furthermore, this result from stable homotopy theory will imply P is n-psuedocoherent, and this will be enough to get the canonical resolution via general nonsense as we let $n \longrightarrow \infty$.

0.5. **Pseudo-coherence**

In an arbitrary abelian category $\mathcal{A}$ with all limits and colimits, admitting a set (class?) of compact projective generators $\mathcal{A}_0$, we say an an object $X \in \mathcal{A}$ is n-pseudocoherent if $\mathrm{Ext}^i_{\mathcal{A}}(X,-) : \mathcal{A} \longrightarrow \mathrm{Ab}$ commutes with all filtered colimits for all $i = 0, 1, ..., n-1$. We say X is pseudocoherent if it is n-pseudocoherent for all $n \geq 1$.

The relevance for this terminology can be seen readily in algebraic geometry. For example, there is a very important reason why in statements of Grothendieck duality one works over coherent rings R and it's category of coherent modules.

Indeed, one can prove that coherent modules M in this case are pseudocoherent, like in the case of Noetherian rings, and this property allows one to work with $\mathrm{RHom}_R(M, -)$ geometrically, as then it is stable under localisation and pullback. When M doesn't have this pseudocoherence property, these intuitive results break apart very quickly. To prove the results above, one shows the following lemma:

Lemma: An object $X \in \mathcal{A}$ is n-pseudocoherent if and only if there is a partial resolution

$$\bigoplus_{j=1}^{k_n} P_{n,j} \to \ldots \to \bigoplus_{j=1}^{k_i} P_{i,j} \to \ldots \to \bigoplus_{j=0}^{k_0} P_{0,j} \to X \to 0$$

where all $P_{i,j} \in \mathcal{A}_0$ are in the fixed set of compact projective generators, and the k_i are nonnegative integers. More precisely, given any shorter partial resolution of X of this form, it can be prolonged to a partial resolution of length n.

We sketch the proof. It is clear the existence of such resolutions implies pseudocoherent. One then shows for $n = 1$. Consider an arbitrary projective resolution:

$$\oplus_{j \in J_1} P_{1,j} \xrightarrow{\phi} \oplus_{j \in J_0} P_{0,j} \longrightarrow X \longrightarrow 0,$$

where $P_{i,j} \in \mathcal{A}_0$. For every finite subsets I_1, I_0 of J_1, J_0 respectively, with $\phi(I_1) \subset I_0$, consider the natural maps:

$$\begin{tikzcd}
\oplus_{j \in J_1} P_{1,j} \arrow[r] \arrow[d] & \oplus_{j \in J_0} P_{0,j} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, dashed] & 0 \\
\oplus_{j \in I_1} P_{1,j} \arrow[r] \arrow[u, bend right] & \oplus_{j \in I_0} P_{0,j} \arrow[r] \arrow[u, bend right] & X_I \arrow[r] \arrow[u, dashed, bend right] & 0
\end{tikzcd}$$

where the down arrows on the right side are splittings, and the dashed arrows are induced via the universal property. Taking colimits on the bottom over all subsets I_1, I_0, one obtains the identity morphism $\mathrm{id} : X \longrightarrow X$ being split by some morphism $X \longrightarrow X_I$, due to $\mathrm{Hom}_A(X, -)$ commuting with colimits. Hence, X is a retract of X_I, the latter of which is compact, and is therefore compact as well.

To show for arbitrary n, we begin by noting that X is at least $n-2$ psedocoherent, giving a resolution of length $n-2$. At the $n-2$ level, one takes the kernel, which is in fact 1 pseudocoherent, and uses the above result to extend the complex. In particular, the following 2 out of 3 properties are useful:

Lemma: Consider an exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$. If B, C are n-pseudocoherent, then A is $n-1$-pseudocoherent. If A, B are n-pseudocoherent, then C is n-pseudocoherent.

0.6. **The key lemma**

Let $\mathcal{A}$ be an abelian category as above. Let $X \in \mathcal{A}$ and assume given a complex

$$C : \ldots \to P_i \to \ldots \to P_0 \to X \to 0$$

$\mathcal{A}$ such that all $P_i \in \mathcal{A}$ are compact projective. If $H_i(C)$ is $n-i-1$ pseudocoherent for all $i = 0, \ldots, n-1$, then X is n-pseudocoherent.

This can be proved by the lemmas listed in the previous section.

0.7. **The proof**

The claim is that the functor Latt $\longrightarrow$ Ab taking $\psi : P \longmapsto P$ is pseudocoherent. We proceed by induction on it being n-pseudocoherent. The partial resolution $\mathbb{Z}[P^2] \longrightarrow \mathbb{Z}[P] \longrightarrow P \longrightarrow 0$ shows ψ is 1-pseudocoherent. Assume ψ is n-1-pseudocoherent. Consider the complex:

$$\mathbb{Z}[BP][-n] \longrightarrow P \longrightarrow 0.$$

By stable homotopy theory, $H_i\mathbb{Z}[BP][-n] = M_i \otimes_{\mathbb{Z}} P$, where M_i are finite abelian groups, so each $H_i(\mathbb{Z}[B\psi][-n])$ is n-1 pseudocoherent,[3] for $i = 0, \ldots, n-1$. We then conclude ψ is n-pseudocoherent, by the lemma above.

0.8. **References:**

1. *Derived functors of the divided power functor*, Lawrence Breen, Roman Mikhailov and Antoine Touze, 2014, arxiv:1312.5676.

2. *Lectures on Condensed Mathematics*, Peter Scholze, 2019

[3]To see this, we have a presentation $\mathbb{Z}^{n_{i_1}} \longrightarrow \mathbb{Z}^{n_{i_0}}$ for finite integers n_{i_0}, n_{i_1} of M_i, one then considers the constant functor c_{M_i} : Latt $\longrightarrow$ Ab. One remarks that our presentation gives a resolution of c_{M_i} by direct sums of the compact projective $h_0^{\mathbb{Z}}$, so c_{M_i} is pseudcoherent. By nonsense, $c_{M_i} \otimes \psi$ is $n-1$-pseudocoherent when ψ is.