Monopoles and Dehn twists on contact 3-manifolds

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Abstract

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In this dissertation, we study the isotopy problem for a certain three-dimensional contactomorphism which is supported in a neighbourhood of an embedded 2-sphere with standard characteristic foliation. The diffeomorphism which underlies it is the Dehn twist on the sphere, and therefore its square becomes smoothly isotopic to the identity. The main result of this dissertation gives conditions under which any iterate of the Dehn twist along a non-trivial sphere is not contact isotopic to the identity. This provides the first examples of exotic contactomorphisms with infinite order in the contact mapping class group, as well as the first examples of exotic contactomorphisms of 3-manifolds with $b_1 = 0$. The proof crucially relies on the construction of an invariant for families of contact structures in monopole Floer homology which generalises the Kronheimer–Mrowka–Ozsváth–Szabó contact invariant, together with the nice interaction between this families invariant and the U map in Floer homology. This is based on material that appeared in [63, 24].

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To Michael Zhao (1995–2018)

Chapter 1: Introduction

Throughout this dissertation, all the 3-manifolds we consider are assumed **closed, connected and oriented unless otherwise stated**. A (positive, co-oriented) *contact structure* on a 3-manifold Y is a co-oriented 2-plane field on Y which is *maximally non-integrable* in the following sense: for any 1-form α with $\xi = \ker\alpha$ we have that $\alpha \wedge d\alpha$ is a positive volume form on Y. A 1-form α such that $\xi = \ker\alpha$ as a co-oriented distribution is a called a *contact form* for ξ.

1.1 Main results

1.1.1 Dehn twists on contact 3-manifolds

A fundamental problem in contact topology is to understand the isotopy classes of contact diffeomorphisms (usually called *contactomorphisms*) of a contact manifold. The following is a longstanding open question in all dimensions:

Question 1.1. *Do there exist exotic contactomorphisms with infinite order as elements in the contact mapping class group?*

In this dissertation we answer this question in the *affirmative* in dimension *three*. We also provide the first known examples of exotic contactomorphisms of 3-manifolds with $b_1 = 0$.

We consider a contact 3-manifold given by the connected sum of two contact 3-manifolds $(Y_\#, \xi_\#) := (Y_-, \xi_-)\#(Y_+, \xi_+)$. Recall that the connected sum is built by removing Darboux balls $B_\pm \subset Y_\pm$ and gluing the complements $Y \setminus B_\pm$ by an orientation-reversing diffeomorphism of their boundary spheres which preserves their characteristic foliations. Reparametrisation of one of the spheres provides a $\mathrm{U}(1)$ worth of choices for gluing, and thus $(Y_\#, \xi_\#)$ naturally belongs in a *family* of contact 3-manifolds

$$(Y_\#, \xi_\#) \to \mathcal{Y}_\# \to \mathrm{U}(1).$$

The monodromy of this family is realised by a contactomorphism of $(Y_\#, \xi_\#)$, well-defined up to contact isotopy. This contactomorphism is a *local symmetry*, as it is supported in arbitrarily small neighbourhoods of the separating sphere $S_\#$ on the "neck" of the connected sum, and its underlying diffeomorphism is the usual *Dehn twist* on $S_\#$. We denote this contactomorphism $\tau_{S_\#}$ and call it the *contact Dehn twist* on $S_\#$. Because $\pi_1\mathrm{SO}(3) = \mathbb{Z}/2$ we have that the 2-fold iterate $\tau_{S_\#}^2$ is *smoothly* isotopic to the identity, but it remains the

Question 1.2. *Is $\tau_{S_\#}^2$ contact isotopic to the identity?*

Associated to the contact structures $\xi_\pm$ we have their Kronheimer–Mrowka contact invariants $\mathbf{c}(\xi_\pm) \in \widetilde{HM}_*(-Y_\pm)$ [47][46]. These are canonical elements (defined up to sign) in the "to" flavor of the monopole Floer homology of $-Y_\pm$. Below we provide some background on this. Our main result is:

Theorem 1.1. *Let $(Y_\pm, \xi_\pm)$ be irreducible contact 3-manifolds. Suppose that the Kronheimer–Mrowka contact invariants $\mathbf{c}(\xi_\pm; \mathbb{Q})$ (taken with coefficients in $\mathbb{Q}$) do not lie in the image of the U-map*

$$U : \widetilde{HM}_*(-Y_\pm; \mathbb{Q}) \to \widetilde{HM}_*(-Y_\pm; \mathbb{Q}).$$

Then

- *(A) The contact Dehn twist $\tau_{S_\#}^2$ is not contact isotopic to the identity and neither are its k-fold iterates $\tau_{S_\#}^k$ for any $k \neq 0$.*
- *(B) If the Euler classes of $\xi_\pm$ vanish, then $\tau_{S_\#}^2$ is formally contact isotopic to the identity.*

In other words, Theorem 1.1(A) asserts that $\tau_{S_\#}^2$ generates an infinite cyclic subgroup$\approx \mathbb{Z}$ of

$$\mathrm{Ker}\big(\pi_0\mathrm{Cont}(Y, \xi) \to \pi_0\mathrm{Diff}(Y)\big). \tag{1.1}$$

In turn, part (B) asserts that this contactomorphism is *exotic* (this is stronger than the statement that it is smoothly isotopic to the identity, see §2.1.3).

Remark 1.1. In fact, we establish a stronger result: the contactomorphism $\tau^2_{S_\#}$ has infinite order as an element in the *abelianisation* of the group (1.1).

Remark 1.2. For comparison with Theorem 1.1, whenever either of $(Y_\pm, \xi_\pm)$ is the tight $S^1 \times S^2$ or a quotient of tight (S^3, ξ) (e.g. the lens spaces $L(p,q)$ or the Poincaré sphere $\Sigma(2,3,5)$) then the squared contact Dehn twist $\tau^2_{S_\#}$ of $(Y_\#, \xi_\#)$ is contact isotopic to the identity, see Lemmas 3.10-3.11.

Remark 1.3. We also note that the conclusion of item (B) of Theorem 1.1 does not use the assumptions that $Y_\pm$ are irreducible or the condition $\mathbf{c}(\xi) \notin \mathrm{Im}U$.

A crucial step towards Theorem 1.1 is the following *relative* version of it. We consider a *Darboux ball* B of a contact manifold (Y, ξ_{st})). That means that B is the image of a contact embedding $(\mathbb{B}^3, \xi = \ker(dz - ydx)) \hookrightarrow (Y, \xi)$ of the standard unit contact 3-ball. Let $(\mathring{Y}, \xi)$ be the compact manifold with boundary obtained from Y by removing B. Then $(\mathring{Y}, \xi)$ is a contact manifold with *convex sphere boundary*. There is a contact Dehn twist along a sphere *parallel to the boundary* of $\mathring{Y}$ which we denote $\tau_{\partial B} \in \pi_0\mathrm{Cont}(\mathring{Y}, \xi)$, where $\mathrm{Cont}(\mathring{Y}, \xi)$ stands for the group of contactomorphisms of $\mathring{Y}$ which *restrict to the identity on the boundary*. We have the following

Theorem 1.2. *Suppose (Y, ξ) is an irreducible contact 3-manifold and that $\mathbf{c}(\xi; \mathbb{Q}) \notin \mathrm{Im}U$. Then*

(A) The contact Dehn twist $\tau^2_{\partial B}$ is not contact isotopic to the identity rel. $\partial\mathring{Y}$ and neither are its k-fold iterates $\tau^k_{\partial B}$ for any $k \neq 0$.

(B) If the Euler class of ξ vanishes (over the closed manifold Y), then $\tau^2_{\partial B}$ is formally contact isotopic to the identity rel. $\partial\mathring{Y}$.

Going beyond irreducible 3-manifolds or sums of two irreducible 3-manifolds we have the following result. Let (Y, ξ) be a *tight* 3-manifold. By a classical result of Colin [10] (see also [42, 13]) we have a unique connected sum decomposition

$$(Y, \xi) \cong (Y_0, \xi)\# \cdots \#(Y_N, \xi_N)$$

into tight contact 3-manifolds (Y_j, ξ_j), where each piece Y_j is a prime 3-manifold. Let $n+1 \leq N$ be the number of prime summands (Y_j, ξ_j) such that $\mathbf{c}(\xi_j; \mathbb{Q}) \notin \mathrm{Im} U$ and the Euler class of ξ_j vanishes. Let $C(Y, \xi)$ (resp. $\Xi(Y, \xi)$) be the space of contact structures (resp. co-oriented 2-plane fields) on Y in the path-component of ξ.

Theorem 1.3. *With (Y, ξ) as above, when $n \geq 1$ there is a $\mathbb{Z}^n$ subgroup in the kernel of*

$$\pi_1 C(Y, \xi) \to \pi_1 \Xi(Y, \xi)$$

which induces a $\mathbb{Z}^n$ summand in the first singular homology $H_1\big(C(Y, \xi); \mathbb{Z}\big)$.

In particular, we can give examples (see §1.2 below) where the exotic summand $\mathbb{Z}^n$ exhibited in Theorem 1.3 is arbitrarily large.

We also note that the n homologically independent loops of contact structures that we detect in Theorem 1.3 yield under the natural map

$$\pi_1 C(Y, \xi) \to \pi_0 \mathrm{Cont}(Y, \xi)$$

the contact Dehn twists on each of the n spheres which separate the $n+1$ prime summands (Y_j, ξ_j). However, we are unable to establish that the corresponding (squared) Dehn twists are non-trivial when $n \geq 2$. See Remark 3.5.

1.1.2 Monopole invariants for families of contact structures

The technical core of the dissertation is the construction of an invariant for families of contact structures on a 3-manifold using the monopole Floer homology groups.

1.1.2.1 Monopole Floer homology and the contact invariant

We provide some basic background for the results in this section. For a quick introduction to Kronheimer and Mrowka's monopole Floer homology groups we recommend [51, 46] and for a detailed treatment the monograph [49]. Here we just comment briefly on a few formal aspects.

Consider a pair $(Y, \mathfrak{s})$ consisting of a 3-manifold Y together with spin-c structure $\mathfrak{s}$ (in this dissertation the only spin-c structure that will be relevant is that induced by a contact structure ξ, denoted $\mathfrak{s}_\xi$). Associated to $(Y, \mathfrak{s})$ there are various *monopole Floer homology groups*, which are modules over a chosen commutative unital ring R (which we may hide from the notation when not essential). The ones relevant to us are the "to" and "tilde" flavors: $\widecheck{HM}_*(Y, \mathfrak{s})$ and $\widetilde{HM}_*(Y, \mathfrak{s})$. The former arises "formally" as the S^1-equivariant Morse homology of the Chern–Simons–Dirac functional. An algebraic manifestation of this equivariant nature is that $\widecheck{HM}_*(Y, \mathfrak{s})$ carries a module structure over the polynomial algebra $R[U]$ (i.e. the S^1-equivariant cohomology of a point, $H^*_{S^1}(\text{point}) = R[U]$) and U decreases grading by two. In turn, the "tilde" flavor should be regarded as the (non-equivariant) Morse homology, and thus is an $H_*(S^1) = R[\chi]/(\chi^2)$-module, with χ raising degree by one. A standard *Gysin exact triangle* relates the two groups:

$$\cdots \xrightarrow{p} \widecheck{HM}_*(Y, \mathfrak{s}) \xrightarrow{U} \widecheck{HM}_{*-2}(-Y, \mathfrak{s}) \xrightarrow{j} \widetilde{HM}_{*-1}(-Y, \mathfrak{s}) \xrightarrow{p} \cdots$$

and the map χ is recovered from this by $\chi = jp$. A common feature of all flavors of the monopole groups of $(Y, \mathfrak{s})$ is a canonical grading by the set of homotopy classes of co-oriented plane fields ξ inducing the spin-c structure $\mathfrak{s}$, which we denote $\pi_0\Xi(Y, \mathfrak{s})$ and which carries a natural $\mathbb{Z}$-action. When $c_1(\mathfrak{s})$ is torsion, then there is a natural $\mathbb{Z}$-equivariant map $\pi_0\Xi(Y, \mathfrak{s}) \to \mathbb{Q}$ which leads to an absolute $\mathbb{Q}$-grading on the monopole Floer groups of $(Y, \mathfrak{s})$.

More generally, extending the $R[U]$-module structure we have that $\widecheck{HM}_*(-Y, \mathfrak{s})$ is a module over the graded R-algebra

$$\mathbb{A}(R) = R[U] \otimes_{\mathbb{Z}} \Lambda^*\big(H_1(Y; \mathbb{Z})/\text{torsion}\big) \tag{1.2}$$

where $H_1(Y; \mathbb{Z})/\text{torsion}$ lowers degree by 1.

The *contact invariant* $\mathbf{c}(\xi)$ is an element of $\widecheck{HM}_{[\xi]}(-Y, \mathfrak{s}_\xi)$ which is well-defined up to a sign, and is canonically attached to a contact structure ξ on Y. It was defined by Kronheimer, Mrowka, Ozsváth and Szabó in [46], but its definition goes back essentially to the earlier paper [47]. Ozsváth and Szabó gave a definition of $\mathbf{c}(\xi)$ in Heegaard-Floer homology [66]. Under the isomorphism between the monopole and Heegaard-Floer groups [50, 11] the contact invariants are shown to agree. The invariant $\mathbf{c}(\xi)$ enjoys several nice properties, a few of which are:

- $\mathbf{c}(\xi) = 0$ if (Y, ξ) is overtwisted [62, 66]
- $\mathbf{c}(\xi; R) \neq 0$ for $R = \mathbb{Q}$ and $\mathbb{Z}/2$ if (Y, ξ) admits a strong symplectic filling [67, 17]
- $\mathbf{c}(\xi, \mathbb{Z}/2)$ is natural under symplectic cobordisms [17] (see also [62, 55]): if (W, ω) is a symplectic cobordism $(Y_1, \xi_1) \rightsquigarrow (Y_2, \xi_2)$ (here the convex end is (Y_2, ξ_2)) then

$$\widetilde{HM}(-W, \mathfrak{s}_\omega; \mathbb{Z}/2)\mathbf{c}(\xi_2; \mathbb{Z}/2) = \mathbf{c}(\xi_1; \mathbb{Z}/2)$$

- $U \cdot \mathbf{c}(\xi) = 0$ (this is clear from the Heegaard-Floer point of view [66]; in the monopole case this follows as a particular case of our Theorem 1.5 below).

1.1.2.2 Motivating question

We discuss first the motivation for our construction. Let (Y, ξ) be a contact 3-manifold and $p \in Y$ be a chosen point. Consider the *evaluation map*

$$ev : C(Y, \xi) \to S^2 \tag{1.3}$$

which sends a contact structure ξ' to its plane $\xi'(p)$ at the point p, with S^2 regarded as the space of co-oriented 2-planes in $T_pY \approx \mathbb{R}^3$. The map ev is a *fibration*. If $B \subset (Y, \xi)$ is a Darboux ball centered at p, the fibre of ev is homotopy equivalent to the subspace $C(Y, \xi, B) \subset C(Y, \xi)$ consisting of contact structures which agree with ξ over B (i.e. those contact structures which look like the standard one $dz - ydx$ over the ball B). We ask the following

Question 1.3. *When does the evaluation map $ev : C(Y, \xi) \to S^2$ admit a homotopy section ? i.e. a map $s : S^2 \to C(Y, \xi)$ such that $ev \circ s$ is homotopic to the identity.*

We will see that this lifting problem is closely tied with the isotopy problem for the contact Dehn twist considered above. As an application of our invariant for families of contact structures we can obstruct the existence of a section of the evaluation map:

Theorem 1.4. *If* $ev : C(Y,\xi) \to S^2$ *admits a homotopy section, then* $\mathbf{c}(\xi;\mathbb{Q}) \in \mathrm{Im}U$.

1.1.2.3 The families contact invariant

We now describe the formal properties of our invariant for families of contact structures. Throughout we fix a coefficient ring R which we assume is commutative and unital. The most basic version of our families invariant is a map of R-modules

$$\mathbf{Fc} : H_*(C(Y,\xi);\Lambda_R) \to \widetilde{HM}_*(-Y,\mathfrak{s}_\xi;R)$$

where $H_*(C(Y,\xi);\Lambda_R)$ is the singular homology group of $C(Y,\xi)$ with coefficients in a certain *local system* Λ_R of free R-modules of rank 1 over the space $C(Y,\xi)$. We have $\Lambda_R = \Lambda_{\mathbb{Z}} \otimes_{\mathbb{Z}} R$ where $\Lambda_{\mathbb{Z}}$ is the local system of $\mathbb{Z}$-modules associated to the determinant line bundle of certain family of Fredholm operators parametrised by $C(Y,\xi)$ (see Definition 4.15). In particular, if the characteristic of R is two, then the local system Λ_R is trivial.

If we choose one of the two generators of the $\mathbb{Z}$-module $\Lambda_{\mathbb{Z}}(\xi)$ given by the fiber of $\Lambda_{\mathbb{Z}}$ over the point ξ, then this fixes the sign of the usual contact invariant $\mathbf{c}(\xi;R) \in \widetilde{HM}_*(-Y,\mathfrak{s}_\xi;R)$. In addition, it also picks out a preferred generator, denoted by 1_R, for the R-module $H_0(C(Y,\xi);\Lambda_R)$. The element $1_{\mathbb{Z}}$ is either non-torsion or has order two, according as to whether the local system $\Lambda_{\mathbb{Z}}$ is trivial over $C(Y,\xi)$ or not, respectively.

In analogy with the monopole Floer homology groups, we will see that $H_*(C(Y,\xi);\Lambda_R)$ can also be endowed with a natural module structure over the graded algebra $\mathbb{A}(R)$ given in (1.2). In particular, the action of U on $H_*(C(Y,\xi);\Lambda_R)$ is defined in terms of the evaluation map (1.3) and the usual cap product

$$U : H_*(C(Y,\xi);\Lambda_R) \to H_{*-2}(C(Y,\xi);\Lambda_R) \quad T \mapsto T \cap ev^*([S^2]^\vee).$$

We refer to Definition 5.5 for the full action of $\mathbb{A}(R)$ on $H_*(C(Y,\xi);R)$.

Remark 1.4. It follows that $U^2 = 0$ on $H_*(C(Y,\xi);\Lambda_R)$, so the latter is really a module over the graded R-algebra

$$R[U]/(U^2) \otimes_{\mathbb{Z}} \Lambda^*\big(H_1(Y;\mathbb{Z})/\text{torsion}\big).$$

The main technical tool that we develop in this dissertation is the following

Theorem 1.5. *There exists a "families contact invariant" given by a collection of R-module maps*

$$\mathbf{Fc} : H_j(C(Y,\xi);\Lambda_R) \to \widetilde{HM}_{[\xi]+j}(-Y,\mathfrak{s}_\xi;R) \quad , \quad j \geq 0 \tag{1.4}$$

which are natural with respect to orientation preserving diffeomorphisms and satisfy the following properties:

(A) The $j = 0$ *map recovers the usual contact invariant:* $\mathbf{Fc}(1_R) = \mathbf{c}(\xi;R)$.

(B) $\mathbf{Fc}$ *is a map of graded* $\mathbb{A}(R)$*-modules:* $\mathbf{Fc}(a \cdot T) = a \cdot \mathbf{Fc}(T)$ *for* $a \in \mathbb{A}(R)$ *and* $T \in H_*(C(Y,\xi);\Lambda_R)$.

Remark 1.5. *Naturality.* The above assertion on naturality has the following meaning. Let f be an orientation-preserving diffeomorphism of Y, and let ξ_1 be the contact structure obtained by pulling back another one ξ_0, $f^*\xi_0 = \xi_1$. By pulling back we have a homeomorphism $F = f^* : C(Y,\xi_0) \xrightarrow{\cong} C(Y,\xi_1)$. The assertion is that then there is a canonical isomorphism of local systems $\eta : \Lambda_{\mathbb{Z}} \xrightarrow{\cong} F^*\Lambda_{\mathbb{Z}}$ such that the following diagram (where the vertical arrows are isomorphisms) commutes

$$\begin{array}{ccc} H_*(C(Y,\xi_0);\Lambda_R) & \xrightarrow{\mathbf{Fc}_0} & \widetilde{HM}_*(-Y,\mathfrak{s}_{\xi_0};R) \\ \Big\downarrow{\scriptstyle (F,\eta)_*} & & \Big\downarrow{\scriptstyle f_*} \\ H_*(C(Y,\xi_1);\Lambda_R) & \xrightarrow{\mathbf{Fc}_1} & \widetilde{HM}_*(-Y,\mathfrak{s}_{\xi_1};R). \end{array}$$

Remark 1.6. *Criterion for triviality of* Λ_R. It is unclear to the author whether Λ_R can be non-trivial. However, a simple criterion is available:

Corollary 1.6. *Suppose the contact invariant* $\mathbf{c}(\xi;\mathbb{Z}) \in \widetilde{HM}(-Y,\mathfrak{s}_\xi;\mathbb{Z})$ *is not* 2*-torsion, i.e.* $2\mathbf{c}(\xi;\mathbb{Z}) \neq 0$*. Then* $\Lambda_\mathbb{Z}$ *is trivial.*

Proof. By Theorem 1.5(A) it follows that $\mathbf{Fc}(1_\mathbb{Z})$ is not 2-torsion, and hence that $H_0(C(Y,\xi);\Lambda_\mathbb{Z})$ is isomorphic to $\mathbb{Z}$ rather than $\mathbb{Z}/2\mathbb{Z}$. Hence $\Lambda_\mathbb{Z}$ is trivial. □

This criterion applies in many cases of interest. For instance, whenever the contact structure admits a *strong symplectic filling*, in which case one has $\mathbf{c}(\xi;\mathbb{Q}) \neq 0$ already [67].

Remark 1.7. *Sign-ambiguity*. Even if the local system $\Lambda_\mathbb{Z}$ over $C(Y,\xi)$ is trivial, there is no canonical choice of generator of the $\mathbb{Z}$-module $\Lambda_\mathbb{Z}(\xi)$ for a given contact structure ξ. In fact, Lin–Ruberman–Saveliev [53] show that there is no way of fixing the sign so that the usual contact invariant $\mathbf{c}(\xi)$ becomes natural with respect to orientation-preserving diffeomorphisms of Y. Indeed, they show that the unique tight contact structure on $Y = -\Sigma(2,3,7)$ admits a contactomorphism f which reverses the sign of $\mathbf{c}(\xi;\mathbb{Z})$ (i.e. $f_*\mathbf{c}(\xi;\mathbb{Z}) = -\mathbf{c}(\xi;\mathbb{Z})$). We also note that the local system $\Lambda_\mathbb{Z}$ is trivial in this example, because this contact structure has a strong symplectic filling.

1.1.2.4 The U-map and families of contact structures

We now describe a refinement of Theorem 1.5 in the case of the action of $U \in \mathbb{A}(R)$. For the remainder of §1.1.2 we assume that the local system $\Lambda_\mathbb{Z}$ over $C(Y,\xi)$ is *trivial* (recall once more the criterion which ensures this, Corollary 1.6) and fix a trivialization (i.e. a choice of generator of the $\mathbb{Z}$-module $\Lambda_\mathbb{Z}(\xi)$) so that the families invariant gives a map

$$\mathbf{Fc} : H_*(C(Y,\xi)) \to \widetilde{HM}_*(-Y,\mathfrak{s}_\xi).$$

Going back to our motivating Question 1.3, observe that the existence of a homotopy section of ev is equivalent to the surjectivity of the *degree map* $\deg := ev_* : \pi_2 C(Y,\xi) \to \pi_2 S^2 = \mathbb{Z}$. The latter is defined also at the level of homology and Theorem 1.5(B) gives us the following

Formula 1.1. *If $T \in H_2(C(Y,\xi))$, then $U \cdot \mathbf{Fc}(T) = \deg(T) \cdot \mathbf{c}(\xi)$.*

Proof of Theorem 1.3. If $\mathbf{c}(\xi;\mathbb{Q}) = 0$ the statement becomes trivial. If $\mathbf{c}(\xi;\mathbb{Q}) \neq 0$ then $\Lambda_{\mathbb{Z}}$ is trivial by Corollary 1.6. A homotopy section $s : S^2 \to C(Y,\xi)$ of ev would yield a family $T := s_*[S^2] \in H_2(C(Y,\xi);\mathbb{Q})$ with $\deg(T) = 1$. Then by Formula 1.1 we have $\mathbf{c}(\xi;\mathbb{Q}) = U \cdot \mathbf{Fc}(T) \in \mathrm{Im} U$. □

Going beyond Question 1.3, one could ask how the homotopy type of the space $C(Y,\xi)$ differs from that of $C(Y,\xi,B)$. Often the latter has "simpler" topology. For example, for the tight contact structure ξ on S^3 one has $C(S^3,\xi) \simeq \mathrm{U}(2)$ whereas $C(S^3,\xi,B) \simeq \{*\}$ [22]. At the homological level, the passage from $C(Y,\xi,B)$ to $C(Y,\xi)$ amounts to understanding how cycles in the total space of the fibration ev intersect with the fibres, and this is encoded into the *Wang exact triangle* for the fibration (1.3) (easily assembled from the Serre spectral sequence)

$$\cdots \longrightarrow H_*(C(Y,\xi)) \xrightarrow{U_B} H_{*-2}(C(Y,\xi,B)) \xrightarrow{\chi} H_{*-1}(C(Y,\xi,B)) \xrightarrow{\iota_*} \cdots$$

In geometric terms, the map U_B acts on a generic cycle in $C(Y,\xi)$ by taking its intersection with the fibre of (1.3), and ι_* is the inclusion of the fibre. The map χ is the differential in the E^2 page of the spectral sequence. The map $H_*(C(Y,\xi)) \xrightarrow{U} H_{*-2}(C(Y,\xi))$ defined earlier can be recovered from the diagram above as the composition $U = \iota_* \circ U_p$.

On the Seiberg–Witten gauge-theory side one can find a structure analogous to the evaluation map $ev : C(Y,\xi) \to S^2$. The space of *irreducible* configurations modulo gauge transformations $\mathcal{B}^*(Y,\mathfrak{s}_\xi)$ also carries a partially-defined evaluation map

$$\mathcal{B}^*(Y,\mathfrak{s}_\xi) \dashrightarrow \mathbb{P}(S_p) \cong \mathbb{C}P^1 = S^2 \tag{1.5}$$

which assigns to the class of a configuration (B,Ψ) the complex line in the spinor bundle fibre $S_p \approx \mathbb{C}^2$ spanned by Ψ at the point p. The relevance of this evaluation map is its close relation with U-action on the Floer theory. Indeed, the action of U on the Floer homology is defined as a sort of cap product with the first Chern class of a canonical complex line bundle $\mathcal{U} \to \mathcal{B}^*(Y, \mathfrak{s}_\xi)$, with (1.5) arising as the map to $\mathbb{C}P^1$ determined by a certain "pencil" of hyperplanes in the class of the line bundle $\mathcal{U}$. Thus, resembling the contact case, this operation corresponds geometrically to taking intersections of moduli of Floer trajectories with the fibres of (1.5). Similarly to the Wang long exact sequence, we on the Floer theory we have the *Gysin exact triangle* (see §1.1.2.1). The connection between the two evaluation maps (1.3) and (1.5) is seen by a certain map which assigns canonical irreducible configurations to contact structures

$$f : C(Y, \xi) \longrightarrow \mathcal{B}^*(Y, \mathfrak{s}_\xi).$$

Under the familiar identification $S^2 = \mathbb{C}P^1$ coming from spin geometry, the map f intertwines our two evaluation maps (1.3) and (1.5). On a heuristic level, one should regard the families contact invariant **Fc** as the "map induced by f in homology", with Floer homology interpreted as the middle dimensional homology of $\mathcal{B}^*(Y, \mathfrak{s}_\xi)$ (one should be able to formalise this by working at the level of spectra, but we don't pursue this direction in this dissertation). At this point, Theorem 1.5(B) and the following refinement should be regarded as algebraic manifestations of the basic phenomenon just described:

Theorem 1.7. *Associated to any closed contact* 3*-manifold* (Y, ξ) *with trivial local system* $\Lambda_\mathbb{Z}$ *there is a natural diagram which is commutative (up to signs)*

$$\begin{array}{ccccccccc}
\xrightarrow{p} & \widetilde{HM}_*(-Y, \mathfrak{s}_\xi) & \xrightarrow{U} & \widetilde{HM}_{*-2}(-Y, \mathfrak{s}_\xi) & \xrightarrow{j} & \widetilde{HM}_{*-1}(-Y, \mathfrak{s}_\xi) & \xrightarrow{p} \\
& \mathbf{Fc}\big\uparrow & & \mathbf{Fc}\cdot\iota_*\big\uparrow & & \widetilde{\mathbf{Fc}}\big\uparrow & \\
\xrightarrow{\iota_*} & H_*(C(Y, \xi)) & \xrightarrow{U_B} & H_{*-2}(C(Y, \xi, B)) & \xrightarrow{\chi} & H_{*-1}(C(Y, \xi, B)) & \xrightarrow{\iota_*}
\end{array}$$

where the top row is the Gysin exact triangle, the bottom row is the Wang exact triangle of the fibration (1.3) and $\widetilde{\mathbf{Fc}}$ *is another "families contact invariant" that we construct in §5.3.*

Some observations are in order:

- As a particular case, Theorem 1.7 recovers a property about the contact invariant $\mathbf{c}(\xi)$ which is well-known from the Heegaard–Floer point of view: that $U \cdot \mathbf{c}(\xi) = 0$ and we have a canonical element $\widetilde{\mathbf{c}}(\xi) := \widetilde{\mathbf{Fc}}(1) \in \widetilde{HM}_{[\xi]}(-Y, \mathfrak{s}_\xi)$ such that $p\widetilde{\mathbf{c}}(\xi) = \mathbf{c}(\xi)$. Conjecturally, the invariant $\widetilde{\mathbf{c}}(\xi)$ corresponds to the Heegaard–Floer contact invariant that takes values in $\widehat{HF}(-Y, \mathfrak{s}_\xi)$ defined in [66].

- Recall that $\widetilde{HM}_*(-Y, \mathfrak{s}_\xi)$ is an $R[\chi]/(\chi^2)$ module, and so it $H_*(C(Y, \xi, B))$. It follows from Theorem 1.7 that the invariant $\widetilde{\mathbf{Fc}}$ is a map of $R[\chi]/(\chi^2)$ modules:

$$\widetilde{\mathbf{Fc}} \cdot \chi = \chi \cdot \widetilde{\mathbf{Fc}}.$$

  In particular, we deduce from this and the diagram that

$$\mathbf{c}(\xi) \in \mathrm{Im}U \text{ \textit{if and only if} } \chi\widetilde{\mathbf{c}}(\xi) = 0.$$

1.2 Examples

1.2.1 Elementary examples

We first discuss some simple examples where $\mathbf{c}(\xi) \in \mathrm{Im}U$.

Example 1.1. *ADE singularities.* Consider the flat hyperkähler structure (g, I_1, I_2, I_3) on $\mathbb{R}^4$. The radial vector field $v = x\partial_x + y\partial_y + z\partial_z + w\partial_w$ in $\mathbb{R}^4$ is Liouville for all symplectic structures in the family $\omega_t = \sum_{i=1}^3 t_i g(I_i \cdot, \cdot)$ parametrised by $t \in S^2$ (i.e. $\mathcal{L}_v \omega_t = \omega_t$) and v is transverse to $S^3 \subset \mathbb{R}^4$. Thus there is a family of contact forms α_t on S^3 given by $\alpha_t = \iota_v \omega_t$ which provides a section of ev on tight S^3. Since this family of contact structures is SU(2)-invariant, we have also constructed a section of ev on the quotients of tight S^3 by a finite subgroup $\Gamma \subset \mathrm{SU}(2)$. The contact manifolds S^3/Γ are precisely the the links of the ADE singularities (which include e.g. the lens spaces $L(p, p-1)$ or the Poincaré sphere $\Sigma(2, 3, 5)$). Let ξ be any contact structure in the S^2-family $\xi_t = \ker\alpha_t$. We have[1] $\widetilde{HM}_*(-S^3/\Gamma, \mathfrak{s}_\xi; \mathbb{Z}) \cong \mathbb{Z}[U, U^{-1}]/U\mathbb{Z}[U]$ and $\mathbf{c}(\xi) = 1$. If T denotes the S^2-family of contact structures given by the ξ_t then from Theorem 1.5(B) we have $\mathbf{Fc}(T) = U^{-1}$, and $U \cdot \mathbf{Fc}(T) = \mathbf{c}(\xi)$.

Example 1.2. *Tight* $S^1 \times S^2$. Let ξ_{tight} be a tight contact structure on $S^1 \times S^2$. We consider two families of contact structures in $C(Y, \xi_{\text{tight}})$. First, consider the family $T_2 \in H_2(C(S^1 \times S^2, \xi_{\text{xi}})$ of contact structures ξ_t parametrised by $t \in S^2$ given by the kernels of $\alpha_t = \sum_{i=1}^3 t_i\alpha_i$ where

$$\alpha_1 = zd\theta + xdy - ydx \ , \ \alpha_2 = xd\theta + ydz - zdy \ , \ \alpha_3 = yd\theta + zdx - xdz.$$

It is a simple exercise to check that this family provides a section for the evaluation map. Secondly, consider the family $T_1 \in H_1(C(S^1 \times S^2, \xi_{\text{tight}})$ given by the following loop ξ_s of contact structures. Let R_θ be the three-dimensional rotation in the xy plane by θ angles. The loop $\theta \in S^1 \mapsto R_{2\theta}$ represents the trivial element in $\pi_1\mathrm{SO}(3) = \mathbb{Z}/2$ and there is (up to homotopy) a unique homotopy $h : S^1 \times [0, 1] \to \mathrm{SO}(3)$ from the constant loop to it. For $s \in [0, 1]$ let $r_s \in \mathrm{Diff}(S^1 \times S^2)$ be given by $r_s(\theta, x, y, z) = (\theta, h(\theta, s)(x, y, z))$. If we set $\xi_s = (r_s)_*\xi_1$ then this defines a loop since $\tau := r_1$ is a contactomorphism of ξ_1 (the *squared contact Dehn twist* on $\{0\} \times S^2 \subset (S^1 \times S^2, \xi_1)$).

As a $\mathbb{Z}[U]$ module we have

$$\widetilde{HM}_*(-S^1 \times S^2, \mathfrak{s}_{\xi_{\text{tight}}}) \cong \mathbb{Z}[U, U^{-1}]/U\mathbb{Z}[U] \otimes_{\mathbb{Z}} H_*(S^1; \mathbb{Z})$$

where we denote by v the generator of $H_1(S^1; \mathbb{Z})$. We have $\mathbf{c}(\xi_{\text{tight}}) = 1$. The action of $[S^1] \in H_1(S^1 \times S^2; \mathbb{Z})/\text{torsion} = \mathbb{Z}$ on Floer homology is given by $a \otimes v^{2j+1} \mapsto a \otimes v^{2j}$ for $j \geq 0$ and zero otherwise. We then have $\mathbf{Fc}(T_2) = U^{-1}$ and $U \cdot \mathbf{Fc}(T) = \mathbf{c}(\xi_{\text{tight}})$. In turn, for the family T_1 one can calculate using Definition 5.5 that $[S^1] \cdot T_1 = 2 \in \mathbb{Z} = H_0(C(S^1 \times S^2, \xi_{\text{tight}}))$, from which it follows

[1]The absolute $\mathbb{Q}$-gradings in Floer homology for the examples in this section are taken **shifted** so that the contact invariant $\mathbf{c}(\xi)$ is in degree 0. Also, all identities involving contact invariants are understood to hold **up to signs**.

by Theorem 1.5(B) that $\mathbf{Fc}(T_1) = 2v$. We also have

$$\widetilde{HM}_*(-S^1 \times S^2) \cong \mathbb{Z}_{(0)} \oplus \mathbb{Z}_{(1)}.$$

Because of the surjectivity of the degree map, we have $H_1(C(S^1 \times S^2, \xi_{\text{tight}}, B)) \cong H_1(C(S^1 \times S^2, \xi_{\text{tight}}))$. Then $\widetilde{\mathbf{c}}(\xi_{\text{tight}})$ generates the summand $\mathbb{Z}_{(0)}$ and $\widetilde{\mathbf{Fc}}(T_1)$ generates $2 \cdot \mathbb{Z}_{(1)} \subset \mathbb{Z}_{(1)}$.

Example 1.3. (*Evaluation of* 2*-plane fields*) We can compare Question 1.3 with the corresponding one at the level of co-oriented plane fields. In this case, the natural evaluation map $\Xi(Y, \xi) \to S^2$ on co-oriented 2-plane fields (also a fibration) admits a section as long as the Euler class of ξ vanishes. Indeed, in this case we may identify $\Xi(Y, \xi)$ with the space $\text{Map}_0(Y, S^2)$ of null-homotopic smooth maps $Y \to S^2$. The evaluation mapping becomes identified with the obvious evaluation mapping on this latter space; and clearly this fibration admits a section given by the constant maps $Y \to S^2$. This is the reason why we have the formal triviality property in Theorem 1.1(B) and Theorem 1.2(B).

The homotopy type of $\Xi(Y, \xi)$ is often well-understood. For instance, whenever Y is an integral homology S^3 then $\Xi(Y, \xi) \simeq \text{Map}_0(S^3, S^2)$ [37].

1.2.2 Examples with $\mathbf{c}(\xi) \notin \text{Im}U$

We now give examples of irreducible contact 3-manifolds (Y, ξ) such that $\mathbf{c}(\xi) \notin \text{Im}U$, many of which also have vanishing Euler class.

Example 1.4. (Links of singularities) The simplest example is the Brieskorn sphere

$$\Sigma(p, q, r) = \left\{(x, y, z) \in \mathbb{C}^3 \mid x^p + y^q + z^r = 0 \text{ and } |x|^2 + |y|^2 + |z|^2 = \epsilon\right\}$$

where $\epsilon \in \mathbb{R}_{>0}$ is small and $p, q, r \geq 1$ are integers with $1/p + 1/q + 1/r < 1$, equipped with the contact structure ξ_{sing} induced from the Brieskorn singularity. More generally, we could take any isolated normal surface singularity germ (X, o) and let (Y, ξ_{sing}) be the contact manifold arising as the *link* of the singularity. Neumann [65] proved that the 3-manifold Y is irreducible. Provided that Y is also a rational homology sphere, then the following are equivalent statements, as proved by Bodnár–Plamenevskaya [4] and Némethi [64]:

(a) $\mathbf{c}(\xi_{\text{sing}}) \notin \text{Im}U$

(b) Y is not an L-space

(c) (X, o) is not a rational singularity.

For instance, all Seifert fibered integral homology spheres excluding S^3 or the Poincaré sphere $\Sigma(2,3,5)$ carry a contact structure ξ_{sing} of this sort. Indeed the Seifert fibered integral homology spheres are given by the manifolds $\Sigma(p_1, p_2, \ldots, p_n)$, where $p_i \geq 2$ are pairwise coprime integers and $n \geq 3$. The manifold $\Sigma(p_1, p_2, \ldots, p_n)$ is the link of the weighted-homogeneous isolated singularity $f_1 = \ldots = f_{n-2} = 0$ with $f_j = \sum a_{ij} x_i^{p_i}$ for sufficiently general coefficients $a_{ij} \in \mathbb{C}$. By [54, 61] none of these are L-spaces, except the Poincaré sphere $\Sigma(2,3,5)$.

To spell out one concrete example, for the Brieskorn sphere $\Sigma(2,3,7)$ we have

$$\widecheck{HM}_*(-\Sigma(2,3,7), \mathfrak{s}_{\xi_{\text{sing}}}) \cong \mathbb{Z} \oplus \mathbb{Z}[U, U^{-1}]/U\mathbb{Z}[U]$$

and the U action is trivial on the $\mathbb{Z}$ summand. In this case one has $\mathbf{c}(\xi) = 1 \in \mathbb{Z}$ and hence $\mathbf{c}(\xi) \notin \text{Im}U$. As a $\mathbb{Z}[\chi]/(\chi^2)$ module we have

$$\widetilde{HM}_*(-\Sigma(2,3,7), \mathfrak{s}_{\xi_{\text{sing}}}) \cong \mathbb{Z}[\chi]/(\chi^2)$$

with $\widetilde{\mathbf{c}}(\xi_{\text{sing}}) = 1$ and $\widetilde{\mathbf{Fc}}(O_{\xi_{\text{sing}}}) = \chi$, where $O_{\xi_{\text{sing}}} := \chi \cdot [\xi_{\text{sing}}] \in H_1(C(\Sigma(2,3,7), \xi_{\text{sing}}, B))$.

Example 1.5. Several surgeries on the Figure Eight knot are hyperbolic (hence irreducible) and support contact structures with $\mathbf{c}(\xi) \notin \text{Im}U$. Contact structures on these manifolds have been classified by Conway and Min [12].

Example 1.6. All but one of the $\frac{n(n-1)}{2}$ tight contact structures supported on $-\Sigma(2,3,6n-1)$ up to isotopy, classified by Ghiggini and Van Horn-Morris [28].

1.2.3 Exotic overtwisted phenomena in 1-parametric families

Let (Y, ξ) be such that $\mathbf{c}(\xi) \notin \mathrm{Im}U$ and ξ has vanishing Euler class. Let $B \subset (Y, \xi)$ be a Darboux ball. From this, one can produce overtwisted contact manifolds by modifying (Y, ξ) by a Lutz Twist inside B, or by taking the connected sum (using B) with an overtwisted contact manifold (M, ξ_{ot}). In either case, the squared contact Dehn twist on the boundary of B becomes isotopic to the identity in this new overtwisted manifold, by an application of Eliashberg's h-principle for overtwisted contact structures [19]. However, this has surprising implications (see §3.3 for the precise statement)

Proposition 1.8. *(A) There exist overtwisted contact* 3*-manifolds that have an exotic loop of Lutz Twist embeddings.*

(B) There exist overtwisted contact 3*-manifolds that have an exotic loop of standard sphere embeddings.*

In other words, (A) says that the h-principle for codimension 0 isocontact embeddings of S^1-embedded families of overtwisted disks fails in 1-parametric families, see [36, 20]. To the best of our knowledge this is the first example of this nature. On the other hand, (B) says that the h-principle for standard spheres [23] in tight contact 3-manifolds fails in the overtwisted case.

The first known exotic phenomena regarding overtwisted disks in overtwisted contact 3-manifolds are due to Vogel [79]. He has proved that the space of overtwisted disks in certain overtwisted 3-sphere is disconnected and used this to construct an exotic loop of overtwisted contact structures.

1.3 Context

1.3.1 H-principles

As with symplectic topology, an ubiquitous theme of contact topology is the contrast between two types of behaviours: flexible (similar to differential topology) and rigid (similar to algebraic geometry). Beyond the tight-overtwisted dichotomy, 3-dimensional contact topology would seem to be dominated by *flexibility*, due to the following h-principle of Eliashberg and Mishachev:

Theorem 1.9 ([22]). *Let* $(\mathbb{B}^3, \xi_{\mathrm{st}} = \mathrm{Ker}(dz - ydx))$ *be the standard contact unit* 3*-ball. Then the inclusion* $\mathrm{Cont}(\mathbb{B}^3, \xi_{\mathrm{st}}) \to \mathrm{Diff}(\mathbb{B}^3)$ *is a homotopy equivalence.*

Here $\mathrm{Cont}(\mathbb{B}^3, \xi)$ is the group of contactomorphisms of Y fixing a neighbourhood of $\partial\mathbb{B}^3$, and likewise for the group of diffeomorphisms $\mathrm{Diff}(\mathbb{B}^3)$. To give some context, the analogous statement that $\mathrm{Diff}(\mathbb{B}^3) \to \mathrm{Homeo}(\mathbb{B}^3)$ is a homotopy equivalence is equivalent to the Smale conjecture in dimension 3, a deep result proved by Hatcher [41]. Then an argument due to Cerf [9] shows that the Smale conjecture implies that $\mathrm{Diff}(Y) \to \mathrm{Homeo}(Y)$ is a homotopy equivalence for all 3-manifolds. Thus, at the π_0-level, Theorem 1.5 is in sharp contrast with the above.

Remark 1.8. We also note in passing that in four-dimensional symplectic topology the statement analogous to the h-principle of Eliashberg and Mishachev is false: for the standard symplectic $(\mathbb{R}^4, \omega = dx \wedge dy + dz \wedge dw)$ the inclusion

$$\mathrm{Symp}_c(\mathbb{R}^4, \omega) \to \mathrm{Diff}_c(\mathbb{R}^4)$$

is not a homotopy equivalence. This follows from Gromov's result on the contractibility of $\mathrm{Symp}_c(\mathbb{R}^4, \omega)$ [35] combined with Watanabe's recent disproof of the 4-dimensional Smale Conjecture [80].

1.3.2 Gompf's contact Dehn twist

We will see (§3.1) that the contact Dehn twist is well-defined on a (co-oriented) sphere $S \subset (Y, \xi)$ with a *tight neighbourhood*. To the author's knowledge, this contactomorphism was first considered by Gompf on the non-trivial sphere in the tight $S^1 \times S^2$. Gompf observed that τ_S and its iterates are not contact isotopic to the identity. Ding and Geiges [14] later established that τ_S^2 generates all smoothly trivial contact mapping classes (see also [60]). Gironella [29] has recently studied higher dimensional analogues of Gompf's contactomorphism. However, all iterates of Gompf's τ_S and Gironella's generalisations happen to be *formally non-trivial* already, and hence *not* exotic.

1.3.3 Finite order exotic contactomorphisms

The previously known exotic three-dimensional contactomorphisms have *finite* order and the underlying 3-manifolds have $b_1 \geq 3$. These were detected on torus bundles by Geiges and Gonzalo [26], who used an essentially elementary argument to reduce the problem to the Giroux–Kanda classification of tight contact structures on T^3. This was reproved using contact homology by Bourgeois [6], who also found more exotic contactomorphisms in Legendrian circle bundles over surfaces of positive genus. In the latter case, those contactomorphisms have been shown to generate the group (1.1) by Geiges, Klukas [27], Giroux and Massot [33]. Unlike the squared Dehn twists, these exotic contactomorphisms are all given by global symmetries. The paradigmatic example is the following:

Example 1.7 ([26, 6])**.** Consider the 3-torus T^3 with the fillable contact structure $\xi_1 = \mathrm{Ker}\big(\cos\theta dx - \sin\theta dy\big)$. By passing to n-fold covers $T^3 \to T^3$, $(\theta, x, y) \mapsto (n\theta, x, y)$ we obtain contact structures ξ_n on T^3. By a classical result of Giroux and Kanda [32, 45] the contact structures ξ_n $(n \geq 1)$ are pairwise not contactomorphic and give all the tight contact structures on T^3. When $n \geq 2$ the deck transformations of the n-fold cover $T^3 \to T^3$ generate all the exotic contactomorphisms of (T^3, ξ_n).

1.3.4 Other Exotic Dehn twists

Dehn twists have been a common source of exotic phenomena in topology:

(a) Let $Y_\# = Y_- \# Y_+$ be the sum of two aspherical 3-manifolds $Y_\pm$. By a result of McCullough [57] (see also [40]) it follows that the kernel of $\pi_0 \mathrm{Diff}(Y_\#) \to \mathrm{Out}(\pi_1 Y_\#)$ is $\cong \mathbb{Z}_2$, generated by the smooth Dehn twist on the separating sphere.

(b) Seidel [73] used Lagrangian Floer homology to detect exotic four-dimensional symplectomorphisms with infinite order in the symplectic mapping class group, given by squared Dehn twists on Lagrangian spheres. He later generalised these results to higher dimensions [72, 71]. See also the recent work of Smirnov [75, 76] using Seiberg–Witten gauge theory.

(c) Kronheimer and Mrowka [48] have proved that the smooth Dehn twist on the separating sphere in the connected sum of two copies of the smooth 4-manifold underlying a $K3$ surface is not smoothly isotopic to the identity, even if it is topologically. For this they employ the Bauer-Furuta homotopical refinement of the Seiberg–Witten invariants of 4-manifolds. See also [52].

1.4 Outline of the proofs

1.4.1 Theorem 1.1(A)

The proof of Theorem 1.1(A) combines rigid obstructions arising from monopole Floer homology together with flexibility results. We outline here a proof which is simpler than the one we present in detail later during the dissertation. In particular, the proof here does not yield the stronger conclusion that the class of $\tau_{S_\#}^2$ is non-trivial in the abelianisation of (1.1). We will also need the stronger argument to obtain the closely related Theorem 1.3.

1.4.1.1 Theorem 1.2 $\implies$ Theorem 1.1

Consider two *tight* irreducible contact manifolds $(Y_\pm, \xi_\pm)$. Recall that non-vanishing of the contact invariant implies tightness. Recall also that their sum $(Y_\#, \xi_\#)$ is obtained by removing two Darboux balls $B_\pm \subset Y_\pm$ and gluing the boundary spheres in an orientation-reversing and characteristic-foliation preserving fashion. Let $\mathrm{CEmb}\big(S^2,(Y_\#,\xi_\#)\big)_{S_\#}$ be the space of co-oriented *convex* embeddings $S^2 \hookrightarrow (Y_\#,\xi_\#)$ with *standard characteristic foliation*, in the isotopy class of the separating sphere $S_\#$. The group of contactomorphisms of $(Y_\#,\xi_\#)$ acts transitively on this space and yields a fibration[2]

$$\begin{aligned}\mathrm{Cont}(Y_\#,\xi_\#,S_\#) \to \mathrm{Cont}(Y_\#,\xi_\#) &\to \mathrm{CEmb}\big(S^2,(Y_\#,\xi_\#)\big)_{S_\#}. \\ f &\mapsto f(S_\#)\end{aligned} \tag{1.6}$$

From the long exact sequence of homotopy groups, a contactomorphism f of $(Y_\#,\xi_\#)$ fixing the sphere $S_\#$ is contact isotopic to the identity (not necessarily fixing $S_\#$) precisely when it arises as the monodromy in (1.6) of a loop of sphere embeddings. It thus becomes essential to understand the topology of the sphere embedding space. This brings us to the following h-principle type result, which asserts that the topological complexity of this space only comes from reparametrisations of the source:

Theorem 1.10 ([23]). *If $(Y_\pm,\xi_\pm)$ are irreducible and tight then the reparametrisation map provides a homotopy equivalence* $\mathrm{U}(1) \xrightarrow{\simeq} \mathrm{CEmb}\big(S^2,(Y_\#,\xi_\#)\big)_{S_\#}$.

In the smooth case, the result analogous to the above was proved by Hatcher [38]. Theorem 1.10 follows easily from Hatcher's result combined with the h-principle for standard convex spheres due to Fernández–Martínez-Aguinaga–Presas [23], the latter being an application of the h-principle for $(\mathbb{B}^3,\xi_{\mathrm{st}})$ of Eliashberg–Mishachev [22].

With these ingredients in place, the proof of Theorem 1.5(A) goes as follows. The monodromy in (1.6) of the standard loop in $\mathrm{U}(1)$ is given by the product of Dehn twists $\tau_{\partial B_-}\tau_{\partial B_+}$ (see Lemma 3.5) where $\tau_{\partial B_\pm}$ is the boundary parallel Dehn twist on $Y_\pm \setminus B_\pm$ extended over $Y_\#$ as the identity. The contact Dehn twist $\tau_{S_\#}$ agrees with the image of $\tau_{\partial B_-}$ (or $\tau_{\partial B_+}^{-1}$) in $\pi_0\mathrm{Cont}(Y_\#,\xi_\#)$. Because the manifolds $(Y_\pm \setminus B_\pm,\xi_\pm)$ have infinite order contact Dehn twists $\tau_{\partial B_\pm}$ rel. $\partial B_\pm$ by Theorem 1.2,

[2] Strictly speaking, we should replace $\mathrm{Cont}(Y_\#,\xi_\#)$ with the subgroup consisting of contactomorphisms which preserve the isotopy class of the co-oriented sphere $S_\#$.

then for all $k \neq 0$ the class $\tau_{\partial B_-}^k \in \pi_0\mathrm{Cont}(Y_\#, \xi_\#, S_\#)$ is not an iterate of $\tau_{\partial B_-}\tau_{\partial B_+}$ or its inverse. It follows that $\tau_{S_\#}$ and its iterates are not contact isotopic to the identity in $(Y_\#, \xi_\#)$.

1.4.1.2 Theorem 1.2

Given a Darboux ball B in a contact 3-manifold (Y, ξ) then the isotopy problem for the boundary parallel Dehn twist $\tau_{\partial B}$ can be recast as a lifting problem. Namely, when Y is *aspherical* (i.e. irreducible and with infinite fundamental group) then $\tau_{\partial B}^2$ is isotopic to the identity rel. B precisely when the evaluation map $ev : C(Y, \xi) \to S^2$ admits a (homotopy) section (see Corollary 3.7). Now, the condition $\mathbf{c}(\xi; \mathbb{Q}) \notin \mathrm{Im}U$ together with the irreducibility assumption on Y implies the aspherical property. Finally, the existence of a section is impossible by $\mathbf{c}(\xi; \mathbb{Q}) \notin \mathrm{Im}U$ because of the obstruction coming from Theorem 1.4. The result follows.

1.4.2 Outline of the construction of the families contact invariant

We summarise in this section the construction of the invariants $\mathbf{Fc}$ and $\widetilde{\mathbf{Fc}}$ and sketch the proof of Theorem 1.5.

1.4.2.1 The invariant $\mathbf{Fc}$

We begin with some general observations. Let X be a 4-manifold together with a non-degenerate 2-form ω i.e. ω^2 is a volume form. We use ω^2 to orient X. Choose an almost complex structure J compatible with ω, which by definition gives a metric $g = \omega(., J.)$. The space of choices of J is contractible. The structure J equips X with a spin-c structure, i.e. a lift of the SO(4)-frame bundle of X along the map $\mathrm{Spin}^c(4) \to \mathrm{SO}(4)$. In differential-geometric terms this yields rank-two complex unitary bundles $S^\pm \to X$ and Clifford multiplication $\rho : TX \to \mathrm{Hom}(S^+, S^-)$ satisfying the "Clifford identity" $\rho(v)^*\rho(v) = g(v, v)\mathrm{Id}$. We follow the notation and conventions from §1 in [49] and we assume the reader is familiar with these.

The Clifford action of the 2-form ω on S^+ splits the bundle S^+ into $\mp 2i$ eigen-subbundles of rank 1. These are given by $S^+ = E \oplus EK_J^{-1}$, where K_J is the canonical bundle of (X, J) and E is a complex line bundle which is easily verified to be trivial. Choose a unit length section Φ_0 of E. A simple calculation shows that there is a unique spin-c connection A_0 on S^+ such that $\nabla_{A_0}\Phi_0$ is a 1-form with values in the $+2i$ eigenspace EK_J^{-1}. At this point, the symplectic condition comes in through the following calculation involving the coupled Dirac operator $D_{A_0} : \Gamma(S^+) \to \Gamma(S^-)$

Lemma 1.11 (Taubes [77]). *The non-degenerate* 2*-form* ω *is symplectic (i.e.* $d\omega = 0$*) if and only if* $D_{A_0}\Phi_0 = 0$.

We now bring in a smoothly varying family of symplectic structures ω_t parametrised by a smooth manifold $U \ni t$, with each ω_t in the same deformation class as ω. Again, we equip the ω_t's with compatible almost complex structures J_t varying smoothly, which provide us with a family of metrics g_t. From our original Clifford bundle $(S^{\pm}, \rho)$ we canonically obtain new ones as follows. The bundles $S^{\pm}$ remain the same but new Clifford structures ρ_t are obtained by setting $\rho_t = \rho \circ b_t$ where b_t is the canonical isometry $(TX, g_t) \xrightarrow{\cong} (TX, g)$ (the unique isometry which is positive and symmetric with respect to g_t). The Clifford action of ω_t again decomposes S^+ into eigenspaces $S^+ = E_t \oplus E_t K_{J_t}^{-1}$. Each E_t is trivializable individually but the family $(E_t)_{t\in U}$ might give a non-trivial line bundle over $U \times X$. When U is contractible then we may choose a family of trivialising sections Φ_t of E_t with unit length, and as before these determine unique spin-c connections A_t with $D_{A_t}\Phi_t = 0$. Then, associated to our family (ω_t, J_t) and the choices of Φ_t we have a family of "deformed" Seiberg–Witten equations on X given by

$$\begin{aligned}
&\frac{1}{2}\rho_t(F_A^+) - (\Phi\Phi^*)_0 = \frac{1}{2}\rho_t(F_{A_t}^+) - (\Phi_t\Phi_t^*)_0 \\
&D_A\Phi = D_{A_t}\Phi_t.
\end{aligned}$$

For each $t \in U$ this is an equation on the pair (A, Φ), where A is a connection on $\Lambda^2 S^+$ and Φ is a section of S^+. In this "deformed" version of the equations the configurations (A_t, Φ_t) solve the equation for the parameter t.

We apply now the above considerations to a special case. Let (Y, ξ) be a closed contact 3-manifold with a contact form α, and let (X, ω) be the *symplectisation* $X = [1, +\infty) \times Y$, with the exact symplectic form $\omega = d(\frac{s^2}{2}\alpha)$. The structure J is chosen to be invariant under the Liouville flow, and the associated Riemannian metric on X is conical. We now bring into the picture a family of contact structures ξ_t parametrised by $t \in U = \Delta^n$, to which we would like to associate an element in the Floer chain complex of $-Y = \partial X$. Here Δ^n is the standard n-simplex. We equip our family ξ_t with corresponding contact forms α_t. This gives a family ω_t of symplectic structures on X.

The construction now proceeds by forming a manifold Z^+ by gluing the cylinder $Z = (-\infty, 0] \times Y$ with the symplectic manifold X. We extend all metrics g_t over to Z^+ in such a way that they all agree with a fixed translation-invariant metric on the cylinder Z. Then the bundle S^+, together with its splitting $S^+ = E \oplus EK_J^{-1}$, extends over Z^+ naturally in a translation-invariant manner. The U-family of metrics and spin-c structures thus constructed on Z^+ are independent of $t \in U$ over Z, so we have effectively trivialised our data over the cylinder end $Z \subset Z^+$. In order to extend the Seiberg–Witten equations over Z^+ we cut off the perturbation term on the right-hand side of the equations so that it vanishes on the cylinder end Z. This way, we have a U-parametric family of Seiberg–Witten equations over Z^+, and natural boundary conditions for these equations (modulo gauge) are

- on the cylinder Z solutions should approach a translation-invariant solution $\mathfrak{a}$ (a generator of the "to" Floer complex $\check{C}(-Y, \mathfrak{s}_\xi)$, i.e. $\mathfrak{a}$ is an irreducible or boundary stable monopole on $-Y$)
- on the symplectic end X solutions should approach the configuration (A_t, Φ_t).

This way we obtain parametrised moduli spaces of solutions

$$\pi : M([\mathfrak{a}], \Delta^n) \to \Delta^n.$$

By introducing suitable perturbations we may achieve the necessary transversality and $M([\mathfrak{a}], \Delta^n)$ will be C^1-manifolds of finite dimension. At this point we note that, because of the gauge-invariance of the equations, a different choice of trivialisations Φ_t would yield diffeomorphic moduli spaces. The connected components of $M([\mathfrak{a}], \Delta^n)$ where the index of π is $-n$ consist of a finite number of isolated points lying over values in the interior of Δ^n, and a signed count of these points gives an integer $\#M([\mathfrak{a}], \Delta^n) \in \mathbb{Z}$. We organise these counts into a Floer chain $\psi(\Delta^n)$

$$\psi(\Delta^n) = \sum_{[\mathfrak{a}]} \#M([\mathfrak{a}], \Delta^n) \cdot [\mathfrak{a}] \in \check{C}_*(-Y, \mathfrak{s}_\xi).$$

The assignment $\Delta^n \mapsto \psi(\Delta^n)$ can be made into a chain map

$$\psi : C_*(C(Y, \xi)) \to \check{C}_*(-Y, \mathfrak{s}_\xi)$$

from the complex of singular chains on $C(Y, \xi)$. Passing to homology yields the families invariant **Fc**.

1.4.2.2 The invariant $\widetilde{\mathbf{Fc}}$ *and Theorem 1.5*

In terms of the "to" Floer complex $\check{C}_*$, the "tilde" Floer complex can be defined by taking the mapping cone of (a suitable chain level version of) the U map. We have $\widetilde{C}_*(Y, \mathfrak{s}) = \check{C}_*(Y, \mathfrak{s}) \oplus \check{C}_{*-1}(Y, \mathfrak{s})$ with differential given by the matrix (ignoring signs)

$$\widetilde{\partial} = \begin{pmatrix} \check{\partial} & 0 \\ U & \check{\partial} \end{pmatrix}.$$

If a family $T \in H_n(C(Y, \xi))$ is in the image of $\iota_* : H_n(C(Y, \xi, B)) \to H_n(C(Y, \xi))$ then we show that $U \cdot \mathbf{Fc}(\beta) = 0$. At the chain level this is witnessed by a canonical chain homotopy θ:

$$U \cdot \psi \circ \iota_* = \check{\partial}\theta + \theta\partial. \tag{1.7}$$

From this we build the chain map

$$\widetilde{\psi} = (\psi \circ \iota_*, \theta) : C_*(C(Y, \xi, B)) \to \widetilde{C}_*(-Y, \mathfrak{s}_\xi)$$

which yields $\widetilde{\mathbf{Fc}}$ in homology. The chain homotopy θ is roughly constructed as follows. We introduce a new parameter $s \in \mathbb{R}$ and let $p \in Y$ be the center of the ball B. Consider the moduli space

$$\mathcal{M}([\mathfrak{a}], \Delta^n) \to \mathbb{R} \times \Delta^n$$

consisting of quadruples (A, Φ, t, s) such that (A, Φ, t) solve the previous set of equations and boundary conditions subject to the further constraint that at the point $(s, p) \in \mathbb{R} \times Y \cong Z^+$ the spinor Φ lies in the second component of the splitting $S^+ = E \oplus EK_J^{-1}$. By a simple modification of this construction one can again achieve transversality and ensure that the $\mathcal{M}([\mathfrak{a}], \Delta^n)$ are C^1-manifolds of finite dimension. Then we set

$$\theta(\Delta^n) = \sum_{[\mathfrak{a}]} \#\mathcal{M}([\mathfrak{a}], \Delta^n) \cdot [\mathfrak{a}].$$

Theorem 1.5(A) just follows by the construction, and (B) is established by carefully analysing the "boundary at infinity" of the 1-dimensional components of the moduli $\mathcal{M}([\mathfrak{a}], \Delta^n)$. The essential point is the following. When Δ^n parametrises a family in $\mathcal{C}(Y, \xi, B)$ then the moduli space is compact as we take the parameter $s \to +\infty$. As $s \to -\infty$ then a boundary appears with connected components (roughly) of the form $M([\mathfrak{a}], U, [\mathfrak{b}]) \times M([\mathfrak{b}], \Delta^n)$, where $M([\mathfrak{a}], U, [\mathfrak{b}])$ are the moduli spaces that one counts to define the U map. The remaining source of non-compactness comes from usual breaking of Floer trajectories. From this one establishes (1.7). If instead Δ^n parametrises a family in the full space of contact structures $\mathcal{C}(Y, \xi)$, then the boundary of the moduli space as $s \to +\infty$ is instead given by

$$M([\mathfrak{a}], \Delta_*^{n-2})$$

where Δ_*^{n-2} is the submanifold (with corners) of Δ^n obtained (essentially) as the preimage under $\Delta^n \to \mathcal{C}(Y, \xi)$ of a fiber of the evaluation map $ev : \mathcal{C}(Y, \xi) \to S^2$. From this one establishes Theorem 1.5(B).

1.5 Structure of the exposition

In Chapter 1 we present the main results of this dissertation, together with examples, relevant context, and sketches of the main arguments. This is based on material which appeared in [63, 24].

In Chapter 2 we discuss general aspects related the main players of this dissertation: spaces of contact structures, contactomorphisms, convex spheres, etc. We also include here an h-principle type result for the space of contact structures on a connected sum. This is based on material which appeared in [24].

In Chapter 3 we introduce the contact Dehn twist, explore various topological aspects related to this contactomorphism, and we end by discussing the proofs of Theorems 1.2, 1.1 and 1.3 assuming the main technical result of this dissertation, namely Theorem 1.5. This is based on material which appeared in [24].

In Chapter 4 we present the construction of the families contact invariant **Fc**, from which Theorem 1.5(A) follows immediately by construction. This is based on material which appeared in [63].

In Chapter 5 we discuss the algebraic structures on the homology of the space of contact structure and present the proof of Theorem 1.5(B). We also discuss the refinement of this result given in Theorem 1.7 and construct the "tilde" version of the families contact invariant $\widetilde{\mathbf{Fc}}$. This is based on material which appeared in [63]

In the course of Chapters 4 and 5 various details on transversality, compactness and orientations of Seiberg–Witten moduli spaces are omitted. These are relegated to the Appendix.

Chapter 2: Topology of families of contact structures

2.1 Background

This section introduces the main players in this dissertation: spaces of contact structures, contactomorphisms, embeddings, etc.

Remark 2.1. In this dissertation a "fibration" will mean a "Serre fibration". A "homotopy equivalence" will mean a "weak homotopy equivalence". However, the latter distinction isn't important: the various infinite dimensional spaces that we consider are Fréchet manifolds, hence they have the homotopy type of countable CW complexes [69, 59] and Whitehead's Theorem applies.

2.1.1 Notation

Let (Y, ξ) be a closed contact 3-manifold. We always assume Y is connected and oriented, and ξ co-oriented and positive. Occasionally we will allow Y to be compact with non-empty boundary, in which case we assume that ∂Y is *convex* for the contact structure ξ and we fix a collar neighbourhood $C = (-1, 0] \times \partial Y$ of ∂Y. We quickly introduce here some of the spaces that will be relevant in the dissertation, all of which are equipped with the Whitney C^∞ topology:

- We denote by $\mathrm{Emb}\big(\mathbb{B}^3, Y\big)$ the space of orientation-preserving smooth embeddings $\phi : \mathbb{B}^3 \hookrightarrow Y$ of the closed unit ball (avoiding the closure of C, if $\partial Y \neq \emptyset$). Let $\mathrm{Emb}\big((\mathbb{B}^3, \xi_{\mathrm{st}}), (Y, \xi)\big)$ be the subspace consisting of contact embeddings of the standard contact unit ball. Such embeddings will be referred to as *Darboux balls* in (Y, ξ). Darboux's theorem asserts that for any interior point p of a contact manifold we may find such ϕ with $\phi(0) = p$. We will often incur in abuse of notation by referring to a Darboux ball only by its image $B := \phi(\mathbb{B}^3)$.

- We denote by $\mathrm{Diff}(Y)$ the group of orientation-preserving diffeomorphisms, and by $\mathrm{Diff}(Y, B)$ the subgroup consisting of those which fix a Darboux ball B pointwise. By $\mathrm{Diff}_0(Y)$ and $\mathrm{Diff}_0(Y, B)$ we denote the subgroups consisting of those which are smoothly isotopic to the identity (rel. B in the second case). We denote by $\mathrm{Cont}(Y) \subset \mathrm{Diff}$ the subgroup of co-orientation preserving contactomorphisms of (Y, ξ), and by $\mathrm{Cont}(Y, B)$ the subgroup consisting of those which fix a Darboux ball B pointwise. By $\mathrm{Cont}_0(Y)$ and $\mathrm{Cont}_0(Y, B)$ we denote the subgroups consisting of those which are *smoothly* isotopic to the identity (rel. B in the second case).

- We denote by $C(Y, \xi)$ the space of contact structures on Y in the path-component of ξ. When $\partial Y \neq \emptyset$ then we also require that they agree with ξ over C. Given a Darboux ball B in (Y, ξ) we denote by $C(Y, \xi, B)$ the subspace consisting of contact structures ξ' for which the coordinate ball B is a Darboux ball for (Y, ξ') (i.e. $\xi = \xi'$ over B).

- We denote by $\mathrm{Fr}(Y)$ the principal $(\mathrm{SO}(3) \simeq)\mathrm{GL}^+(3)$-bundle over Y of oriented frames in TY, and by $\mathrm{CFr}(Y)$ the principal $(\mathrm{U}(1) \simeq)\mathrm{CSp}^+(2, \mathbb{R})$-bundle over Y of co-oriented frames in ξ. By the smooth and contact versions of the Disk Theorem[1] we have homotopy equivalences

$$\begin{aligned} \mathrm{Emb}(\mathbb{B}^3, Y) &\xrightarrow{\simeq} \mathrm{Fr}(Y) \simeq Y \times \mathrm{SO}(3) \\ \phi &\mapsto (d\phi)_0(e_1, e_2, e_3) \\ \mathrm{Emb}((\mathbb{B}^3, \xi_{\mathrm{st}}), (Y, \xi)) &\xrightarrow{\simeq} \mathrm{CFr}(Y, \xi) \simeq Y \times \mathrm{U}(1) \\ \phi &\mapsto (d\phi)_0(e_1, e_2). \end{aligned} \tag{2.1}$$

- An embedding $e : S^2 \hookrightarrow (Y, \xi)$ is a *standard convex embedding* (or just "standard embedding") if its oriented characteristic foliation $(e^*\xi) \cap TS^2$ coincides with the characteristic foliation of the boundary sphere $e_0 : S^2 = \partial \mathbb{B}^3 \hookrightarrow (\mathbb{R}^3, \xi_{\mathrm{st}})$ of the unit ball $\mathbb{B}^3$. In fact, by this property we obtain a (homotopically) unique contact embedding of a neighbourhood of

[1] The key point in the contact case is that $\varphi_t(x, y, z) := (tx, ty, t^2 z)$ is a contactomorphism of $(\mathbb{R}^3, \xi_{\mathrm{st}})$ for every $t > 0$, so the proof in the contact case follows along the same lines as in the smooth case (see [25], Theorem 2.6.7).

$e_0(S^2) \subset (\mathbb{R}^3, \xi_{st})$ inside (Y, ξ) such that e_0 is identified with e. We recall that the north pole of e is then a positive elliptic point and the south pole a negative elliptic point. See Figure 2.1. We denote by $\mathrm{Emb}(S^2, Y)$ the space of co-oriented embeddings of 2-spheres. By $\mathrm{CEmb}\big(S^2, (Y, \xi)\big)$ we denote the subspace consisting of standard convex spheres. More generally, recall that a surface $\Sigma \subset (Y, \xi)$ is *convex* [30][25] if there exists a contact vector field on a neighbourhood which is transverse to Σ.

- We denote by $\mathrm{Cont}(Y, \xi, S)$ the subgroup of contactomorphisms which fix a standard convex sphere S pointwise, and likewise for $\mathrm{Diff}(Y, S)$.

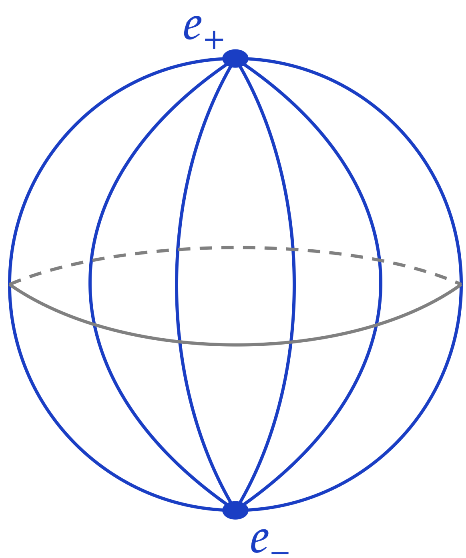

Figure 2.1: Depiction of the characteristic foliation (blue) of a standard sphere, together with the positive (resp. negative) elliptic points at the north (resp. south) poles.

2.1.2 Standard fibrations

Next, we review how the spaces introduced above relate to each other through various natural fibrations. Some of the material from this section is treated in [33] in greater detail.

2.1.2.1 Diffeomorphisms acting on contact structures

By an application of Gray's stability Theorem (a.k.a Moser's argument) [25] with parameters one can show

Lemma 2.1. *The action $f \mapsto f_*\xi$ of the group of diffeomorphisms on a fixed contact structure ξ gives a fibration*

$$\mathrm{Cont}_0(Y,\xi) \to \mathrm{Diff}_0(Y) \to C(Y,\xi). \tag{2.2}$$

Similarly, there is fibration

$$\mathrm{Cont}_0(Y,\xi,B) \to \mathrm{Diff}_0(Y,B) \to C(Y,\xi,B). \tag{2.3}$$

By (2.2), understanding the homotopy type of the space of contact structures $C(Y,\xi)$ and the group of contactomorphisms $\mathrm{Cont}_0(Y,\xi)$ is essentially equivalent, since the homotopy type of $\mathrm{Diff}_0(Y)$ is often well-understood (e.g. for all prime 3-manifolds by now).

2.1.2.2 Contactomorphisms acting on Darboux balls

By an application of the contact isotopy extension Theorem [25] with parameters we have

Lemma 2.2. *The action $f \mapsto f(B)$ of the group of contactomorphisms on a fixed Darboux ball $B \subset Y$ gives a fibration*

$$\mathrm{Cont}(Y,\xi,B) \to \mathrm{Cont}(Y,\xi) \to \mathrm{Emb}((\mathbb{B}^3,\xi_{\mathrm{st}}),(Y,\xi)). \tag{2.4}$$

Similarly, there is a fibration

$$\mathrm{Diff}(Y,B) \to \mathrm{Diff}(Y) \to \mathrm{Emb}(\mathbb{B}^3,Y). \tag{2.5}$$

2.1.2.3 Evaluation of contact structures at a point

Fix a Darboux ball $B \subset Y$ with center $0 \in Y$. By regarding the 2-sphere S^2 as the space of co-oriented planes in the tangent space T_0B we obtain the *evaluation map*

$$ev : C(Y, \xi) \to S^2 \quad , \quad \xi' \mapsto \xi'(0). \tag{2.6}$$

The following result is well-known but we provide a proof:

Lemma 2.3. *The evaluation map (1.3) is a fibration. The inclusion $C(Y, \xi, B) \to (ev)^{-1}(\xi(0))$ is a homotopy equivalence.*

Proof. Let $\mathbb{B}^j$ be the unit j-disk and consider a homotopy $[0, 1] \times \mathbb{B}^j \to S^2$, $(t, u) \mapsto \sigma_{t,u}$, together with a lift of the time zero map $\{0\} \times \mathbb{B}^j \to C(Y, \xi)$, $u \mapsto \xi_u$ i.e. at the point $0 \in B$ we have $\xi_u(0) = \sigma_{0,u}$. We must find a family of contact structures $\xi_{t,u}$ with $\xi_{t,u}(0) = \sigma_{t,u}$ and $\xi_{0,u} = \xi_u$.

Let $v_{t,u} \in S(T_0B) = S^2$ be the unit normal (with respect to the standard flat metric on B) to the plane $\sigma_{t,u}$. Since the action of SO(3) on S^2 gives a fibration $\mathrm{SO}(3) \to S^2$, $A \mapsto Ae_3$, then we may find $A_{t,u} \in \mathrm{SO}(3)$ such that $A_{t,u}e_3 = v_{t,u}$. Differentiating $A_{t,u}$ in t we get a vector field on $V_{t,u}$ on $\mathbb{R}^3$. After cutting off $V_{t,u}$ outside the unit ball $B \subset Y$ we regard $V_{t,u}$ as an u-family of t-dependent vector fields on Y whose associated flows (starting at time $t = 0$) we denote ϕ^t_u. We obtain contact structures $\xi_{t,u} := (\phi^t_u)_* \xi_u$ with the desired property, which in fact agree with ξ outside $B \subset Y$.

For the second part, let $\xi_u = \mathrm{Ker}\alpha_u$ be a family of contact structures parametrised by a sphere $S^j \ni u$ and with $\xi_u(0) = \xi(0)$. We must deform rel. 0 this family of contact structures to another family which agrees with ξ over the Darboux ball B. By the parametric version of Darboux's Theorem we obtain a family of disk embeddings $\phi_u : \mathbb{B}^3 \hookrightarrow Y$ with $\phi_u(0) = 0 \in B$ and $(d\phi_u)_0 = \mathrm{id}$ which are Darboux balls for ξ_u. By (2.1) we may deform the family of embeddings ϕ_u to the original embedding B, and this deformation may be followed by an isotopy $f_{u,t}$. The contact structures $(f_{u,1})_* \xi_u$ now agree with ξ over B. □

2.1.2.4 Contactomorphisms act on standard convex spheres

Again, an application of the contact isotopy extension Theorem gives

Lemma 2.4. *The action $f \mapsto f(S)$ of the group of contactomorphisms on a fixed standard convex sphere $S \subset Y$ gives a fibration*

$$\mathrm{Cont}(Y, \xi, S) \to \mathrm{Cont}(Y, \xi) \to \mathrm{CEmb}\big(S^2, (Y, \xi)\big) \tag{2.7}$$

Similarly, there is a fibration

$$\mathrm{Diff}(Y, S) \to \mathrm{Diff}(Y) \to \mathrm{Emb}(S^2, Y). \tag{2.8}$$

Remark 2.2. The above statement isn't quite precise. For either (2.7) or (2.8), the downstairs projection is not surjective in general, so strictly speaking we only have a fibration over a union of connected components of the right-hand side. We will make no further comment on this point from now on.

2.1.3 Formal triviality and exoticness

Here we collect basic material that we need related to the notion of a formal contactomorphism. The material in this section should be well-known to experts but we did not find a convenient reference.

2.1.3.1 Formal contact structures and contactomorphisms

For a 3-manifold Y, the flexible analogue[2] of a contact structure is a 2-*plane field* i.e. a codimension 1 distribution $\xi \subset TY$. All 2-planes in a 3-manifold are assumed to be co-oriented from now on, as we've been assuming with contact structures. Let $\Xi(Y, \xi)$ denote the path-component

[2]In general, if Y has dimension $2n + 1 \geq 3$ one should define $\Xi(Y, \xi)$ as the space of codimension 1 hyperplane fields in TY *equipped with* a $\mathrm{U}(n)$ structure.

of a fixed 2-plane field ξ in the space of all such. If ξ is a contact structure we have a natural inclusion map $C(Y,\xi) \to \Xi(Y,\xi)$. The correct flexible analogue of a contactomophism is:

Definition 2.1. A *formal contactomorphism* of (Y,ξ) (where ξ is a 2-plane field) is a pair $(f, \{\phi^s\}_{0\leq s\leq 1})$ such that $f \in \mathrm{Diff}(Y)$ and $\{\phi^s\}_{0\leq s\leq 1}$ is a homotopy through vector bundle isomorphisms $\phi^s : TY \xrightarrow{\cong} f^*TY$ such that $\phi^0 = df$ and ϕ^1 preserves the 2-plane field ξ.

The group of formal contactomorphisms of (Y,ξ) is denoted $\mathrm{FCont}(Y,\xi)$. When ξ is a contact structure there is the obvious inclusion map $\mathrm{Cont}(Y,\xi) \to \mathrm{FCont}(Y,\xi)$ given by $f \mapsto (f, df)$ (where df denotes the constant homotopy at df).

A homotopy class in $\pi_j\mathrm{Cont}(Y,\xi)$ is said to be *formally trivial* if it lies in the kernel of $\pi_j\mathrm{Cont}(Y,\xi) \to \pi_j\mathrm{FCont}(Y,\xi)$. If, in addition, such a homotopy class is non-trivial in $\pi_j\mathrm{Cont}(Y,\xi)$ then we call it *exotic*. Similar terminology applies for families of contact structures.

2.1.3.2 A flexible analogue of (2.2)

We introduce a flexible counterpart of the fibration (2.2). This is done via fibrant replacement of the map $\mathrm{Diff}_0(Y) \to \Xi(Y,\xi)$, $f \mapsto f^*\xi$. That is, we decompose this map as the composite of a homotopy equivalence $\mathrm{Diff}_0(Y) \xrightarrow{\simeq} \mathrm{FDiff}_0(Y)$ and a fibration $\mathrm{FDiff}_0(Y) \to \Xi(Y,\xi)$. Here $\mathrm{FDiff}(Y)$ is the topological group which consists of pairs $(f, \{\phi^t\}_{0\leq t\leq 1})$ where $f \in \mathrm{Diff}(Y)$ and $\{\phi^t\}_{0\leq t\leq 1}$ is a homotopy of vector bundle isomorphisms $\phi^t : TY \xrightarrow{\cong} f^*TY$ such that $\phi^0 = df$. By $\mathrm{FDiff}_0(Y)$ we denote the identity component. Clearly the inclusion induces a homotopy equivalence $\mathrm{Diff}(Y) \simeq \mathrm{FDiff}(Y)$. Define a mapping

$$\begin{aligned} \mathrm{FDiff}_0(Y) &\to \Xi(Y,\xi) \\ (f, \{\phi^t\}) &\mapsto \phi^1(\xi) \end{aligned} \tag{2.9}$$

Lemma 2.5. *Let ξ be a 2-plane field on a compact oriented 3-manifold Y. Then the mapping (2.9) is a fibration with fiber $\mathrm{FCont}_0(Y,\xi)$. Thus, for a contact structure ξ we have a commuting* *diagram of fibrations inducing a homotopy equivalence of total spaces*

$$\begin{array}{ccccc} \mathrm{FCont}_0(Y,\xi) & \longrightarrow & \mathrm{FDiff}_0(Y) & \longrightarrow & \Xi(Y,\xi) \\ \uparrow & & \simeq\uparrow & & \uparrow \\ \mathrm{Cont}_0(Y,\xi) & \longrightarrow & \mathrm{Diff}_0(Y) & \longrightarrow & C(Y,\xi) \end{array}$$

Corollary 2.6. *Let (Y,ξ) be a contact 3-manifold. If $\beta \in \pi_j C(Y,\xi)$ is formally trivial, then so is its image in $\pi_{j-1}\mathrm{Cont}_0(Y,\xi)$ under the connecting map of the fibration (2.2).*

Proof of Lemma 2.5. It suffices to check the Cerf-Palais fibration criterion (see [70], Theorem A): that for every $\widetilde{\xi} \in \Xi(Y,\xi)$ the mapping $\mathrm{FDiff}_0(Y) \to \Xi(Y,\xi)$ given by $(f,\{\phi^t\}) \mapsto \phi^1(\widetilde{\xi})$ has a section $s : U \to \mathrm{FDiff}_0(Y)$ defined on a neighbourhood U of $\widetilde{\xi}$. Without loss of generality $\widetilde{\xi} = \xi$. We let U be a contractible neighbourhood of ξ and we fix a deformation retraction $h : [0,1]\times U \to U$ to ξ i.e. if $h_t := h(t,\cdot)$ we have $h_1 = \mathrm{id}_U$, $h_t(\xi) = \xi$ for all t, and $h_0(U) = \{\xi\}$. Let ξ_u be the plane field represented by the point $u \in U$. Let $\mathrm{Aut}_0(TY)$ be the identity component in the group of automorphisms of the vector bundle TY covering the identity. The key point is:

Claim 2.1. *The mapping $\mathrm{Aut}_0(TY) \to \Xi(Y,\xi)$ given by $\phi \mapsto \phi(\xi)$ admits a section over the open U.*

We establish the Claim. We may find a family of isomorphisms $i_u : \xi \xrightarrow{\cong} \xi_u$ of oriented vector bundles over Y, since U is contractible. If u_0 is the point representing the plane field ξ then we may assume $i_{u_0} = \mathrm{id}_\xi$. Choosing a metric on Y we obtain identifications $TY = \xi_u \oplus \mathbb{R}$ for all $u \in U$. This gives us a family of isomorphisms $\phi_u : TY = \tilde{\xi} \oplus \mathbb{R} \xrightarrow{i_u \oplus \mathrm{id}_{\mathbb{R}}} \xi_u \oplus \mathbb{R} = TY$ varying continuously with $u \in U$, and $\phi_{u_0} = \mathrm{id}_{TY}$. Thus the ϕ_u provide a section over U of $\mathrm{Aut}_0(TY) \to \Xi(Y,\xi)$. The Claim follows. Finally, we define the required section s by $s(u) = (\mathrm{id}_Y, \{\phi_{h_t(u)}\})$. □

The homotopy type of the space $\Xi(Y,\xi)$ is often easy to understand, unlike that of $C(Y,\xi)$.

Example 2.1. Let Y be any integral homology 3-sphere, and ξ a 2-plane field on Y. Let ξ_{st} be any contact structure on S^3 (say, the tight one). By a result of Hansen [37] there is a homotopy equivalence $\Xi(S^3, \xi_{\mathrm{st}}) \simeq \Xi(Y, \xi)$. From this one easily calculates

$$\pi_j \Xi(Y, \xi) \approx \pi_j S^2 \times \pi_{j+3} S^2.$$

2.2 The space of tight contact structures on a connected sum

In this section we study the space of tight contact structures on connected sums using tools from h-principles. The main result is Theorem 2.7, which will be an ingredient in the proof of Theorems 1.1 and 1.3. It is, in essence, a "families version" of a classical result of Colin [10].

2.2.1 Main result

Consider $n+1$ tight contact 3-manifolds (Y_j, ξ_j), $j = 0, \ldots, n$ with $n \geq 1$. Let $(Y_\#, \xi_\#)$ be their connected sum, which we build as follows. We fix Darboux balls $B_{0-} \subset Y_0$, $B_{n+} \subset Y_n$ and for each $0 < j < n$ we fix two Darboux balls $B_{j\pm} \subset Y_j$. Then the connected sum $(Y_\#, \xi_\#)$ is formed by gluing in the following order

$$\big(Y_0 \setminus B_{0-}\big) \bigcup_{\partial B_{0-} = -\partial B_{1+}} \big(Y_1 \setminus (B_{1+} \cup B_{1-})\big) \cdots \bigcup_{\partial B_{(n-1)-} = -\partial B_{n+}} \big(Y_n \setminus B_{n+}\big)$$

where one glues $\partial B_{(j-1)-}$ and ∂B_{j+} by an *orientation-reversing* diffeomorphism which *preserves the oriented characteristic foliation*. It is because of the latter requirement that the connected sum $Y_\#$ inherits a contact structure $\xi_\#$. We will denote by $e_j : S^2 \hookrightarrow (Y_\#, \xi_\#)$, $j = 1, \ldots, n$, the embedding of the j^{th} separating standard sphere in the connected sum $(Y_\#, \xi_\#)$. Denote by s_j the south pole on the j^{th} sphere, regarded as a point in $e_j(S^2) \subset Y_\#$.

We will denote by $\mathrm{Tight}(Y, B)$ the space of tight contact structures on Y that are fixed on a Darboux ball B and by $\mathrm{Tight}(Y, B, B')$ the subspace of $\mathrm{Tight}(Y, B)$ given by contact structures that are fixed on a second Darboux ball B' disjoint from B. A classical result of Colin [10] asserts that the contact manifold $(Y_\#, \xi_\#)$ is tight, and we have a well-defined map

$$\#_{n+1} : \mathrm{Tight}(Y_0, B_{0-}) \times \prod_{j=1}^{n-1} \mathrm{Tight}(Y_j, B_{j+}, B_{j-}) \times \mathrm{Tight}(Y_n, B_{n+}) \to \mathrm{Tight}(Y_\#). \tag{2.10}$$

On the other hand, the evaluation map of each tight contact structure on Y at the south poles s_j defines a fibration

$$ev_{n+1} : \mathrm{Tight}(Y_\#) \to (S^2)^n. \tag{2.11}$$

The fiber $\mathcal{F}$ of ev_{n+1} over $(\xi_\#(s_j))$ has the homotopy type of the space of tight contact structures on $Y_\#$ that agree with $\xi_\#$ over n disjoint Darboux balls $B_{\# j}$ around s_j. Therefore, there is a natural inclusion

$$i_\# : \mathrm{Tight}(Y_0, B_{0-}) \times \prod_{j=1}^{n-1} \mathrm{Tight}(Y_j, B_{j+}, B_{j-}) \times \mathrm{Tight}(Y_n, B_{n+}) \hookrightarrow \mathcal{F}.$$

We establish the following h-principle for families of tight contact structures on connected sums:

Theorem 2.7. *The inclusion $i_\#$ is a homotopy equivalence.*

Remark 2.3. Since S^2 is simply connected we deduce from the long exact sequence in homotopy groups of (2.11) that

$$\pi_0\big(\mathrm{Tight}(Y_\#)\big) \cong \prod_{j=0}^{n} \pi_0\big(\mathrm{Tight}(Y_j)\big)$$

which is the classical result of Colin [10].

Remark 2.4. Note how the homotopy-equivalence

$$C(Y_0, B_0, \xi_0) \times C(Y_1, B_1, \xi_1) \simeq C(Y_0 \# Y_1, B_\#, \xi_0 \# \xi_1)$$

(for tight contact structures ξ_0, ξ_1) has an analogue in monopole Floer homology. Namely, there's a *connected sum formula* for the "tilde" flavor (proved in Heegaard Floer theory [68]): for a field $\mathbb{F}$

$$\widetilde{HM}(-Y_0, \mathfrak{s}_{\xi_0}; \mathbb{F}) \otimes_{\mathbb{F}} \widetilde{HM}(-Y_1, \mathfrak{s}_{\xi_1}; \mathbb{F}) \cong \widetilde{HM}(-(Y_0 \# Y_1), \mathfrak{s}_{\xi_0 \# \xi_1}; \mathbb{F}).$$

2.2.2 The space of standard convex spheres in a tight contact 3-manifold

The next ingredient in the proof of Theorem 2.7 is an h-principle for standard convex embeddings in tight contact 3-manifolds due to Fernández–Martínez-Aguinaga–Presas [23] (see also the author's paper [24] with Eduardo Fernández, where a different proof is presented).

Consider the space of smooth embeddings $\mathrm{Emb}(\sqcup_j S^2, Y)$ of n-disjoint spheres and the corresponding subspace of standard spheres $\mathrm{CEmb}(\sqcup_j S^2, (Y,\xi))$. Fix also an arbitrary standard embedding $e : \sqcup S^2 \to (Y,\xi)$ and consider the subspaces $\mathrm{Emb}(\sqcup_j S^2, Y, \sqcup_j s_j)$ of embeddings that agree with e on an open neighbourhood $\sqcup_j U_j$ of the south pole s_j of each sphere. Similarly, consider the analogous subspace of standard embeddings $\mathrm{CEmb}(\sqcup_j S^2, (Y,\xi), \sqcup_j s_j)$.

Theorem 2.8 ([23])**.** *Assume that (Y,ξ) is tight. Then the inclusion $\mathrm{CEmb}(\sqcup_j S^2, (Y,\xi), \sqcup_j s) \hookrightarrow \mathrm{Emb}(\sqcup_j S^2, Y, \sqcup_j s_j)$ is a homotopy equivalence.*

Two key facts are exploited in the proof of this result which require the tightness condition. First, the h-principle of Eliashberg-Mishachev (Theorem 1.9). Secondly, that the space of convex spheres in (Y,ξ) with a fixed characteristic foliation and tight neighbourhood is C^0-dense inside the space of smoothly embedded spheres **when the contact 3-manifold is tight**, because of Giroux's Genericity and Realisation Theorems and Giroux's Tightness Criterion [30, 31]. We also have:

Corollary 2.9 ([23])**.** *Assume that (Y,ξ) is tight. For every $k \geq 1$ the natural homomorphism*

$$\pi_k(\mathrm{SO}(3)^n, \mathrm{U}(1)^n) \to \pi_k(\mathrm{Emb}(\sqcup_j S^2, Y), \mathrm{CEmb}(\sqcup_j S^2, (Y,\xi)))$$

induced by reparametrisation on the source is an isomorphism.

Proof of Corollary 2.9. Note that there is a natural map of fibrations given by the evaluation at the n south poles:

$$\begin{array}{ccccc} \mathrm{Emb}(\sqcup_j S^2, Y, \sqcup_j s_j) & \longrightarrow & \mathrm{Emb}(\sqcup_j S^2, Y) & \longrightarrow & \mathrm{Fr}_n(Y) \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{CEmb}(\sqcup_j S^2, (Y,\xi), \sqcup_j s_j) & \longrightarrow & \mathrm{CEmb}(\sqcup_j S^2, (Y,\xi)) & \longrightarrow & \mathrm{CFr}_n(Y,\xi) \end{array}$$

in which the vertical maps are inclusions. Here, the base $\mathrm{Fr}_n(Y)$ is the space of framings over n different points of M, that is, the total space of a fiber bundle over the configuration space $\mathrm{Conf}_n(Y)$ with fiber $\approx \mathrm{GL}^+(3)^n$, and likewise for $\mathrm{CFr}_n(Y,\xi)$ but with contact frames. Observe that the map between the fibers is a homotopy equivalence because of Theorem 2.8, so that the homomorphism induced by the evaluation map

$$\begin{aligned}\pi_k(\mathrm{Emb}(\sqcup_j S^2, Y), \mathrm{CEmb}(\sqcup_j S^2, (Y,\xi))) &\to \pi_k(\mathrm{Fr}_n(Y), \mathrm{CFr}_n(Y,\xi)) \\ &\cong \pi_k(\mathrm{SO}(3)^n, \mathrm{U}(1)^n)\end{aligned}$$

is an isomorphism and defines an inverse to the reparametrisation map. This concludes the proof. □

From the above we may deduce Theorem 1.10. We first discuss its smooth counterpart. The relevant reference on this topic is Hatcher's work [38]. Let $Y_\# = Y_- \# Y_+$ with $Y_\pm$ now *irreducible*. Let $\mathrm{Emb}(S^2, Y_\#)_{S_\#} \subset \mathrm{Emb}(S^2, Y_\#)$ be the subspace of smooth co-oriented embeddings $S^2 \hookrightarrow Y_\#$ isotopic to a fixed given one $S_\#$, and let

$$\mathcal{S} = \mathrm{Emb}(S^2, Y_\#)_{S_\#}/\mathrm{Diff}(S^2)$$

be the space of *unparametrised* co-oriented non-trivial spheres. Hatcher [38] proved that $\mathcal{S}$ is contractible. We also have a fibration

$$\mathrm{SO}(3) \simeq \mathrm{Diff}(S^2) \to \mathrm{Emb}(S^2, Y_\#)_{S_\#} \to \mathcal{S}$$

and hence

$$\mathrm{Emb}(S^2, Y_\#)_{S_\#} \simeq \mathrm{SO}(3).$$

Proof of Theorem 1.10. Immediate from the commuting diagram

$$\begin{array}{ccc} \mathrm{CEmb}\big(S^2, (Y_\#, \xi_\#)\big)_{S_\#} & \longrightarrow & \mathrm{Emb}\big(S^2, Y_\#\big)_{S_\#} \\ \uparrow & & \simeq\uparrow \\ \mathrm{U}(1) & \longrightarrow & \mathrm{SO}(3) \end{array}$$

combined with Corollary 2.9. □

2.2.3 Proof of Theorem 2.7

Let K be a compact parameter space and $G \subseteq K$ a subspace. It is enough to prove that: if $\xi^k \in \mathcal{F}$ is a K-family of tight contact structures on $Y_\#$ that coincide with $\xi_\#$ over the n Darboux balls $B_{\# j}$ and such that $\xi^k \in \mathrm{Im}(i_\#)$ for $k \in G$, then there exists a homotopy of tight contact structures ξ^k_t, $t \in [0,1]$, such that

- $\xi^k_0 = \xi^k$,
- $\xi^k_t = \xi^k$ for $k \in G$ and
- $\xi^k_1 \in \mathrm{Im}(i_\#)$.

The key point is to observe that $\xi^k \in \mathrm{Im}(i_\#)$ if and only if the embeddings $e_j : S^2 \hookrightarrow (Y_\#, \xi^k)$ are standard for $j = 1, \ldots, n$. For a given tight contact structure ξ denote by

$$\mathcal{CE}_\xi := \mathrm{CEmb}(\sqcup_{j=1}^n S^2, (Y_\#, \xi), \sqcup_{j=1}^n s_j))$$

the space of standard embeddings of n disjoint spheres that coincide with (e_j) over a neighbourhood of the south poles (s_j), and by

$$\mathcal{E} := \mathrm{Emb}(\sqcup_{j=1}^n S^2, Y_\#, \sqcup_{j=1}^n s_j))$$

the analogous space of smooth embeddings. Consider the space $\mathcal{X}$ of pairs (ξ, e_t) where $\xi \in \mathcal{F}$ and $e_t \in \mathcal{E}$, with $t \in [0,1]$, is a homotopy of embeddings with $e_0 = e$ and $e_1 \in \mathcal{CE}_\xi$. There is a natural forgetful map

$$p : \mathcal{X} \to \mathcal{F}, (\xi, e_t) \mapsto \xi,$$

which is in fact a fibration because of Lemma 2.1. By Theorem 2.8 we know that the inclusion $\mathcal{CE}_\xi \to \mathcal{E}$ is a homotopy equivalence. Therefore, the fibers of the previous fibration are contractible.

This is enough to conclude the proof. Indeed, our initial family ξ^k is given by a map $j : K \to \mathcal{F}$ and the pullback fibration $j^*\mathcal{X} \to K$ has a well-defined section over $G \subseteq K$ given by the constant isotopy $e_t^k = e$, $(k, t) \in G \times [0, 1]$. Since the fiber of this fibration is contractible we can extend this section over K obtaining a section e_t^k, $(k, t) \in K \times [0, 1]$. Then we apply the smooth isotopy extension theorem to this family of embeddings to find an isotopy $\varphi_t^k \in \mathrm{Diff}(Y_\#)$, $(k, t) \in K \times [0, 1]$, such that

- $\varphi_0^k = \mathrm{Id}$,
- φ_t^k is the identity over a neighbourhood of the south poles (s_j),
- $\varphi_t^k \circ e = e_t^k$,
- $\varphi_t^k = \mathrm{Id}$ for $(k, t) \in G \times [0, 1]$.

The homotopy of contact structures $\xi_t^k = (\varphi_t^k)^*\xi^k$ solves the problem since now $e = (\varphi_1^k)^{-1} \circ e_1^k$ is standard for $(\varphi_t^k)^*\xi^k$ because e_1^k is standard for ξ^k. The proof is complete. □

Chapter 3: The three-dimensional contact Dehn twist

3.1 Contact Dehn twists on spheres

In this section we define the contact Dehn twist on a sphere in several equivalent ways, establish some key properties and discuss some examples when its square is isotopic to the identity.

3.1.1 The contact Dehn twist

Let (Y, ξ) be a contact 3-manifold, and $S \subset Y$ be a co-oriented embedded sphere. Provided S has a tight neighbourhood, we can associate to S a contactomorphism τ_S well-defined in $\pi_0\mathrm{Cont}(Y, \xi)$. We discuss this construction now.

3.1.1.1 Local model

We start by discussing the local picture. Consider the contact 3-manifold $Y_0 = [-1, 1] \times S^2$ with the tight contact structure $\xi_0 = \mathrm{Ker}(\alpha_0)$ where $\alpha_0 = zds + \frac{1}{2}xdy - \frac{1}{2}ydx$. Here s is the standard coordinate on $[-1, 1]$ and x, y, z coordinates on $\mathbb{R}^3$ restricted onto the unit sphere S^2. Consider the sphere $S_0 = \{0\} \times S^2 \subset Y_0$. We now describe the contact Dehn twist τ_{S_0} on the sphere S_0.

We choose a smooth function $\theta : [-1, 1] \to [0, 2\pi]$ with $\theta(s) \equiv 0$ near $s = -1$ and $\theta(s) = 2\pi$ near $s = 1$. Let R_φ be the counterclockwise rotation in the xy plane with angle φ. Consider the diffeomorphism $\widetilde{\tau}_{S_0}$ of Y_0 given by a smooth Dehn twist along S_0

$$\widetilde{\tau}_{S_0}(s, x, y, z) = (s, R_{\theta(s)}(x, y), z).$$

Since $\pi_1\mathrm{SO}(3) = \mathbb{Z}/2$ it follows that the squared Dehn twist $\widetilde{\tau}^2_{S_0}$ is smoothly isotopic to the identity rel. ∂Y_0. We don't quite have a contactomorphism of (Y_0, ξ_0) since

$$\widetilde{\tau}_{S_0}^* \alpha_0 = \alpha_0 + \frac{\theta'(s)}{2}(x^2 + y^2)ds.$$

However, consider the naive interpolation from α_0 to $\widetilde{\tau}_{S_0}^* \alpha_0$

$$\alpha_t = \alpha_0 + t\frac{\theta'(s)}{2}(x^2 + y^2)ds$$

and observe that

Lemma 3.1. *For any $t \in [0, 1]$ the form α_t is a contact form.*

Proof. A straightforward calculation shows $\alpha_t \wedge d\alpha_t = \alpha_0 \wedge d\alpha_0 > 0$. □

Thus, by Gray stability (a.k.a Moser's argument) [25] the deformation of contact structures $\xi_t = \mathrm{Ker}(\alpha_t)$ is realised by an isotopy f_t i.e. $f_0 = \mathrm{id}$ and $(f_t)^* \xi_t = \xi_0$. Since the forms α_t don't depend on t near ∂Y_0 we may further assume that $f_t = \mathrm{id}$ near ∂Y_0. We then replace $\widetilde{\tau}_{S_0}$ with $\tau_{S_0} := \widetilde{\tau}_{S_0} \circ f_1$ and the latter is a contactomorphism of (Y_0, ξ_0). We also have that that the support of τ_{S_0} can be made arbitrarily close to the sphere S_0 by choosing $\theta(s)$ appropriately. Then, for any $\epsilon \in (0, 1]$ we have a well-defined isotopy class of contact Dehn twist

$$\tau_{S_0} \in \pi_0 \mathrm{Cont}([-\epsilon, \epsilon] \times S^2, \xi_0).$$

It is worth pointing out the following

Lemma 3.2. *The group* $\mathrm{Cont}(Y_0, \xi_0)$ *is homotopy equivalent to* $\Omega \mathrm{U}(1) \simeq \mathbb{Z}$. *It is generated by the contact Dehn twist τ_{S_0}.*

Proof. Gluing a Darboux ball B to (Y_0, ξ_0) gives back the standard contact ball $(\mathbb{B}^3, \xi_{\mathrm{st}})$. Thus, from the fibration (2.4) we have a map of fiber sequences

$$\begin{array}{ccccc}
\mathrm{Cont}(Y_0,\xi_0) & \longrightarrow & \mathrm{Cont}(\mathbb{B}^3,\xi_{\mathrm{st}}) & \longrightarrow & \mathrm{Emb}\big((\mathbb{B}^3,\xi_{\mathrm{st}}),(\mathbb{B}^3,\xi_{\mathrm{st}})\big) \\
\uparrow & & \simeq\uparrow & & \simeq\uparrow \\
\Omega\mathrm{U}(1) & \longrightarrow & \{*\} & \longrightarrow & \mathrm{U}(1)
\end{array}$$

where the middle homotopy equivalence follows from the h-principle of Eliashberg and Mishachev[22]. The first assertion now follows. For the second assertion, we need to show that the generator $1 \in \pi_1\mathrm{U}(1)$ maps to the class of the contact Dehn twist τ_{S_0} under the connecting map.

We first describe the contact Dehn twist on S_0 more conveniently in terms of the coordinates on the ball $\mathbb{B}^3 = B \cup Y_0$. Recall that the standard contact structure on $\mathbb{B}^3$ is $\xi_{\mathrm{st}} = \mathrm{Ker}\alpha_{\mathrm{st}}$ where $\alpha_{\mathrm{st}} = dz + \frac{1}{2}xdy - \frac{1}{2}ydx$. Choose a smooth function $\theta : [0,1] \to [0,2\pi]$ with $\theta = 0$ near 0 and $\theta = 2\pi$ near 1. Let $r^2 := x^2+y^2+z^2$ be the radius squared function on $\mathbb{B}^3$. Then the diffeomorphism of $\mathbb{B}^3$ given by

$$\widetilde{\tau}(x,y,z) := (R_{\theta(r^2)}(x,y),z)$$

does not quite preserve the contact structure, but

$$(\widetilde{\tau})^*\alpha_{\mathrm{st}} = \alpha_{\mathrm{st}} + \frac{1}{2}(x^2+y^2)\theta'(r^2)d(r^2).$$

As in Lemma 3.1, the obvious interpolation that takes the second term in the above identity to zero gives a path of *contact* forms, and as in §3.1.1.2 we may canonically deform $\widetilde{\tau}$ to a contactomorphism τ_{S_0} in the isotopy class of the contact Dehn twist on S_0.

Consider now a homotopy of maps $\theta_t : [0,1] \to [0,2\pi]$ with with θ_t constant near 1 (with value 2π), such that $\theta_0 = \theta$ and θ_1 is the constant function with value 2π. We obtain an isotopy through diffeomorphisms of $\mathbb{B}^3$ (fixing a neighbourhood of the boundary $\partial\mathbb{B}^3$, but not the smaller ball B!) given by

$$\widetilde{\tau}_t(x,y,z) := (R_{\theta_t(r^2)}(x,y),z)$$

such that $\widetilde{\tau}_0 = \widetilde{\tau}$ and $\widetilde{\tau}_1 = \mathrm{id}$. Again, by observing that for each t the obvious interpolation from $(\widetilde{\tau}_t)^*\alpha_{\mathrm{st}}$ and α_{st} gives a path of contact forms, we may canonically deform the isotopy $\widetilde{\tau}_t$ to a *contact* isotopy τ_t with $\tau_0 = \tau_{S_0}$ and $\tau_1 = \mathrm{id}$.

Now, the path of contactomorphisms τ_{1-t} from the identity to $\tau_{\partial B}$ induces a *loop* of Darboux balls $(\tau_{1-t})(B)$ in the class of the generator $1 \in \mathbb{Z} = \pi_1 \mathrm{Emb}\big((\mathbb{B}^3, \xi_{\mathrm{st}}), (\mathbb{B}^3, \xi_{\mathrm{st}})\big)$. From this the required result now follows. $\square$

Likewise, we have a firm hold on the topology of the space of standard spheres in our local picture. Denote by $e_0 : S^2 \hookrightarrow Y_0$ the embedding of $S_0 \subset Y_0$.

Lemma 3.3. *The map induced by reparametrisation of* e_0

$$\mathrm{U}(1) \to \mathrm{CEmb}(S^2, (Y_0, \xi_0)) \quad , \quad \theta \mapsto e_0 \circ r_\theta$$

is a homotopy equivalence. Here $r_\theta(x, y, z) = (R_\theta(x, y), z)$*. Under the connecting homomorphism of the fibration (2.7) the generator of* $\pi_1 \mathrm{U}(1) = \mathbb{Z}$ *maps to the class*

$$(\tau_{S_{-1/2}})^{-1} \tau_{S_{1/2}} \in \pi_0 \mathrm{Cont}(Y_0, \xi_0, S_0).$$

Proof. We have the following map of fiber sequences, with homotopy equivalences on the fiber and total space by Lemma 3.2

$$\begin{array}{ccccc}
\mathrm{Cont}(Y_0, \xi_0, S_0) & \longrightarrow & \mathrm{Cont}(Y_0, \xi_0) & \longrightarrow & \mathrm{CEmb}(S^2, (Y_0, \xi_0)) \\
\simeq\uparrow & & \simeq\uparrow & & \uparrow \\
\Omega\mathrm{U}(1) \times \Omega\mathrm{U}(1) & \longrightarrow & \Omega\mathrm{U}(1) & \longrightarrow & \mathrm{U}(1)
\end{array}$$

This establishes both assertions. $\square$

3.1.1.2 General case

The robustness of our local picture allows us to consider contact Dehn twists in more general settings. We fix a 3-manifold (Y, ξ) together with a co-oriented *standard convex sphere* $S \subset Y$ i.e.

an embedded sphere whose characteristic foliation agrees with that of $S_0 \subset Y_0$ in the local model. It follows that neighbourhoods of $S \subset Y$ and $S_0 \subset Y_0$ are contactomorphic in a (homotopically) canonical fashion [30, 25], and by making the support of τ_{S_0} sufficiently close to S_0 we may therefore implant τ_{S_0} into (Y, ξ) as a compactly supported contactomorphism τ_S, which we refer to as the *contact Dehn twist* on the co-oriented standard convex sphere $S \subset Y$. The class of τ_S in $\pi_0\text{Cont}(Y, \xi)$ only depends on the isotopy class of S in the space of co-oriented standard convex spheres, defining a map of sets

$$\pi_0\text{CEmb}(S^2, (Y, \xi)) \to \pi_0\text{Cont}(Y, \xi) \quad , \quad S \mapsto \tau_S$$

The contactomorphism τ_S makes sense more generally whenever $S \subset Y$ is a just a convex co-oriented sphere with a *tight neighbourhood* U (but not necessarily having standard characteristic foliation). Indeed, by Giroux's Criterion [31] the dividing set of S is connected. Then by Giroux's Realisation theorem, we may find a smooth isotopy of sphere embeddings S_t whose image lies in the tight neighbourhood U, $S_0 = S$ and S_1 is a *standard* convex sphere, to which we associate the Dehn twist τ_{S_1} by the previous construction. A different choice of isotopy S'_t may yield a different standard convex sphere S'_1. The two spheres (S_1 and S'_1)) are isotopic within U as *standard* convex spheres by a result of Colin ([10], Proposition 10), so the contact Dehn twists τ_{S_1} and $\tau_{S'_1}$ are contact isotopic. Therefore, we have a well defined contact Dehn twist $\tau_S \in \pi_0\text{Cont}(Y, \xi)$ associated to the convex sphere S with tight neighbourhood U. In fact, since any *smooth* sphere can be made convex by a small isotopy [30], this construction defines a map

$$\pi_0\text{Emb}_{\text{tight}}(S^2, (Y, \xi)) \to \pi_0\text{Cont}(Y, \xi) \quad , \quad S \mapsto \tau_S$$

where $\text{Emb}_{\text{tight}}(S^2, (Y, \xi))$ stands for the space of *smooth* co-oriented embeddings $S^2 \subset Y$ which admit a tight neighbourhood. In particular, if (Y, ξ) is tight (globally) then τ_S only depends up to contact isotopy on the *smooth* isotopy class of the co-oriented sphere S.

The following particular case will play an essential role in this article, so we emphasize it now. Consider a Darboux ball $B = \phi(\mathbb{B}^3)$ in a contact manifold (Y, ξ). Associated to an exterior sphere (i.e. contained in the complement $Y \setminus B$) parallel to ∂B we have a well defined contact Dehn twist which fixes B pointwise. By abuse in notation and for convenience we denote this contactomorphism by $\tau_{\partial B}$ even if the Dehn twist is not on the sphere ∂B. This defines a map of sets

$$\pi_0\mathrm{Emb}\big((\mathbb{B}^3, \xi_{\mathrm{st}}), (Y, \xi)\big) \to \pi_0\mathrm{Cont}(Y, \xi, B) \quad , \quad B \mapsto \tau_{\partial B}.$$

The following convenient description of $\tau_{\partial B}$ follows from the local calculation in the proof of Lemma 3.2.

Lemma 3.4. *The Dehn twist* $\tau_{\partial B} \in \pi_0\mathrm{Cont}(Y, \xi, B)$ *agrees with the image of* $1 \in \mathbb{Z}$ *under the map*

$$\mathbb{Z} = \pi_1\mathrm{U}(1) \to \pi_1\mathrm{Emb}((\mathbb{B}^3, \xi_{\mathrm{st}}), (Y, \xi)) \to \pi_0\mathrm{Cont}(Y, \xi, B)$$

where the first map is induced by the reparametrisation map

$$\mathrm{U}(1) \to \mathrm{Emb}\big((\mathbb{B}^3, \xi_{\mathrm{st}}), (Y, \xi)\big) \quad , \quad \theta \mapsto \phi \circ r_\theta$$

and the second map is the connecting map in the long exact sequence of the fibration (2.4).

If $S = e(S^2) \subset (Y, \xi)$ is a co-oriented standard convex sphere, let $S_\pm$ be two parallel copies of S given by pushing S forward and backward. By the local calculation in Lemma 3.3 we have:

Lemma 3.5. *The product of Dehn twists* $(\tau_{S_-})^{-1}\tau_{S_+} \in \pi_0\mathrm{Cont}(Y, \xi, S)$ *agrees with the image of* $1 \in \mathbb{Z}$ *under the map*

$$\mathbb{Z} = \pi_1\mathrm{U}(1) \to \pi_1\mathrm{CEmb}\big(S^2, (Y, \xi)\big) \to \pi_0\mathrm{Cont}(Y, \xi, S)$$

where the first map is induced by the reparametrisation map

$$\mathrm{U}(1) \to \mathrm{CEmb}\big(S^2, (Y,\xi)\big) \quad , \quad \theta \mapsto e \circ r_\theta$$

and the second map is the connecting map in the long exact sequence of the fibration (2.7).

3.1.2 The Dehn twist and the evaluation map

We move on to study a *relative* version of the isotopy problem for the Dehn twist. Consider the Dehn twist $\tau_{\partial B}$ on an exterior sphere parallel to the boundary ∂B of a Darboux ball, as in the previous section. We will now rephrase the problem of whether $\tau_{\partial B}^2$ defines the trivial class in $\pi_0\mathrm{Cont}_0(Y,\xi,B)$ as a *lifting* problem.

The main player is the evaluation mapping $ev : C(Y,\xi) \to S^2$ defined by (1.3), which is a fibration (Lemma 2.3). If $\delta : \pi_2 S^2 \to \pi_1 C(Y,\xi,B)$ is the connecting map in the homotopy long exact sequence, then we have a distinguished class

$$O_\xi := \delta(1) \in \pi_1 C(Y,\xi,B) \tag{3.1}$$

which, by construction, is the *obstruction class* to finding a homotopy section of ev (i.e. a map $s : S^2 \to C(Y,\xi)$ such that $ev \circ s : S^2 \to S^2$ has degree one):

ev admits a homotopy section if and only if $O_\xi = 0$.

We relate the problem of finding a section of ev to the triviality of the Dehn twist $\tau_{\partial B}^2$ as follows. Consider the connecting map $\delta' : \pi_1 C(Y,\xi,B) \to \pi_0\mathrm{Cont}_0(Y,\xi,B)$ of the fibration (2.3). The key observation is the following:

Proposition 3.6. *The class $\delta'(O_\xi) \in \pi_0\mathrm{Cont}_0(Y,\xi,B)$ agrees with the* ***squared*** *contact Dehn twist $\tau_{\partial B}^2$.*

Proof. Consider first the case when (Y,ξ) is the contact unit ball $(\mathbb{B}^3, \xi_{\mathrm{st}} = \mathrm{Ker}(dz + \frac{1}{2}x dy - \frac{1}{2} y dx))$ and $B \subset \mathbb{B}^3$ a subball of smaller radius with center at 0. The fibrations from §2.1.2 fit into a commuting diagram

$$\begin{array}{ccccc} C(\mathbb{B}^3,\xi_{\mathrm{st}},B) & \longrightarrow & C(\mathbb{B}^3,\xi_{\mathrm{st}}) & \xrightarrow{ev} & S^2 \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{Diff}_0(\mathbb{B}^3,B) & \longrightarrow & \mathrm{Diff}_0(\mathbb{B}^3) & \longrightarrow & \mathrm{Emb}(\mathbb{B}^3,\mathbb{B}^3)\simeq \mathrm{SO}(3) \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{Cont}_0(\mathbb{B}^3,\xi_{\mathrm{st}},B) & \longrightarrow & \mathrm{Cont}_0(\mathbb{B}^3,\xi_{\mathrm{st}}) & \longrightarrow & \mathrm{Emb}((\mathbb{B}^3,\xi_{\mathrm{st}}),(\mathbb{B}^3,\xi_{\mathrm{st}}))\simeq \mathrm{U}(1) \end{array}$$

In the third vertical fiber sequence the map $\pi_2 S^2 = \mathbb{Z} \to \pi_1 \mathrm{U}(1) = \mathbb{Z}$ is multiplication by 2. From the diagram we see that the image of $O_{\xi_{\mathrm{st}}} \in \pi_1 C(\mathbb{B}^3,\xi_{\mathrm{st}},B)$ in $\pi_0\mathrm{Cont}_0(\mathbb{B}^3,\xi_{\mathrm{st}},B)$ can be alternatively calculated as the image of $2 \in \mathbb{Z} = \pi_1\mathrm{U}(1)$ in $\pi_0\mathrm{Cont}_0(\mathbb{B}^3,\xi_{\mathrm{st}},B)$. From Lemma 3.4 this is the class of $\tau^2_{\partial B}$.

For an arbitrary (Y,ξ) and a Darboux ball $B \subset Y$ the result then follows from the previous local calculation by extending the contact embedding $B \hookrightarrow Y$ to a contact embedding $B \subset \mathbb{B}^3 \hookrightarrow Y$, and considering the commuting diagram

$$\begin{array}{ccccc} \pi_2 S^2 & \longrightarrow & \pi_1 C(\mathbb{B}^3,\xi_{\mathrm{st}},B) & \longrightarrow & \pi_0\mathrm{Cont}_0(\mathbb{B}^3,\xi_{\mathrm{st}},B) \\ \downarrow & & \downarrow & & \downarrow \\ \pi_2 S^2 & \longrightarrow & \pi_1 C(Y,\xi,B) & \longrightarrow & \pi_0\mathrm{Cont}_0(Y,\xi,B). \end{array}$$

□

Corollary 3.7. *Suppose Y is aspherical (i.e. irreducible and with infinite fundamental group). Then $\tau^2_{\partial B}$ is isotopic to the identity rel. B if and only if the evaluation mapping (1.3) admits a homotopy section (i.e. the obstruction class O_ξ vanishes).*

Proof. By the fibration (2.3) we have the exact sequence

$$\pi_1\mathrm{Diff}_0(Y,B) \to \pi_1 C(Y,\xi,B) \to \pi_0\mathrm{Cont}_0(Y,\xi,B)$$

so by Proposition 3.6 the result will follow from $\pi_1\mathrm{Diff}(Y,B) = 0$. Let us now explain why the latter group vanishes. By the fibration (2.5) we have an exact sequence

$$1 \to \pi_1 \mathrm{Diff}(Y,B) \to \pi_1 \mathrm{Diff}(Y) \to \pi_1 \mathrm{Fr}(Y) \cong \pi_1 Y \times \mathbb{Z}_2 \to \pi_0 \mathrm{Diff}(Y,B).$$

Here to have a 1 on the left we use $\pi_2 Y = 0$ (which follows from Y being aspherical). Suppose for the moment that $\pi_1 \mathrm{Diff}(Y) \to \pi_1 Y$ was injective. This would give us, by the second exact sequence above, that the Dehn twist $\tau_{\partial B}$ is non-trivial in $\pi_0 \mathrm{Diff}(Y,B)$ because the class $(0,1) \in \pi_1 Y \times \mathbb{Z}_2$ is not in the image of $\pi_1 \mathrm{Diff}(Y)$. Thus from the exact sequence we see that $\pi_1 \mathrm{Diff}(Y,B) = 0$, as required. Finally, because Y is an aspherical 3-manifold, then $\pi_1 \mathrm{Diff}_0(Y) \to \pi_1 Y$ is indeed injective. This follows from the calculation of the homotopy type of the group $\mathrm{Diff}_0(Y)$ for all aspherical[1] 3-manifolds. More precisely, the papers [39, 41, 38, 44, 58] cover all aspherical 3-manifolds with the exception of the non-Haken infranil manifolds (see [58] for a nice summary). The latter consist of the non-trivial S^1-bundles over T^2, which are covered by [1]. In all these cases $\mathrm{Diff}_0(Y)$ has the homotopy type of $(S^1)^k$ where k is the rank of the center of $\pi_1 Y$ and $\pi_1 \mathrm{Diff}(Y) \to \pi_1 Y$ is the inclusion of the center. The proof is now complete. □

3.1.3 Formal triviality of $\tau^2_{\partial B}$

We continue in the setting of the previous section, and we show

Lemma 3.8. *Suppose the Euler class of ξ vanishes. Then both the loop of contact structures given by the obstruction class $O_\xi \in \pi_1 C(Y,\xi,B)$ and the squared Dehn twist $\tau^2_{\partial B} \in \pi_0 \mathrm{Cont}_0(Y,\xi,B)$ are formally trivial rel. B.*

Proof. On the space of co-oriented plane fields we have an analogous evaluation mapping (a fibration also, in fact)

$$\Xi(Y,\xi) \to S^2 \quad , \quad \xi' \mapsto \xi'(0).$$

[1]For the irreducible 3-manifolds with finite fundamental group, the calculation of the homotopy type of $\mathrm{Diff}_0(Y)$ has also been completed [43, 3, 2]. Thus, the homotopy type of $\mathrm{Diff}_0(Y)$ is known for all prime 3-manifolds.

When the Euler class of ξ vanishes then we may identify $\Xi(Y,\xi)$ with the space $\mathrm{Map}_0(Y,S^2)$ of null-homotopic smooth maps $Y \longrightarrow S^2$. The evaluation mapping becomes identified with the obvious evaluation mapping on this latter space. Clearly this fibration admits a section given by the constant maps $Y \to S^2$. Hence, the corresponding obstruction class vanishes, and hence

$$O_\xi \in \mathrm{Ker}\big(\pi_1 C(Y,\xi,B) \to \pi_1 \Xi(Y,\xi,B)\big)$$

so O_ξ is formally trivial. From the rel. B analogue of Corollary 2.6 it follows that $\tau^2_{\partial B}$ is formally trivial also. □

3.1.4 Behaviour of O_ξ under sum

We proceed by discussing how the obstruction class O_ξ from (3.1) interacts with formation of connected sums.

First we briefly review a convenient model for the contact connected sum, following [25]. Let $(Y_\pm, \xi_\pm)$ be two contact 3-manifolds with Darboux balls $B_\pm \subset Y_\pm$ with coordinates x, y, z. On $B_\pm$ the contact structures look standard

$$\xi_\pm|_{B_\pm} = \mathrm{Ker}(dz + \frac{1}{2}xdy - \frac{1}{2}ydx).$$

Definition 3.1. The *connected sum* of contact manifolds

$$(Y_\#, \xi_\#) := (Y_-, \xi_-)\#(Y_+, \xi_+)$$

is defined as follows. On $\mathbb{R}^4$ with coordinates (x,y,z,t) and symplectic form $\omega_{\mathrm{st}} = dx\wedge dy + dz\wedge dt$, we have a Liouville vector field $v_{\mathrm{st}} = \frac{1}{2}x\partial_x + \frac{1}{2}y\partial_y + 2z\partial_z - t\partial_t$ which on the hypersurfaces $\{t = \pm 1\}$ induces the contact structure $\ker(\iota_{v_{\mathrm{st}}}\omega_{\mathrm{st}}) = \mathrm{Ker}(\pm dz + \frac{1}{2}xdy - \frac{1}{2}ydx)$. Attach a smooth 1-handle $H := [-1,1] \times \mathbb{B}^3$ by an embedding $H \hookrightarrow \mathbb{R}^4$ that connects the hyperplanes $\{t = \pm 1\}$ by gluing $\{\pm 1\} \times \mathbb{B}^3 \subset \partial H$ in a standard manner with $\mathbb{B}^3 \subset \{t = \pm 1\}$. The embedding of H must be such that v_{st} is transverse to the boundaries of H.

Next, we identify $\mathbb{B}^3 \subset \{t = \pm 1\}$ with Darboux balls $B_\pm \subset Y_+$ (where the identification with B_- is by the orientation-reversing map $(x, y, z) \mapsto (x, y, -z)$). Thus, we can glue the boundary piece $[-1, 1] \times \partial\mathbb{B}^3 \subset H$ to $Y_- \setminus B_- \sqcup Y_+ \setminus B_+$ and this yields the manifold $Y_\#$ together with a contact structure $\xi_\#$ that restricts to $\xi_\pm$ over $Y_\pm \setminus B_\pm$.

We will fix a third Darboux ball $B_\# \subset Y_\#$ inside the neck region $[-1, 1] \times \partial\mathbb{B}^3 \subset Y_\#$. We also have natural inclusions $C(Y_\pm, \xi_\pm, B_\pm) \subset C(Y_\#, \xi_\#, B_\#)$. We consider their induced maps on π_1

$$(-)\#\xi_+ : \pi_1 C(Y_-, \xi_-, B_-) \to \pi_1 C(Y_\#, \xi_\#, B_\#)$$

$$\xi_-\#(-) : \pi_1 C(Y_+, \xi_+, B_+) \to \pi_1 C(Y_\#, \xi_\#, B_\#)$$

Proposition 3.9. *The obstruction class $O_{\xi_\#} \in \pi_1 C(Y_\#, \xi_\#, B_\#)$ is given by*

$$O_{\xi_\#} = (O_{\xi_-}\#\xi_+) \cdot (\xi_-\#O_{\xi_+}).$$

Proof. Consider the contact manifold $(\mathbb{B}^3, \xi_{\mathrm{st}} = \ker(dz + \frac{1}{2}xdy - \frac{1}{2}ydx))$ and let $B \subset \mathbb{B}^3$ be a smaller Darboux ball with center $0 \in \mathbb{B}^3$. We first describe an explicit loop in the class $O_{\xi_{\mathrm{st}}} \in \pi_1 C(\mathbb{B}, \xi_{\mathrm{st}}, B)$. Let $q : [0, 1] \times S^1 \to \mathrm{SO}(3)/\mathrm{U}(1)$ be a map such that $q(r, 0) = q(r, 1) = [\mathrm{id}]$ and the induced map $S(S^1) \to \mathrm{SO}(3)/\mathrm{U}(1)$ from the unreduced suspension of S^1 is a homeomorphism. By the homotopy lifting property of $\mathrm{SO}(3) \to \mathrm{SO}(3)/\mathrm{U}(1)$ we may find matrices $A_{r,\varphi} \in \mathrm{SO}(3)$ (with $(r, \varphi) \in [0, 1] \times S^1$) such that $q(r, \varphi) = [A_{r,\varphi}]$ and $A_{0,\varphi} = A_{r,0} = \mathrm{Id}$. Consider the vector field $V_{r,\varphi} = \partial_\varphi A_{r,\varphi}$ on $\mathbb{R}^3$, which we regard as an r-family of φ-dependent vector fields. Cut off $V_{r,\varphi}$ outside B and let ϕ_r^φ be the induced flow (starting at time $\varphi = 0$) with $\varphi \in \mathbb{R}$ now, which we regard as a flow on $\mathbb{B}^3$ supported in B. Then $\xi_{r,\varphi} := (\phi_r^\varphi)_* \xi_{\mathrm{st}}$ gives a family of contact structures in $C(\mathbb{B}^3, \xi_{\mathrm{st}})$ parametrised by $(r, \varphi) \in [0, 1] \times \mathbb{R}$. Because $\mathrm{U}(1) \subset \mathrm{SO}(3)$ acts by contactomorphisms of ξ_{st} then we see that the family $\xi_{r,\varphi}$ is in fact parametrised by $(r, \varphi) \in [0, 1] \times S^1/\{0\} \times S^1 \cong \mathbb{B}^2$ and $\xi_{r,0} = \xi_{\mathrm{st}}$. Evaluating the $\mathbb{B}^2$-family $\xi_{r,s}$ at the point $0 \in B$ yields a map from $S^2 = \mathbb{B}^2/\partial\mathbb{B}^2$ into $S(T_0 B) = S^2$ which represents the class of $1 \in \mathbb{Z} = \pi_2 S^2$. The loop of contact structures $\xi_{1,\varphi}$, which lies in $C(\mathbb{B}^3, \xi_{\mathrm{st}}, B)$, is therefore a representative of the class $O_{\xi_{\mathrm{st}}}$.

For an arbitrary (Y, ξ) we obtain a representative loop of $O_\xi \in \pi_1 C(Y, \xi, B)$ out of the loop constructed in the previous paragraph extending it by ξ outside $B \subset Y$. Let $\xi^{\pm}_{1,\varphi}$ denote the loops representing $O_{\xi_\pm} \in \pi_1 C(Y_\pm, \xi_\pm, B_\pm)$. Now, on the 1-handle $H = [-1, 1] \times \mathbb{B}^3 \hookrightarrow \mathbb{R}^4$ from Definition 3.1 we have the $\mathbb{B}^2$-family of symplectic structures $\omega_{r,\varphi} := (\phi^{\varphi}_r)_* \omega_{\mathrm{st}}$ and corresponding Liouville vector fields $v_{r,\varphi} := (\phi^{\varphi}_r)_* v_{\mathrm{st}}$ transverse to the boundaries of H. The induced $\mathbb{B}^2$-family of contact structures $\xi_{\#,r,\varphi} \in C(Y_\#, \xi_\#)$ has the property that $\xi_{\#,1,\varphi}$ represents $O_{\xi_\#}$. In a self-evident notation, we have $\xi_{\#,r,\varphi} = \xi^-_{r,\varphi} \# \xi^+_{r,\varphi}$. In particular $\xi_{\#,1,\varphi} = \xi^-_{1,\varphi} \# \xi^+_{1,\varphi}$, which completes the proof.

□

Remark 3.1. In particular, it follows from Propositions 3.9 and 3.6 that $\tau^2_{\partial B_+} \tau^2_{\partial B_-} = \tau^2_{B_\#}$ in $\pi_0 \mathrm{Cont}_0(Y_\#, \xi_\#, \partial B_\#)$.

3.1.5 Examples: trivial Dehn twists

For comparison with Theorem 1.5 we now exhibit examples where the squared Dehn twist on a connected sum becomes trivial as a contactomorphism.

3.1.5.1 Quotients of S^3

Let Γ be a finite subgroup of $\mathrm{U}(2)$. Then Γ preserves the standard contact structure $\xi_{\mathrm{st}} = \ker(\sum_{j=1,2} x_j dy_j - y_j dx_j)$ on the unit 3-sphere S^3, so it descends onto the quotient $M_\Gamma = S^3/\Gamma$. The M_Γ's are the spherical 3-manifolds and include, among others, the lens spaces $L(p, q)$ and the Poincaré sphere $\Sigma(2, 3, 5)$.

Lemma 3.10. *The squared Dehn twist $\tau^2_{\partial B}$ on the boundary of a Darboux ball $B \subset M_\Gamma$ is contact isotopic to the identity rel. B. Hence the squared Dehn twist $\tau^2_{S_\#}$ on the separating sphere $S_\#$ in $(Y, \xi) \# (M_\Gamma, \xi_{\mathrm{st}})$ is contact isotopic to the identity.*

Proof. The center of $\mathrm{U}(2)$ is given by the subgroup $\cong \mathrm{U}(1)$ of diagonal matrices with diagonal (λ, λ) for some $\lambda \in \mathrm{U}(1)$. This subgroup acts on Y_Γ by contactomorphisms and thus also on the space of Darboux balls. This gives a map $\pi_1 \mathrm{U}(1) = \mathbb{Z} \to \pi_1(M_\Gamma \times \mathrm{U}(1)) = \Gamma \times \mathbb{Z}$ which we assert is given by $1 \mapsto (e, 2)$ where $e \in \Gamma$ is the identity element. From Lemma 3.4 and this assertion, the result would follow.

That the component $\mathbb{Z} \to \Gamma$ is trivial follows from $\mathrm{U}(1)$ being the center of $\mathrm{U}(2)$. To verify that $\mathbb{Z} \to \mathbb{Z}$ is multiplication by 2 we need to calculate the change in contact framing under the action of $\mathrm{U}(1)$. We view S^3 as the unit sphere in the quaternions $\mathbb{H} = \mathbb{R}\langle 1, i, j, k\rangle$, so the tangent space at $q \in S^3$ is given by $T_qS^3 = \mathbb{R}\langle iq, jq, kq\rangle$ and the standard contact structure is $\xi_{\mathrm{st}}(q) = \mathbb{R}\langle jq, kq\rangle = \mathbb{C}\langle jq\rangle$. Thus, the frame jq trivializes $\xi_{\mathrm{st}} \cong \mathbb{C}$ as a complex line bundle. The center subgroup $\mathrm{U}(1) \subset \mathrm{U}(2)$ acts on S^3 by $(\lambda, q) \mapsto \lambda q$, and the action of $\mathrm{U}(1)$ on the frame jq is

$$\lambda \cdot jq = j\overline{\lambda}q = \lambda^2 \cdot j(\lambda q)$$

and thus the action on $\xi_{\mathrm{st}} \cong \mathbb{C}$ is by multiplication by λ^2 on the fibres. This establishes our assertion, and hence the proof is complete. □

Remark 3.2. When $\Gamma \subset \mathrm{SU}(2)$, an alternative proof of Lemma 3.10 can be obtained by instead exhibiting a section of $ev : C(M_\Gamma, \xi_{\mathrm{st}}) \to S^2$. The point is that the radial vector field $x\partial_x + y\partial_y + z\partial_z + w\partial_w$ is a Liouville vector field for each of the symplectic forms ω_u, $u \in S^2$, in the flat hyperkähler structure of $\mathbb{R}^4$. The induced S^2-family of contact structures ξ_u on S^3 descends to the quotients M_Γ (with $\Gamma \subset \mathrm{SU}(2)$) and provides a section of ev.

3.1.5.2 Tight $S^1 \times S^2$

Consider the unique tight contact structure on $S^1 \times S^2$, given by $\xi_0 = \mathrm{Ker}(zd\theta + \frac{1}{2}xdy - \frac{1}{2}ydx)$.

Lemma 3.11. *The squared Dehn twist $\tau^2_{\partial B}$ on the boundary of a Darboux ball $B \subset S^1 \times S^2$ is contact isotopic to the identity rel. B. Hence the squared Dehn twist $\tau^2_{S_\#}$ on the separating sphere $S_\#$ in any contact connected sum of the form $(Y, \xi)\#(S^1 \times S^2, \xi_0)$ is contact isotopic to the identity.*

Proof. By considering the subgroup $\mathrm{U}(1) \subset \mathrm{Cont}(S^1 \times S^2, \xi_0)$ given by rotating the S^2 factor along the z-axis one easily checks that $\pi_1(\mathrm{Cont}(S^1 \times S^2, \xi_0) \to \pi_1\mathrm{Emb}((\mathbb{B}^3, \xi_{\mathrm{st}}), (S^1 \times S^2, \xi_0)) \to \pi_1\mathrm{U}(1)$ is surjective, so the result follows. □

Remark 3.3. In turn, the contact Dehn twist on the non-trivial sphere in $(S^1 \times S^2, \xi_0)$ is non-trivial (and with infinite order). However, it is formally non-trivial already and therefore not exotic, see §3.1.6.

3.1.5.3 Sum with an overtwisted contact 3-manifold

Let $(r, \theta, z) \in \mathbb{R}^3$ be cylindrical coordinates. Consider the contact structure ξ_{ot} in $\mathbb{R}^3$ defined by the kernel of

$$\alpha_{\text{ot}} = \cos r dz + r \sin r d\theta.$$

The disk $\Delta_{\text{ot}} = \{(r, \theta, z) \in \mathbb{R}^3 : z = 0, r \leq \pi\}$ is an *overtwisted disk.*

Definition 3.2 (Eliashberg [19])**.** An overtwisted contact 3-manifold is a contact 3-manifold that contains an embedded overtwisted disk.

Let $C(Y, \Delta_{\text{ot}})$ be the space of contact structures in Y with a fixed overtwisted disk $\Delta_{\text{ot}} \subset Y$. Let $\Xi(Y, \Delta_{\text{ot}})$ be the space of co-oriented plane fields in Y tangent to Δ_{ot} at the point $0 \in \Delta_{\text{ot}}$. A foundational result of Eliashberg, generalised in higher dimensions by Borman, Eliashberg and Murphy, is

Theorem 3.12 (Eliashberg [19, 5])**.** *The inclusion*

$$C(Y, \Delta_{\text{ot}}) \to \Xi(Y, \Delta_{\text{ot}})$$

is a homotopy equivalence.

Remark 3.4. A relative version Eliashberg's h-principle is available. Suppose $A \subseteq Y \setminus \Delta_{\text{ot}}$ is compact and $Y \backslash A$ is connected. Given a family of co-oriented plane fields $\xi^k \in \Xi(Y, \Delta_{\text{ot}})$ that is contact over an open neighbourhood of A there exists a homotopy rel. A from ξ^k to a family of contact structures.

Using Eliashberg's h-principle we obtain

Lemma 3.13. *Let (Y, ξ) be a contact 3-manifold with vanishing Euler class. Then, for every overtwisted contact 3-manifold (M, ξ_{ot}) the squared contact Dehn twist $\tau^2_{S_\#}$ in $(Y, \xi)\#(M, \xi_{\text{ot}})$ is contact isotopic to the identity.*

Proof. Let $B \subset (Y, \xi)$ be a Darboux ball that we remove when performing the connected sum. By Lemma 3.8 we have that $\tau^2_{\partial B}$ is formally contact isotopic to the identity rel. B. It follows that $\tau^2_{S_\#}$ is formally contact isotopic to the identity on $Y\#M$, in fact relative to a small ball B_{ot} containing an overtwisted disk $\Delta_{\text{ot}} \subset M$. At this point, by Eliashberg's Theorem 3.12 and Lemma 2.5 applied to the contact 3-manifold with convex boundary $(Y\#(M \setminus B_{\text{ot}}), \xi\#\xi_{\text{ot}})$ we see that the group of contactomorphisms fixing Δ_{ot} is homotopy equivalent to the corresponding space of formal contactomorphisms. The result now follows. □

In §3.3 we will see that Lemma 3.13 implies exotic 1-parametric phenomena in overtwisted contact 3-manifolds.

3.1.6 The Reidemeister I Move and Gompf's Contactomorphism

We now describe the contact Dehn twist diagrammatically by means of front projections of Legendrian arcs. This approach is in the spirit of Gompf's description [34] of the contact Dehn twist. For convenience we consider the unit ball $(\mathbb{B}^3, \xi = \ker(dz - ydx))$. Let $Y_0 = [-1, 1] \times S^2$ be the complement in $\mathbb{B}^3$ of a small open ball B_ε around the origin. Consider the standard Legendrian arc $l : [-1, 1] \to \mathbb{B}^3$, $t \mapsto (t, 0, 0)$. Perform two Reidemeister I moves to the Legendrian l to obtain a second Legendrian arc $\hat{l}$. We may assume that $\hat{l}$ coincides with l over the B_ε. The front of these arcs are depicted in Figure 3.1.

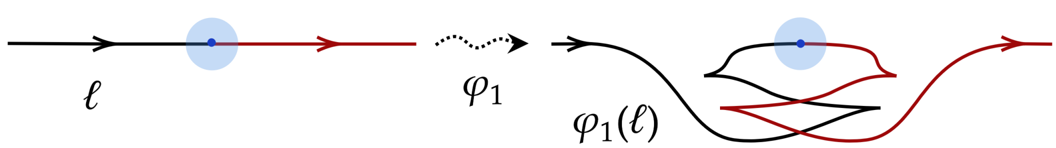

Figure 3.1: Front projection of l and $\hat{l}$. The blue ball represents the small ball $B_\varepsilon \subset \mathbb{B}^3$.

These arcs are Legendrian isotopic, so there exists a contact isotopy $\varphi_t \in \text{Cont}(\mathbb{B}^3, \xi)$ with $\varphi_0 = \text{id}$ and $\varphi_1 \circ l = \hat{l}$. Moreover, φ_1 can be taken to be the identity over B_ε. Therefore, φ_1

gives a contactomorphism τ of the contact manifold with convex boundary (Y_0, ξ). From now on, we will denote the restrictions of l and $\hat{l}$ to the red segments in Figure 3.1 by the same letters for convenience. We have $\tau(l) = \hat{l}$ and the arc $\hat{l}$ is obtained in (Y_0, ξ) from l by a positive stabilization, see Figure 3.2. In particular,

$$\mathrm{rot}(\tau(l)) = \mathrm{rot}(l) + 1.$$

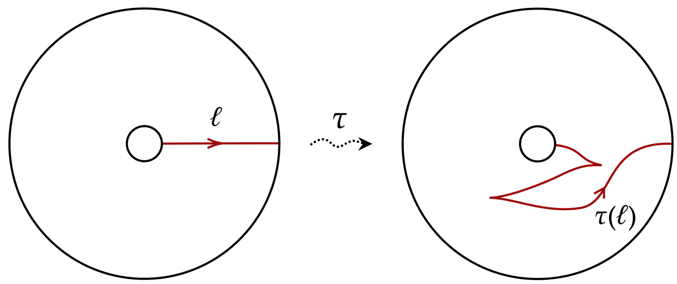

Figure 3.2: The image of l under τ.

It follows that τ is not (formally) contact isotopic to the identity as a contactomorphism of (Y_0, ξ) rel. ∂Y_0. This contactomorphism is contact isotopic to the contact Dehn twist as we have defined it in this section. In fact, any contactomorphism of (Y_0, ξ) can be described just in terms of its effect of l and, therefore, just by means of front projections of Legendrian arcs. The path-connected components of the space $\mathrm{Leg}(Y_0, \xi)$ of unknotted Legendrian embeddings of arcs that coincide with l at the end points can be easily understood:

Lemma 3.14. *The map* $\mathrm{rot} : \pi_0 \mathrm{Leg}(Y_0, \xi) \to \mathbb{Z}$, $L \mapsto \mathrm{rot}(L)$, *is an isomorphism.*

Proof. This is an application of the Theorem of Eliashberg and Fraser [21]. Indeed, given two Legendrian arcs L_1 and L_2 with the same rotation number we can always find another Legendrian arc L' in the ball $(\mathbb{B}^3, \xi)$ in such a way that the concatenations $L' \# L_1$ and $L' \# L_2$ are long Legendrian unknots in the ball. Observe that both have the same rotation number by hypothesis and therefore they differ by a finite number of double stabilizations (pairs of positive and negatives stabilizations). We conclude that L_2 is obtained from L_1, as Legendrian arcs in (Y_0, ξ), by a sequence of double stabilizations. As depicted in Figure 3.3 this shows that both Legendrians are isotopic in (Y_0, ξ). □

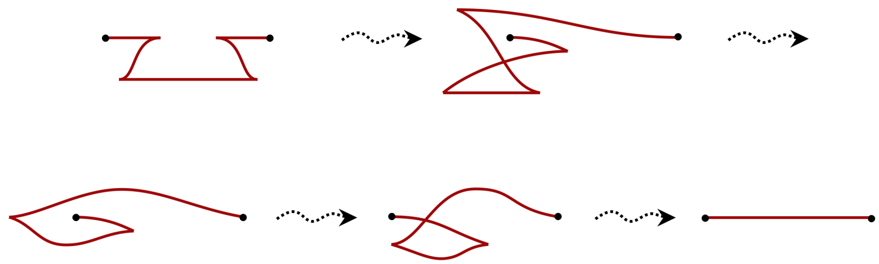

Figure 3.3: Legendrian isotopy from a double stabilization of l to l in (Y_0, ξ).

We conclude the following

Lemma 3.15. *The map* $\mathrm{Cont}(Y_0, \xi) \to \mathrm{Leg}(Y_0, \xi), f \mapsto f \circ l$ *is a homotopy equivalence. In particular,*

$$\pi_0 \mathrm{Cont}(Y_0, \xi) \to \mathbb{Z} \quad , \quad f \mapsto \mathrm{rot}(f \circ l)$$

is an isomorphism. Moreover, the contact Dehn twist is characterized, up to contact isotopy, by the relation

$$\mathrm{rot}(f(l)) = \mathrm{rot}(l) + 1.$$

Proof. This follows by the previous Lemma, the Eliashberg–Mishachev Theorem 1.9 and Hatcher's Theorem [41], since the fiber of $\mathrm{Cont}(Y_0, \xi) \to \mathrm{Leg}(Y_0, \xi)$ can be identified with the contactomorphism group of the complement of a neighbourhood of l, and the latter is a tight 3-ball. □

3.2 Proofs of main results, assuming Theorem 1.5

3.2.1 Diffeomorphisms of connected sums of two irreducible 3-manifolds

We include here a preliminary result that we shall use, Lemma 3.16.

Consider $Y_\# = Y_- \# Y_+$ with $Y_\pm$ *irreducible*. Recall that Hatcher [38] proved

$$\mathrm{Emb}(S^2, Y_\#)_{S_\#} \simeq \mathrm{SO}(3).$$

This has the following useful consequence:

Lemma 3.16. *Suppose that $Y_\pm$ are aspherical (i.e. irreducible and with infinite fundamental group). Then $\pi_1 \mathrm{Diff}(Y_\#) = 0$.*

Proof of Lemma 3.16. From the fibration (2.8) we have an exact sequence

$$\pi_1 \mathrm{Diff}(Y_-, B_-) \times \pi_1 \mathrm{Diff}(Y_+, B_+) \longrightarrow \pi_1 \mathrm{Diff}(Y_\#) \longrightarrow \mathbb{Z}_2 \longrightarrow \pi_0 \mathrm{Diff}(Y_-, B_-) \times \pi_0 \mathrm{Diff}(Y_+, B_+)$$

Under the connecting map, the non-trivial element in $\mathbb{Z}/2$ maps to $\tau_{\partial B_-}\tau_{\partial B_+} \in \pi_0 \mathrm{Diff}(Y_-, B_-) \times \pi_0 \mathrm{Diff}(Y_+, B_+)$. We saw in the proof of Corollary 3.7 that the Dehn twists $\tau_{\partial B_\pm} \in \pi_0 \mathrm{Diff}(Y_\pm, B_\pm)$ are non-trivial and $\pi_1 \mathrm{Diff}(Y_\pm, B_\pm) = 0$. From this and the exact sequence above it now follows that $\pi_1 \mathrm{Diff}(Y_\#) = 0$. □

3.2.2 Proof of Theorem 1.2, assuming Theorem 1.5

We consider the Wang long exact sequence of the fibration $ev : C(Y, \xi) \to S^2$, a portion of which is

$$H_2(C(Y, \xi); \mathbb{Q}) \xrightarrow{\deg} \mathbb{Q} \xrightarrow{\delta} H_1(C(Y, \xi, B); \mathbb{Q}).$$

Because $\mathbf{c}(\xi; \mathbb{Q}) \notin \mathrm{Im} U$, by Formula 1.1 (which follows from Theorem 1.5) we deduce that $\deg = 0$ and thus $\delta : \mathbb{Q} \hookrightarrow H_1(C(Y, \xi, B); \mathbb{Q})$. Recall that $0 \neq \delta(1)$ is the (homological) obstruction class O_ξ.

Since Y is irreducible and $\mathbf{c}(\xi; \mathbb{Q}) \notin \mathrm{Im} U$ then Y must be aspherical (as the irreducible 3-manifolds with finite fundamental group are precisely the quotients of S^3, and for these one has that the map $U : \widecheck{HM}_*(-Y, \mathbb{Q}) \to \widecheck{HM}_*(-Y, \mathbb{Q})$ is onto). It follows then from the fact that $\pi_1 \mathrm{Diff}_0(Y, B) = 0$ (see the proof of Corollary 3.7) and the long exact sequence for the fibration (2.3) that

$$H_1(C(Y, \xi, B); \mathbb{Z}) \cong \mathrm{Ab}\Big(\pi_0 \mathrm{Cont}_0(Y, \xi, B)\Big).$$

Since the obstruction class $0 \neq O_\xi \in H_1(C(Y,\xi,B);\mathbb{Q})$ corresponds to the class of $\tau^2_{\partial B}$ on the right-hand side, we have shown that $\tau_{\partial B}$ has infinite order in the abelianisation of

$$\pi_0\mathrm{Cont}_0(Y,\xi,B) = \mathrm{Ker}\Big(\pi_0\mathrm{Cont}(Y,\xi,B) \to \pi_0\mathrm{Diff}(Y,B)\Big).$$

Thus, Theorem 1.2(A) follows. Lemma 3.8 gives part (B).

3.2.3 Proof of Theorem 1.1, assuming Theorem 1.5

By Theorem 2.7 we have

$$C(Y_\#,\xi_\#,B_\#) \simeq C(Y_-,\xi_-,B_-) \times C(Y_+,\xi_+,B_+)$$

and then by Proposition 3.9 the obstruction class $O_{\xi_\#} \in \pi_1 C(Y_\#,\xi_\#,B_\#)$ to finding a homotopy section of $ev_\# : C(Y_\#,\xi_\#) \to S^2$ corresponds to

$$O_{\xi_\#} \cong (O_{\xi_-}, O_{\xi_+}) \in \pi_1 C(Y_-,\xi_-,B_-) \times \pi_1 C(Y_+,\xi_+,B_+).$$

A portion of the Wang long exact sequence for the fibration $ev_{B_\#}$ is

$$\mathbb{Q} \xrightarrow{\delta} H_1\big(C(Y_-,\xi_-,B_-),\mathbb{Q}\big) \oplus H_1\big(C(Y_+,\xi_+,B_+);\mathbb{Q}\big) \to H_1\big(C(Y_\#,\xi_\#);\mathbb{Q}\big) \to 0$$

where $\delta(1) = O_{\xi_\#} = (O_{\xi_-}, O_{\xi_+})$. Because $\mathbf{c}(\xi_\pm;\mathbb{Q}) \notin \mathrm{Im}U$ then as in the proof of Theorem 1.2 above we deduce that $O_{\xi_\pm}$ are non-trivial in $H_1\big(C(Y_\pm,\xi);\mathbb{Q}\big)$. It follows that the class $(O_{\xi_-},0)$ is not in the image of δ, thus the image of $(O_{\xi_-},0)$ in $H_1\big(C(Y_\#,\xi_\#)\big)$ is non-trivial.

Now, from Lemma 3.16 we have $\pi_1\mathrm{Diff}(Y_\#) = 0$ since $Y_\pm$ are aspherical. Then, by the long exact sequence in homotopy groups of (2.2) it follows that

$$H_1\big(C(Y_\#,\xi_\#);\mathbb{Z}\big) \cong \mathrm{Ab}\Big(\pi_0\mathrm{Cont}_0(Y_\#,\xi_\#)\Big).$$

Under this isomorphism, the non-trivial class $(O_{\xi_-}, 0)$ corresponds to the class of the squared Dehn twist $\tau^2_{S_\#}$ by Proposition 3.6. This completes the proof of Theorem 1.5(A).

We now establish Theorem 1.5(B). By Lemma 3.8 we have that the image of $\tau_{\partial B_\pm}$ in $\pi_0\mathrm{FCont}_0(Y_\pm, \xi_\pm, B_\pm)$ is trivial. Hence, so is the image of $\tau^2_{S_\#}$ in $\pi_0\mathrm{FCont}_0(Y_\#, \xi_\#)$. The proof of Theorem 1.1 is now complete. □

3.2.4 Proof of Theorem 1.1, assuming Theorem 1.5

We write $(Y, \xi) = (Y_0, \xi_0)\#\cdots\#(Y_n, \xi_n)\#(M, \xi_M)$ where (Y_j, ξ_j) are those prime summands such that $\mathbf{c}(\xi_j; \mathbb{Q}) \notin \mathrm{Im}U$ and the Euler class of ξ_j vanishes, and (M, ξ_M) is the sum of the remaining prime summands. We take the latter to be (S^3, ξ_{st}) if there are no prime summands remaining. We choose Darboux balls $B_{0-} \subset Y_0$, $B_{M+} \subset M$ and for $j = 1, \ldots, n$ we choose two Darboux balls $B_{j\pm} \subset Y_j$. We may take the connected sum (Y, ξ) to be built by gluing in the following order

$$\big(Y_0 \setminus B_{0-}\big) \bigcup_{\partial B_{0-} = -\partial B_{1+}} \big(Y_1 \setminus (B_{1+} \cup B_{1-})\big) \cdots \big(Y_k \setminus (B_{n+} \cup B_{n-})\big) \bigcup_{\partial B_{n-} = -\partial B_{M+}} \big(M \setminus B_{M+}\big)$$

with $n + 1$ separating spheres. Consider the evaluation map at the $n + 1$ south poles of the spheres, which provides a fibration

$$\mathcal{F} \to C(Y, \xi) \to (S^2)^{n+1}. \tag{3.2}$$

Theorem 2.7 identifies the fiber as

$$\mathcal{F} \simeq C(Y_0, B_{0-}) \times \Big(\prod_{j=1,\ldots,n+1} C(Y_j, B_{j+} \cup B_{j-})\Big) \times C(M, B_{M+}).$$

Observe that we have homotopy equivalences

$$C(Y_j, B_{j+} \cup B_{j-}) \simeq \Omega S^2 \times C(Y_j, B_{j-}).$$

Indeed, the evaluation map corresponding to the ball B' gives a fibration

$$C(Y,\xi,B\cup B')\to C(Y,\xi,B)\xrightarrow{ev_{B'}} S^2$$

but now the map $ev_{B'}$ is *null-homotopic*, as can be seen by dragging the evaluation point (the center of B') into the first ball B.

With this in place, we now consider the Serre spectral sequence of the fibration (3.2), from which we can assemble an exact sequence of the form

$$\mathbb{Q}^{n+1}\xrightarrow{\delta} H_1(\mathcal{F};\mathbb{Q})\to H_1\big(C(Y,\xi);\mathbb{Q}\big)\to 0.$$

We now give an explicit description of δ. Let 1 stand for the generator of $H_1(\Omega S^2;\mathbb{Q})=\mathbb{Q}$. By a slight variation of Proposition 3.9 we have

$$\begin{aligned}\delta(a_1,\dots,a_{n+1}) &= \big(\,a_1\cdot O_{\xi_0}\,,\,a_1\cdot 1\,,\,a_2\cdot O_{\xi_1}\,,\,a_2\cdot 1\,,\,\dots\,,a_n\cdot 1\,,\,a_{n+1}\cdot O_{\xi_n}\,,\,a_{n+1}\cdot O_{\xi_M}\,\big)\\ &\in H_1\big(\mathcal{F};\mathbb{Q}\big)\end{aligned}$$

By the condition $\mathbf{c}(\xi_j;\mathbb{Q})\notin \mathrm{Im}U$ we deduce as in the proof of Theorem 1.2 above that the classes O_{ξ_j} $(j=0,\dots,n)$ are homologically non-trivial. Hence the n-dimensional subspace of $H_1\big(\mathcal{F};\mathbb{Q}\big)$ given by the elements

$$\big(\,b_1\cdot O_{\xi_0}\,,\,0\,,\,b_2\cdot O_{\xi_1}\,,\,0\,,\,\dots\,,0\,,\,b_n\cdot O_{\xi_{n-1}}\,,\,0\,,\,0\,,\,0\,\big)\quad,\quad (b_j)\in\mathbb{Q}^n$$

injects as a subspace of $H_1\big(C(Y,\xi);\mathbb{Q}\big)$. The proof of the formal triviality assertion is similar to the one given for Theorem 1.1. The proof of Theorem 1.3 is now complete. □

Remark 3.5. When Y is the sum of two aspherical 3-manifolds we have $\pi_1\mathrm{Diff}(Y)=0$ (see Lemma 3.16). In the proof of Theorem 1.1 this allowed us to pass from a non-trival element in $\pi_1 C(Y,\xi)$ to a non-trivial element in $\pi_0\mathrm{Cont}_0(Y,\xi)$ via the fibration (2.2). This is a special situation. For instance, if Y is instead the sum of *at least three* aspherical 3-manifolds then it is known that $\pi_1\mathrm{Diff}(Y)$ is not finitely generated [56]. A better control on $\pi_1\mathrm{Diff}(Y)$ for general Y would allow us to understand whether the exotic loops of contact structures that we find in Theorem 1.3 yield non-trivial contactomorphisms (i.e. the Dehn twists on the corresponding separating spheres).

3.3 Exotic phenomena in overtwisted contact 3-manifolds

In this final section we exhibit examples of 1-parametric exotic phenomena in *overtwisted* contact 3-manifolds.

On a heuristic level, Eliashberg's overtwisted h-principle [19] is based on applying Gromov's h-principle for open manifolds to the complement of a 3-ball and using the overtwisted disk to fill in the ball. In the same spirit of this idea is what we call the "overtwisted escape principle", explained to us by F. Presas, which is a general strategy for proving an h-principle for a family of objects in a contact manifold (Y, ξ). First, perform the connected sum with an overtwisted manifold (M, ξ_{ot}), in order to apply the overtwisted h-principle [19, 5] in the contact 3-manifold $(Y, \xi)\#(M, \xi_{\mathrm{ot}})$. This could be thought of as analogous to opening up the 3-manifold in the previous situation. Secondly, try to isotope the objects for which you want an h-principle so that they avoid ("escape") the overtwisted region $(M, \xi_{\mathrm{ot}}) \setminus B$, where B is a Darboux ball. However, there could be obstructions to carrying out this second step. There are two scenarios: if these obstructions can be sorted out then our initial problem satisfies an h-principle; if not these obstructions should give rise to an exotic phenomenon in the overtwisted contact manifold $(Y, \xi)\#(M, \xi_{\mathrm{ot}})$. In [8] the authors succesfully carry out this procedure to prove an existence h-principle for codimension 2 isocontact embeddings. Next, we will instead start out of a problem in (Y, ξ) which we know is geometrically obstructed a priori, and from this deduce an exotic overtwisted phenomenon.

Let $e : S^2 \to (Y, \xi)$ be a standard embedding into a contact manifold (Y, ξ). A *formal standard embedding* of a sphere into (Y, ξ) is a pair (f, F^s), $s \in [0, 1]$, such that $f \in \mathrm{Emb}(S^2, Y)$ is a smooth embedding and $F^s : TS^2 \to f^*TY$ is a homotopy of vector bundle injections with $F^0 = df$ and $(F^1)^*\xi = e^*\xi \subset TS^2$. We will denote by $\mathrm{FCEmb}(S^2, (Y, \xi))$ the space of formal standard embeddings and by $\mathrm{FCEmb}(S^2,(Y,\xi),s)$ the subspace of formal standard embedding that coincide with e over an open neighbourhood U of the south pole $s\in S^2$.

Let (M,ξ_{ot}) be an overtwisted contact 3-manifold. Consider the overtwisted contact 3-manifold $(Y_\#,\xi_\#)=(Y,\xi)\#(M,\xi_{\mathrm{ot}})$. We will consider the spaces $\mathrm{CEmb}(S^2,(Y_\#,\xi_\#),s)$ and $\mathrm{FCEmb}(S^2,(Y_\#,\xi_\#),s)$ as pointed spaces with base point given by the separating sphere $e:S^2\hookrightarrow(Y_\#,\xi_\#)$. We have a natural inclusion $\mathrm{CEmb}(S^2,(Y_\#,\xi_\#),s)\hookrightarrow\mathrm{FCEmb}(S^2,(Y_\#,\xi_\#),s)$. From our previous discussion and the theory developed in this article we deduce the following

Corollary 3.17. *Assume that (Y,ξ) is irreducible, ξ has vanishing Euler class and $\mathbf{c}(\xi)\notin\mathrm{Im}U$. Then, there exists an element with infinite order in*

$$\mathrm{Ker}\Big(\pi_1\mathrm{CEmb}(S^2,(Y_\#,\xi_\#),s)\to\pi_1\mathrm{FCEmb}(S^2,(Y_\#,\xi_\#),s)\Big).$$

Remark 3.6.

- This should be compared with Theorem 2.8, which in particular asserts that this type of phenomenon does not happen when the underlying contact manifold is tight.
- Under the same assumptions, our proof also yields an element with infinite order in

$$\mathrm{Ker}\Big(\pi_1\mathrm{CEmb}(S^2,(Y_\#,\xi_\#))\to\pi_1\mathrm{FCEmb}(S^2,(Y_\#,\xi_\#))\Big).$$

Proof. Denote by $S_\#=e(S^2)$ the standard separating sphere. Consider the squared Dehntwist $\tau^2_{S^+_\#}$ along a parallel copy $S^+_\#$ of $S_\#$, where we assume that $S^+_\#$ is contained in $(Y,\xi)\backslash B$, where B is the Darboux ball used to perform the connected sum. By the vanishing of the Euler class of ξ there exists a homotopy through formal contactomorphisms joining the identity with $\tau^2_{S^+_\#}$ (Lemma 3.8). It follows from Eliashberg's Theorem 3.12 combined with Lemma 2.5 that we can deform this homotopy (through formal contactomorphisms) to a homotopy φ_t through contactomorphisms with $\varphi_0=\mathrm{id}$ and $\varphi_1=\tau^2_{S^+_\#}$. This process can be done relative to an open neighbourhood of the south pole $e(s)\in(Y\#M,\xi\#\xi_{ot})$, see Remark 3.4. The loop of standard spheres $\varphi_t\circ e$ is formally trivial by construction but geometrically non-trivial. Indeed, the triviality of this loop would imply that $\tau^2_{S^+_\#}$, regarded as a contactomorphism of (Y,ξ), is contact isotopic to the identity rel. B, which is in contradiction with Theorem 1.2. □

Given a contact 3-manifold (Y,ξ) and a transverse knot $K \subset (Y,\xi)$ one can replace a small tubular neighbourhood of K by a *Lutz Twist* $(LT = \mathbb{D}^2 \times S^1, \xi_{\mathrm{ot}})$ to obtain an *overtwisted* contact manifold (Y,ξ_K). Intuitively, the Lutz Twist (LT,ξ_{ot}) is an *embedded* S^1-family of overtwisted disks, see [25] for the precise definitions. We will denote by $\mathrm{LT}(Y,\xi_K)$ the space of contact embeddings $e : (LT,\xi_{\mathrm{ot}}) \hookrightarrow (Y,\xi_K)$, regarded as a based space with basepoint the standard one, and by $\mathrm{FLT}(Y,\xi_K)$ the corresponding space of formal contact embeddings. As before, there is an inclusion map $\mathrm{LT}(Y,\xi_K) \to \mathrm{FLT}(Y,\xi_K)$. The following can be deduced following using the same strategy as above:

Corollary 3.18. *Let (Y,ξ) be a irreducible contact 3-manifold with vanishing Euler class and such that $\mathbf{c}(\xi) \notin \mathrm{Im} U$. Consider a Darboux ball $B \subset (Y,\xi)$ and a transverse knot $K \subset B$. Then, there exists an element with infinite order in*

$$\mathrm{Ker}\Big(\pi_1 \mathrm{LT}(Y,\xi_K) \to \pi_1 \mathrm{FLT}(Y,\xi_K)\Big).$$

Chapter 4: A monopole invariant for families of contact structures

4.1 Families of spin-c structures and irreducible configurations

In this section we discuss preliminary material regarding spin-c structures as they vary in families.

4.1.1 Basic notions about spin-c structures

We let M be an oriented manifold of dimension $n = 2m$ or $n = 2m + 1$. Our case of interest is $n = 4$ or $n = 3$.

Definition 4.1. A *spin-c structure* on M is a triple $\mathfrak{s} = (g, S, \rho)$ consisting of the following data:

(a) a Riemannian metric g on M

(b) a unitary vector bundle $S \to M$ of rank 2^m

(c) a vector bundle map $\rho : T^*M \to \mathrm{Hom}(S, S)$ which is skew-adjoint $\rho(v)^* = -\rho(v)$ and satisfies the *Clifford identity* $\rho(v)^2 = -|v|_g^2 \cdot \mathrm{id}_S$, for all $v \in T^*M$.

The bundle S is referred to as the *spinor bundle* of $\mathfrak{s}$ and its sections are *spinors*; the map ρ is the *Clifford multiplication* of $\mathfrak{s}$.

The Clifford multiplication ρ naturally extends to a vector bundle map from the complexified exterior algebra $\rho : \Lambda^\bullet T^*M \otimes \mathbb{C} \to \mathrm{Hom}(S, S)$ by linearity and the following rule: for a one-form α and a form β

$$\rho(\alpha \wedge \beta) = \frac{1}{2}\big(\rho(\alpha)\rho(\beta) + (-1)^{\deg \beta}\rho(\beta)\rho(\alpha)\big).$$

From the canonical volume element ω determined from the metric g we form the complex volume element $\omega_{\mathbb{C}} = i^{\lfloor \frac{n+1}{2} \rfloor}\omega \in \Gamma(M, \Lambda^n T^*M \otimes \mathbb{C})$. One sees that $\rho(\omega_{\mathbb{C}})^2 = 1$. In the case $n = 2m$ the bundle S decomposes $S = S^+ \oplus S^-$ as the sum of the ± 1-eigensubbundles of $\rho(\omega_{\mathbb{C}})$. Each $S^\pm$ has rank 2^{m-1} and these are referred to as *positive* or *negative* spinor bundles. In the case $n = 2m + 1$ we *require* in the definition of a spin-c structure that $\rho(\omega_{\mathbb{C}})$ acts on S by -1.

If X is an oriented manifold of dimension $2m$ with $\partial X = Y$ and we are given a spin-c structure $\mathfrak{s}_X = (g_X, S_X, \rho_X)$ on X, we can *restrict* it to Y and obtain a spin-c structure $\mathfrak{s}_X|_Y = (g_X|_Y, S_X^+|_Y, \rho_Y)$. Here ρ_Y is defined by $\rho_Y(v) = \rho_X(n)^{-1}\rho_X(v)$, where n stands for the unit outward normal to Y.

We now describe some further differential geometric notions associated with a spin-c structure:

Definition 4.2. A unitary connection A on the unitary bundle $S \to M$ is a *spin-c connection* if the Clifford action $\rho : T^*M \to \mathrm{Hom}(S, S)$ is parallel with respect to the connection on $TM \otimes \mathrm{Hom}(S, S)$ induced by A and the Levi-Civita connection of g.

There is a one-to-one correspondence between spin-c connections on S and unitary connections on the associated line bundle $L = \det S^+$ if $n = 2m$ (and $L = \det S$ if $n = 2m+1$). The connection on L induced by A is denoted by $\hat{A}$, and the correspondence is just $A \mapsto \hat{A}$. Thus, the space of spin-c connections is an affine space over $\Omega^1(M; i\mathbb{R})$.

Definition 4.3. The *Dirac operator* coupled with a spin-c connection A is the differential operator

$$D_A : \Gamma(X, S) \to (X, S)$$

defined by $D_A\Phi = \rho(\nabla_A \Phi)$, where the latter expression denotes the contraction of the T^*X and the S component of ∇_A using the Clifford action ρ.

The differential operator D_A is elliptic and self-adjoint. In the case $\dim M = n = 2m$, the Dirac operator decomposes $D_A = D_A^+ \oplus D_A^-$ as a sum of two elliptic differential operators $D_A^\pm : \Gamma(X, S^\pm) \to \Gamma(X, S^\mp)$.

4.1.2 Changing the metric of a spin-c structure

Given a spin-c structure $\mathfrak{s}_0 = (g_0, S_0, \rho_0)$ and a different Riemannian metric g_1 on M, there is a natural device for producing a new spin-c structure (g_1, S_1, ρ_1). We describe this now following

[7].

Consider first a real finite-dimensional vector space V equipped with two inner products g_0, g_1. Then there is a *canonical linear isometry* $b_{g_1,g_0} : (V, g_0) \xrightarrow{\cong} (V, g_1)$, characterised by the property that it is positive and symmetric with respect to g_0. It is constructed as follows. Write $g_1 = g_0(H\cdot, \cdot)$ for a (unique) symmetric positive endomorphism H of (V, g_0). Then $b_{g_1,g_0} = H^{-1/2}$ is the required isometry. Finally, given two Riemannian metrics g_0, g_1 on a manifold M, the previous construction applies fibrewise to produce an isometry $b_{g_1,g_0} : (TM, g_0) \xrightarrow{\cong} (TM, g_1)$.

Remark 4.1. The canonical isometry satisfies $b^{-1}_{g_1,g_0} = b_{g_0,g_1}$. Unfortunately, in general it is *not* functorial: $b_{g_2,g_1} \circ b_{g_1,g_0} \neq b_{g_2,g_0}$ (see [7]).

This construction allows us to change the metric in a spin-c structure $\mathfrak{s}_0 = (g_0, S_0, \rho_0)$. Given another Riemannian metric g_1, we define $S_1 = S_0$ and $\rho_1 : T^*X \to \mathrm{Hom}(S_0, S_0)$ as $\rho_1(v) = \rho_0(b^*_{g_1,g_0} v)$. This yields a new spin-c structure $\mathfrak{s}_1 = (g_1, S_1, \rho_1)$.

Definition 4.4. Given two spin-c structures $\mathfrak{s}_i = (g_i, S_i, \rho_i)$ $(i = 0, 1)$ on M, an *isomorphism* between them consists of an isomorphism of unitary vector bundles $h : S_0 \xrightarrow{\cong} S$ such that $\rho_1(v) = h \circ \rho_0(b^*_{g_1,g_0} v) \circ h^{-1}$ for all $v \in T^*X$.

It can be shown using Schur's Lemma that set of isomorphism classes[1] of spin-c structures on M is a *torsor* over the cohomology group $H^2(M; \mathbb{Z})$. Given a unitary line bundle Q over M, the action of $c_1(Q) \in H^2(M; \mathbb{Z})$ on the isomorphism class of the spin-c structure $\mathfrak{s} = (g, S, \rho)$ is defined by

$$c_1(Q) \cdot [\mathfrak{s}] = [(g, S \otimes Q, \rho \otimes \mathrm{id}_Q)].$$

4.1.3 Irreducible configurations

The space of configurations (A, Φ), where A is a spin-c connection on S, and $\Phi \in \Gamma(M, S^+)$ (resp. $\Phi \in \Gamma(M, S)$) when $n = 2m$ (resp. $2m + 1$) is denoted by $\mathcal{C}(M, \mathfrak{s})$. We equip $\mathcal{C}(M, \mathfrak{s})$ with

[1]It is clear that "isomorphism" gives a reflexive and symmetric relation on the set of spin-c structures (g, S, ρ); it can be shown using Schur's Lemma that it is also *transitive*, even if $b_{g_2,g_1} \circ b_{g_1,g_0} = b_{g_2,g_1}$ does not hold in general.

the C^∞ topology. We denote by $\mathcal{C}^*(M, \mathfrak{s}) \subset \mathcal{C}(M, \mathfrak{s})$ the open subset of *irreducible configurations*, namely those such that Φ is not identically vanishing on M. Configurations $(A, 0)$ are called *reducible*.

The automorphism group $\mathcal{G}$ of a spin-c structure $\mathfrak{s} = (g, S, \rho)$ is referred to as the *gauge group*. It can be shown using Schur's Lemma that $\mathcal{G}$ agrees with space of smooth mappings $\mathcal{G} = \mathrm{Map}(M, \mathrm{U}(1))$. We make $\mathcal{G}$ into a topological group by equipping it with the C^∞ topology. There is a continuous $\mathcal{G}$-action on $\mathcal{C}(M, \mathfrak{s})$: given $v \in \mathcal{G}$ and configuration (A, Φ) we set

$$v \cdot (A, \Phi) = (A - v^{-1}dv, v\Phi).$$

The $\mathcal{G}$-action is free on $\mathcal{C}^*(X, \mathfrak{s})$, whereas it has stabiliser$\cong \mathrm{U}(1)$ at the reducible configurations.

Definition 4.5. The *configuration space* modulo gauge is the quotient space $\mathcal{B}(X, \mathfrak{s}) = \mathcal{C}(X, \mathfrak{s})/\mathcal{G}$. The subspace $\mathcal{C}^*(X, \mathfrak{s})/\mathcal{G} \subset \mathcal{B}(X, \mathfrak{s})$ is denoted $\mathcal{B}^*(X, \mathfrak{s})$.

The space $\mathcal{B}(X, \mathfrak{s})$ is Hausdorff. If an isomorphism $h : \mathfrak{s}_0 \xrightarrow{\cong} \mathfrak{s}_1$ of two spin-c structures on X is given, there is an induced homeomorphism $\mathcal{B}(X, \mathfrak{s}_0) \xrightarrow{\cong} \mathcal{B}(X, \mathfrak{s}_1)$ given by $(A, \Phi) \mapsto (A_h, h(\Phi))$ where A_h is the unique spin-c connection (for $\mathfrak{s}_1$) such that $(h(A))^t = (A_h)^t$.

4.1.4 Families of spin-c structures and irreducible configurations

We now consider continuously-varying families of spin-c structures $\mathfrak{s}_t = (g_t, S_t, \rho_t)$ on a *fixed* oriented smooth manifold M parametrised by a "nice" connected topological space T. Note that the isomorphism class of the spin-c structures on M given by $[\mathfrak{s}_t]$ is independent of $t \in T$. We denote such a T-family by $\underline{\mathfrak{s}} = (\mathfrak{s}_t)_{t \in T}$. By a T-family of irreducible configurations on M we mean a T-family of spin-c structures $\underline{\mathfrak{s}}$ together with a continuosly varying family of (smooth) irreducible configurations $(A_t, \Phi_t) \in \mathcal{C}^*(M, \mathfrak{s}_t)$. Similarly, we adopt the notation $(\underline{A}, \underline{\Phi})$ for such a family.

Remark 4.2. We will need to work with smoothly-varying families later on (with T a smooth manifold); however, the discussion that follows applies equally well with only minor modifications.

There is an obvious notion of "isomorphism" for two families of spin-c structures (resp. irreducible configurations) parametrised by the same space T: a continuosly varying T-family of isomorphisms of spin-c structures (resp. carrying the irreducible configurations onto one another). Much as before, the set of isomorphism classes of T-families of spin-c structures on M is a torsor over the cohomology group $H^2(M \times T; \mathbb{Z})$. When it comes to families of irreducible configurations, the relevant "moduli functor" is represented by the irreducible configuration space:

Lemma 4.1. *There is a one-to-one correspondence between*

(i) the set of isomorphism classes of T-families of irreducible configurations $(\underline{A}, \underline{\Phi})$ on M with underlying isomorphism class of spin-c structure on M represented by $\mathfrak{s}_M$, and

(ii) the set of continuous maps $\mathrm{Map}\big(T, \mathcal{B}^*(M, \mathfrak{s}_M)\big)$.

The main point is that $\mathcal{B}^*(M, \mathfrak{s}_M)$ parametrises a *universal* family $(\underline{A_\infty}, \underline{\Phi_\infty})$ of irreducible configurations on M. This is constructed as follows. Say $\mathfrak{s}_M = (g, S, \rho)$. The pullback of S over the product $M \times C^*(M, \mathfrak{s}_M)$ is a $\mathcal{G}$-equivariant unitary vector bundle: the action of $v \in \mathcal{G}$ on the fibres of S over $\{m\} \times C^*(M, \mathfrak{s}_M)$ is given by multiplication by $v(m) \in \mathrm{U}(1)$; and the action on the base is the natural action on the second factor. The $\mathcal{G}$-action on the base is free, and passing to the quotient we obtain a unitary vector bundle $\underline{S_\infty}$ over $M \times \mathcal{B}^*(M, \mathfrak{s}_M)$ with a $\mathcal{B}^*(M, \mathfrak{s}_M)$-family of Clifford multiplications. This yields a family of spin-c structures on M parametrised by $\mathcal{B}^*(M, \mathfrak{s}_M)$. Furthermore the tautological family of irreducible configurations on M parametrised by $C^*(M, \mathfrak{s}_M)$ descends to a corresponding family of irreducible configurations $(\underline{A_\infty}, \underline{\Phi_\infty})$ parametrised by $\mathcal{B}^*(M, \mathfrak{s}_M)$.

Conversely, given a family of irreducible configurations $(\underline{A}, \underline{\Phi})$ we construct an associated *classifying map*

$$f_{\underline{A}, \underline{\Phi}} : T \to \mathcal{B}^*(M, \mathfrak{s}_M) \tag{4.1}$$

as follows. For a given $t \in T$ we choose an isomorphism $\mathfrak{s}_t \xrightarrow{\cong} \mathfrak{s}_M$. Using this, we carry the irreducible configuration $(A_t, \Phi_t) \in C^*(M, \mathfrak{s}_t)$ to an irreducible configuration $(A_t', \Phi_t') \in C^*(M, \mathfrak{s}_M)$.

This is done as follows. If $h : S_t \cong S$ is the underlying unitary isomorphism of spinor bundles, then we define $\Phi'_t := h(\Phi_t)$; then A'_t is defined as the unique spin-c connection whose induced connection $\hat{A}'_t$ on $\Lambda^2 S^+$ agrees with the connection $\hat{A}_t$ on $\Lambda^2 S^+_t$ under the isomorphism $\Lambda^2 S^+_t \cong \Lambda^2 S^+$ induced by h. Note that choosing a different isomorphism $\mathfrak{s}_t \cong \mathfrak{s}_M$ only results in a gauge-equivalent irreducible configuration in $C^*(M, \mathfrak{s}_M)$. We then set $f_{\underline{A},\underline{\Phi}}(t) = [(A'_t, \Phi'_t)]$, which is easily verified to give a continuous map as we vary $t \in T$.

Proof of Lemma 4.1. To go from (i) to (ii) we send a T-family of irreducible configurations $(\underline{A}, \underline{\Phi})$ to its classifying map $f_{(\underline{A},\underline{\Phi})}$. For the other direction, if $f : T \to \mathcal{B}^*(M, \mathfrak{s}_M)$ is a continuous map then the universal family of irreducible configurations $(\underline{A_\infty}, \underline{\Phi_\infty})$ parametrised by $\mathcal{B}^*(M, \mathfrak{s}_M)$ can be pulled back along f to produce a T-family of irreducible configurations on M.

The two assignments described above are inverse to each other. Indeed, given a family of irreducible configurations $(\underline{A}, \underline{\Phi})$ there is a *unique* isomorphism of $(\underline{A}, \underline{\Phi})$ with the pullback of the unversal family $(\underline{A_\infty}, \underline{\Phi}_\infty)$ by the map $f_{\underline{A},\underline{\Phi}}$. The uniqueness follows again from the fact that $\mathcal{G}$ acts freely on irreducible configurations. $\square$

The elementary correspondence from Lemma 4.1 implies the following "slogan" which plays a role in the upcoming construction of the families contact invariant:

Slogan 4.1. *One can trade a (possibly non-trivial) T-family of spin-c structures on M carrying a T-family of irreducible configurations for a* ***constant*** *family of spin-c structures on M together with a T-family of irreducible configurations which are only* ***well-defined up to $\mathcal{G}$-action****.*

In concrete terms, what this means is the following. Fix an open cover $T = \bigcup_{i \in I} U_i$ by contractible open sets. Then there is a correspondence between:

(i) the set of isomorphism classes of T-families of irreducible configurations $(\underline{A}, \underline{\Phi})$ on M with underlying isomorphism class of spin-c structure on M represented by $\mathfrak{s}_M$, and

(ii) isomorphism classes of I-tuples of continuous maps $\big((A_i, \Phi_i) : U_i \to C^*(M, \mathfrak{s}_M)\big)_{i \in I}$ such that for each overlap $U_i \cap U_j$ there exists a (unique) continuous map $v_{ji} : U_i \cap U_j \to \mathcal{G}$ such that $v_{ji}(t) \cdot (A_i(t), \Phi_i(t)) = (A_j(t), \Phi_j(t))$.

Let us mention at this point that the role played by certain families of irreducible configurations (coming from families of contact structures) is going to be to provide natural "boundary conditions" for the Seiberg–Witten equations over an end of a non-compact 4-manifold. It will be necessary later to "trivialise" the family of spin-c structures, and the $\mathcal{G}$-ambiguity of the resulting family of irreducible configurations will pose no issue due to the $\mathcal{G}$-invariance of the Seiberg–Witten equation.

4.1.5 Families of irreducible configurations from symplectic and contact structures

4.1.5.1 Symplectic 4*-manifolds*

Let (X, ω) be a symplectic 4-manifold, oriented by the volume form ω^2. We make the auxiliary choice of an ω-compatible almost-complex structure J. This means that the tensor $g = \omega(\cdot, J\cdot)$ defines a Riemannian metric. We refer to such a triple (ω, J, g) as an *almost-Kähler* structure on X.

Definition 4.6. The *canonical spin-c structure* $\mathfrak{s}_{\omega,J,g} = (g, S_{\omega,J,g}, \rho_{\omega,J,g})$ determined from the almost-Kähler structure (ω, J, g) is given by the following data:

- $S^+_{\omega,J,g} = \mathbb{C} \oplus \Lambda^{0,2}_J T^*X$ and $S^-_{\omega,J,g} = \Lambda^{0,1}_J T^*X$, equipped with the hermitian metrics naturally induced from g.
- the Clifford multiplication by $\eta \in T^*X$ has the component $\rho^+_{\omega,J,g}(\eta) : S^+_{\omega,J,g} \to S^-_{\omega,J,g}$ defined by

$$\rho^+_{\omega,J,g}(\eta)(\alpha, \beta) = \sqrt{2}(\eta^{0,1} \wedge \alpha - \iota_{\eta_{0,1}}\beta).$$

Above, ι_X stands for contraction by X on the first component, and $\eta_{0,1}$ is the $(0,1)$-part of the metric dual tangent vector of η. The remaining component of the Clifford action, $\rho^-_{\omega,J,g} : S^-_{\omega,J,g} \to S^+_{\omega,J,g}$ can be recovered from the above, using the fact that $\rho_{\omega,J,g}$ should be skew-adjoint.

A computation shows that the Clifford action of the symplectic form $\rho_{\omega,J,g}(\omega) : S^+_{\omega,J,g} \to S^+_{\omega,J,g}$ is given by $-2i$ on $\mathbb{C}$ and $+2i$ on $\Lambda^{0,2}_J$. Observe that there is a canonical section $\Phi_{\omega,J,g}$ of $S^+_{\omega,J,g}$ given by constant 1 on the $\mathbb{C}$ component.

Lemma 4.2. *[78] There exists a unique spin-c connection $A_{\omega,J,g}$ on $S_{\omega,J,g}$ such that*

$$D^+_{A_{\omega,J,g}} \Phi_{\omega,J,g} = 0.$$

Remark 4.3. Alternatively, $A_{\omega,J,g}$ is uniquely characterised by the property that the covariant derivative $\nabla_{A_{\omega,J,g}} \Phi_{\omega,J,g}$ is a 1-form with values in the subbundle $\Lambda^{0,2}_J T^*X$.

Definition 4.7. The *canonical configuration* associated to (ω, J, g) is the pair $(A_{\omega,J,g}, \Phi_{\omega,J,g}) \in C^*(X, \mathfrak{s}_{\omega,J,g})$.

Thus, the space of almost-Kähler structures on X parametrises a family of irreducible configurations on X.

Remark 4.4. It is a fundamental Theorem of Taubes [78] that the Seiberg–Witten invariant $\mathrm{SW}(\mathfrak{s}_{\omega,J,g}) \in \mathbb{Z}$ of the canonical spin-c structure of a closed symplectic 4-manifold with $b^+(X) > 1$ is non-vanishing. Taubes' proof shows that, for a suitable large perturbation of the Seiberg–Witten equations, the canonical configuration becomes the only solution to the equations, modulo $\mathcal{G}$-action.

4.1.5.2 Contact 3-manifolds

Let (Y, ξ) be a contact 3-manifold. We now choose the auxiliary data of a complex structure j on the contact distribution (inducing the positive orientation) and a (positive) contact form α. We will refer to such a triple as *contact metric structure*. Indeed, given (ξ, α, j) there exists a *unique* Riemannian metric $g_{\xi,\alpha,j}$ on Y characterised by

- $|\alpha|_{g_{\xi,\alpha,j}} = 1$
- $d\alpha = 2 * \alpha$ where $*$ is the Hodge star operator of $g_{\xi,\alpha,j}$

- j is an isometry of $(\xi, g_{\xi,\alpha,j})$.

Observe that the *Reeb vector field* R (determined uniquely by the requirement that $\alpha(R) = 1$ and $d\alpha(R, \cdot) = 0$) is $g_{\xi,\alpha,j}$-orthogonal to the contact plane ξ. It is convenient to regard j as an endomorphism of TY by setting $j(R) = 0$. Then, we can write down explicitly the Riemannian metric $g_{\alpha,j}$ as

$$g_{\xi,\alpha,j} = \alpha \otimes \alpha + \frac{1}{2} d\alpha(\cdot, j\cdot). \tag{4.2}$$

Definition 4.8. The *canonical spin-c structure* $\mathfrak{s}_{\xi,\alpha,j} = (g_{\xi,\alpha,j}, S_{\xi,\alpha,j}, \rho_{\xi,\alpha,j})$ determined from the contact structure ξ and the auxiliary data α, j is given by the following data

- $S_{\xi,\alpha,j} = \mathbb{C} \oplus \langle \alpha \rangle^{\perp}$ where the second factor is the $g_{\xi,\alpha,j}$-orthogonal complement of α inside of T^*Y
- $\rho_{\xi,\alpha,j}(\eta)(x, y) = (i\eta(R)x, -i\eta(R)y) - \sqrt{2}(\eta^{0,1}x - \iota_{\eta_{0,1}} y)$ for $\eta \in T^*Y$.

For the above, note that we can decompose $\eta \in T^*Y$ as $\eta = \eta(R)\alpha + \eta^{1,0} + \eta^{0,1}$ where $\eta^{p,q} \in \langle \alpha \rangle^{\perp} \otimes_{\mathbb{R}} \mathbb{C}$ stands for the (p, q) component of the projection to $\langle \alpha \ \rangle^{\perp}$ of η, using the complex structure j on $\langle \alpha \ \rangle^{\perp}$.

The 3-dimensional contact analogue of Taubes' Theorem about closed symplectic 4-manifolds now states that for a contact structure ξ which admits a strong symplectic filling, the contact invariant $\mathbf{c}(\xi) \in \widetilde{HM}^*(-Y, -\mathfrak{s}_{\xi,\alpha,j})$, is non-vanishing. A monopole Floer proof of this result has recently been given by Echeverria [17].

Given a contact form α for (Y, ξ), by its *symplectization* we will mean the symplectic manifold (with concave boundary) (K, ω) where $K = [1, +\infty) \times Y$ and

$$\omega = d(\frac{s^2}{2}\alpha) = s ds \wedge \alpha + \frac{s^2}{2} d\alpha. \tag{4.3}$$

If we start with a triple (ξ, α, j) on Y, we obtain an almost-Kähler structure (ω, J, g) on K by having J agree with j on $\xi = \ker\alpha$ and setting

$$J(\partial/\partial s) = \frac{1}{s}R \tag{4.4}$$

where R is the Reeb vector field of α. It follows that the Riemannian metric $g = \omega(\cdot, J\cdot)$ over $K = [1, +\infty) \times Y$ is the *cone metric over* $(Y, g_{\xi,\alpha,j})$, namely

$$g = ds^2 + s^2 g_{\xi,\alpha,j}.$$

Lemma 4.3 ([16], Lemma 35)**.** *There is a canonical identification of spin-c structures on* $-Y = \partial K$

$$(\mathfrak{s}_{\omega,J,g})|_{-Y} \cong -\mathfrak{s}_{\xi,\alpha,j}.$$

Above we denote by $-\mathfrak{s}_{\xi,\alpha,j}$ the induced spin-c structure on $-Y$ (obtained by adding a negative sign to the Clifford multiplication).

Definition 4.9. The *canonical configuration* associated to (ξ, α, j) is the pair $(A_{\xi,\alpha,j}, \Phi_{\xi,\alpha,j}) \in C^*(Y, \mathfrak{s}_{\xi,\alpha,j})$ obtained by restriction onto Y of the canonical configurations $(A_{\omega,J,g}, \Phi_{\omega,J,g}) \in C^*(K, \mathfrak{s}_{\omega,J,g})$ associated to the almost-Kähler structure (ω, J, g) on $K = [1, +\infty) \times Y$.

Thus, the space of contact metric structures parametrises a family of irreducible configurations on Y and $K = [1, +\infty) \times Y$.

4.2 Construction of the families contact invariant

In this section we construct the families contact invariant (1.4). There is a Poincaré duality for the Floer groups [[49], §3], $\widetilde{HM}_*(-Y, \mathfrak{s}_{\xi_0}) \cong \widehat{HM}^*(Y, \mathfrak{s}_{\xi_0})$, and the map (1.4) most naturally arises as a map into the latter group: the *from* version of the monopole Floer cohomology groups. We give a rough outline of this construction before going into the details.

First, we equip contact structures with auxiliary structures. Let $C\mathcal{M}(Y, \xi_0)$ be the space of *contact metric structures* (see §4.1.5.2) (ξ, α, j) on Y such that the contact structures ξ and ξ_0 are isotopic. The forgetful projection induces a weak homotopy equivalence $\mathcal{CM}(Y, \xi_0) \simeq \mathcal{C}(Y, \xi_0)$. We will also find it convenient to work within the realm of Banach spaces. A way to do this is by considering triples $(\xi,\ \alpha, j)$ where α (and hence ξ) is only assumed to be of class C^l, and the complex structure j is of class C^{l-1}, for a suitable positive integer l. The metric $g_{\xi,\alpha,j}$ (see (4.2)) determined from the triple (ξ, α, j) is therefore of class C^{l-1}. The space of such triples is a Banach manifold homotopy equivalent to the space of C^∞ triples. From now on, we will reserve the notation $\mathcal{CM}(Y, \xi_0)$ for this more convenient Banach manifold version only.

Associated to each triple in $\mathcal{CM}(Y, \xi_0)$ and each element of a certain Banach space of perturbations $\mathcal{P}$ we consider the Seiberg–Witten monopole equations over a certain non-compact 4-manifold Z^+, with suitable asymptotics over its ends to canonical configurations determined by the contact structures together with critical points of the Chern-Simons functional. This leads to a *universal* moduli space of solutions, which is a Banach manifold equipped with a Fredholm map

$$\mathfrak{M}(Z^+) \xrightarrow{\pi} \mathcal{CM}(Y, \xi_0) \times \mathcal{P}.$$

The moduli space decomposes according to critical points of the Chern-Simons-Dirac functional

$$\mathfrak{M}(Z^+) = \bigcup_{[\mathfrak{a}]} \mathfrak{M}([\mathfrak{a}], Z^+).$$

Given a generic cycle T in $\mathcal{CM}(K, \xi_0) \times \mathcal{P}$ transverse to the Fredholm map π, we count *isolated* points in $\mathfrak{M}(Z^+)$ which lie over T, and this leads to integers $\#(\mathfrak{M}([\mathfrak{a}], Z^+) \cdot T) \in \mathbb{Z}$. Indexing the counts by the critical points $[\mathfrak{a}]$ we obtain a cocycle in the Floer cochain complex

$$\psi(T) = \sum_{[\mathfrak{a}]} \#\big(\mathfrak{M}([\mathfrak{a}], Z^+) \cdot T\big) \cdot [\mathfrak{a}] \in \widehat{C}^*(Y, \mathfrak{s}_{\xi_0}; R).$$

This yields the homomorphism (1.4). In fact, we will be able to define the homomorphism at the chain level.

4.2.1 Differential-geometric aspects

4.2.1.1 The symplectic end and the cylindrical end

We start by discussing the various metric structures that come into the construction.

Remark 4.5. For ease in notation we will denote elements of $\mathcal{CM}(Y, \xi_0)$ by the symbol t. When we need to make reference to it, the contact metric structure on Y associated to t is denoted (ξ_t, α_t, j_t). From now on, we also fix a C^∞ base triple $(\xi_0, \alpha_0, j_0) \in \mathcal{CM}(Y, \xi_0)$.

Let Z^+ be the non-compact 4-manifold $\mathbb{R} \times Y$ with the product orientation. Let $K = [1, +\infty) \times Y \subset Z^+$ and $Z = (-\infty, 0] \times Y$. For each $t \in \mathcal{CM}(Y, \xi_0)$ we have an almost-Kähler structure (ω_t, J_t, g_t) over K obtained from (4.3-4.4), where ω_t is C^{l-1}, J_t is C^l and g_t is C^{l-1}. Recall that g_t is the cone metric over $(Y, g_{\xi_t,\alpha_t,j_t})$, namely $g_t = ds^2 + s^2 g_{\xi_t,\alpha_t,j_t}$.

We now extend the metric g_t from K to the whole of Z^+. Over $Z = (-\infty, 0] \times Y$ the metric g_t agrees with the cylindrical product metric $ds^2 + g_{\xi_0,\alpha_0,j_0}$. We fix the behaviour of the metric g over the region $[0, 1] \times Y$ as follows. Choose a smooth function $\kappa : [0, 1] \to \mathbb{R}_{\geq 0}$ such that $\kappa \equiv 1$ on a neighbourhood of $[0, 1/2]$ and $\kappa \equiv 0$ on a neighbourhood of 1. Then the metric g_t over the region $[0, 1] \times Y$ is defined as $ds^2 + \kappa(t) g_{\xi_0,\alpha_0,j_0} + (1 - \kappa(s)) s^2 g_{\xi_t,\alpha_t,j_t}$.

We will refer to K as the *conical* or *symplectic end* of Z^+, and to Z as the *cylindrical end*. Observe that in this construction the family of metrics g_t restricted over the cylindrical end Z is independent of t (it only depends on the fixed base triple (ξ_0, α_0, j_0)). We will denote by (g_0, ω_0, J_0) the corresponding structures determined by the base triple (ξ_0, α_0, j_0).

4.2.1.2 Families of spin-c structures and canonical configurations

We move on now to discuss families of spin-c structures and irreducible configurations on the non-compact manifold Z^+. The latter will provide us with the right boundary conditions for the Seiberg–Witten equations over the symplectic end later on.

We consider the "trivial" family of spin-c structures on Z^+ parametrised by $\mathcal{CM}(Y, \xi_0)$. Namely, we start with the spin-c structure $\mathfrak{s}_{\omega_0,J_0,g_0}$ on K determined by the almost-Kähler structure (ω_0, J_0, g_0)

(see §4.1.5.1). By Lemma 4.3 we have $\mathfrak{s}_{\omega_0,J_0,g_0}|_{-Y} = -\mathfrak{s}_{\xi_0,\alpha_0,j_0}$. Hence, we may extend the spin-c structure $\mathfrak{s}_{\omega_0,J_0,g_0}$ from K over to the whole of Z^+ in a translation-invariant manner over $Z^+ \setminus K$. We denote this spin-c structure on Z^+ by $\mathfrak{r}_0 = (g_0, S, \rho)$. Finally, by *changing metrics* we obtain a family of spin-c structures on Z^+ parametrised by $C\mathcal{M}(Y, \xi_0)$: for $t \in C\mathcal{M}(Y, \xi_0)$ we set $\mathfrak{r}_t = (g_t, S, \rho_t)$ with $\rho_t = \rho \circ b^*_{g_t,g_0}$. Observe that the spinor bundle of $\mathfrak{r}_t$ is S, independent of t.

We now discuss irreducible configurations on Z^+ parametrised by $C\mathcal{M}(Y, \xi_0)$. The Clifford action of the symplectic structures gives a trace-less skew-adjoint map $\rho_t(\omega_t) : S^+|_K \to S^+|_K$ such that $\rho(\omega_t)^2 = -4 \cdot \mathrm{id}_{S^+|_K}$. This induces a decomposition

$$S^+|_K = E_-(t) \oplus E_+(t)$$

into $\mp 2i$ eigensubbundles (each with with rank 1). Because $\mathfrak{r}_t$ is (non-canonically!) isomorphic over K to the spin-c structure induced from (ω_t, J_t, g_t) it follows that the $-2i$ eigensubbundle admits a trivialisation $E_-(t) \approx \mathbb{C}$ for each t. We warn the reader that, as a bundle over $K \times C\mathcal{M}(Y, \xi_0)$, the bundle E_- need *not* admit a trivialisation.

Let $U \subset C\mathcal{M}(Y, \xi_0)$ be an open contractible subset. Then we may choose a unitary trivialisation of E_- over $K \times U$ and obtain a U-family of nowhere-vanishing spinors $\Phi_t \in \Gamma(K, E_-)$ with pointwise unit length. As in §4.1.5.1 there is a unique U-family of spin-c connections A_t over K such that $D^+_{A_t}\Phi_t = 0$. We refer to the U-family of irreducible configurations (A_t, Φ_t) as the *canonical configurations* over the symplectic end (associated to a given trivialisation of E_- over $K \times U$).

Remark 4.6. By putting together the canonical configurations (A_t, Φ_t) over all $U \subset C\mathcal{M}(Y, \xi_0)$ (and applying a further change of metrics back to g_0) we obtain a continuous map

$$f : C\mathcal{M}(Y, \xi_0) \to \mathcal{B}^*(K, \mathfrak{s}_{\omega_0,J_0,g_0}).$$

The interpretation of this map is clear: the space $C\mathcal{M}(Y, \xi_0)$ parametrises a family of almost-Kähler structures on K, which in turn parametrises a family of irreducible configurations over K

as in §4.1.5.1. The classifying map for this family (in the sense of Lemma 4.1) is the map f. At a heuristic level, the families contact invariant that we construct in this section should be regarded as a sort of "pushforward map in homology" induced by f.

4.2.1.3 *Translation invariance of the canonical configurations*

We now discuss how the canonical configurations over the symplectic end can be made *translation invariant* in a suitable sense, which will become convenient occasionally.

Recall that $g_t = ds^2 + s^2 g_{\xi_t,\alpha_t,j_t}$ is a conical metric over $K = [1,+\infty) \times Y$. Denote by $\overline{g}_t = ds^2 + g_{\xi_t,\alpha_t,j_t}$ the corresponding cylindrical metric. The *rescaling operator* $\mathcal{R}_t : (TK, \overline{g}_t) \to (TK, g_t)$ gives an isometry between the two metrics:

$$\mathcal{R}_t(\partial_s) = \partial_s$$
$$\mathcal{R}_t(v) = \frac{1}{s} v \ , \ v \in TY.$$

The almost complex structure J_t is carried to a translation-invariant almost complex structure $\overline{J}_t = \mathcal{R}_t^{-1} \circ J_t \circ \mathcal{R}_t$, and we have corresponding unitary vector bundle isometries

$$\mathcal{R}_t^* : (\Lambda^{p,q}_{J_t} T^*K, h_t) \to (\Lambda^{p,q}_{\overline{J}_t} T^*K, \overline{h}_t)$$

where h_t and $\overline{h}_t$ stand for the hermitian metrics determined by the pairs (g_t, J_t) and $(\overline{g}_t, \overline{J}_t)$.

At this point, we recall that the spinor bundle $S = S^+ \oplus S^-$ underlying our family of spin-c structures $\mathfrak{r}_t$ is independent of t and has the following form over K (see §4.2.1.2 and §4.1.5.1):

$$S^+ = \mathbb{C} \oplus \Lambda^{0,2}_{J_0} T^*K \ , \ S^- = \Lambda^{1,1}_{J_0} T^*K.$$

The "rescaled" unitary bundle $\overline{S^+} := \mathbb{C} \oplus \Lambda^{0,2}_{\overline{J}_0} T^*K$ (and likewise for $\overline{S^-}$) is identified with the pullback to $K = [1,+\infty) \times Y$ of a bundle over Y, and hence one can speak of translation-invariant sections or connections on this bundle. We have:

Lemma 4.4. *Let (A_t, Φ_t) be a U-family of canonical configurations on K, for a contractible open $U \subset \mathcal{CM}(Y, \xi_0)$. After applying a U-family of smooth gauge transformations $g_t : K \to \mathrm{U}(1)$ to (A_t, Φ_t) we may assume that:*

(i) the sections $\overline{\Phi}_t := \mathcal{R}_0^*\Phi_t \in \Gamma(K, \overline{S^+})$ are translation-invariant, and

(ii) the connections $\overline{A}_t := \mathcal{R}_0^* A_t$ on $\overline{S} = \overline{S^+} \oplus \overline{S^-}$ are translation-invariant.

Definition 4.10. A family of canonical configurations (A_t, Φ_t) over K parametrised by U is in *translation-invariant form* if it satisfies (i)-(ii) above.

Proof of Lemma 4.4. The family of spin-c structures $\mathfrak{r}_t = (g_t, S, \rho_t)$ is isomorphic, via the rescaling operator $\mathcal{R}^*$, to the family of spin-c structures $\overline{\mathfrak{r}}_t = (\overline{g}_t, \overline{S}, \overline{\rho}_t)$ where

$$\overline{\rho}_t(v) = \mathcal{R}_0^*\rho_t((\mathcal{R}_t^*)^{-1}v)(\mathcal{R}_0^*)^{-1}, \;\; v \in T^*K. \tag{4.5}$$

Now, we have a translation-invariant non-degenerate 2-form $\overline{\omega}_t := \mathcal{R}^*\omega_t = ds \wedge \alpha_t + \frac{1}{2}d\alpha_t$, so the $-2i$-eigensubbundle $\overline{E}_- \subset \overline{S^+}$ corresponding to the action of $\overline{\rho}_u(\overline{\omega}_u)$ on $\overline{S^+}$ is also translation-invariant. Thus, in view of (4.5) the assertion (i) is clear: one chooses a trivialisation of $\overline{E}_-$ over $K \times U$ by a translation-invariant unit section $\overline{e} \in \Gamma(K \times U, \overline{E}_-)$ to obtain a U-family of sections $(\mathcal{R}_t^*)^{-1}\overline{e}(\cdot, t) \in \Gamma(K, E_-(t))$ which agree with Φ_t up to gauge transformations. Next, we verify that (ii) holds assuming that Φ_t satisfies (i). Write the covariant derivative with respect to $(\mathcal{R}_0)^*A_t$ as $\frac{d}{ds} + \nabla_{B_t(s)} + c_t(s)ds$, where for each t we have paths of connections $s \mapsto B_t(s)$ and $i\mathbb{R}$-valued functions $s \mapsto c_t(s)$ on Y (parametrised by $s \in [1, +\infty)$). Recall that A_t can be characterised by the property that $\nabla_{A_t}\Phi_t$ is orthogonal to Φ_t (see Remark after Lemma 4.2) we obtain

$$\langle \nabla_{B_t(s)}\overline{\Phi}_t, \overline{\Phi}_t\rangle_{\overline{h_0}} + c_t(s)ds \equiv 0.$$

It follows that both terms above vanish. The vanishing of $\langle \nabla_{B_t(s)}\overline{\Phi}_t, \overline{\Phi}_t\rangle_{\overline{h_0}}$ also gives us that $B_t(s)$ is independent of s since $\overline{\Phi}_t$ is translation-invariant. Thus, (ii) follows. □

4.2.1.4 A basic example

It may be instructive to review the above constructions in a particularly simple case.

Consider the flat hyperkähler structure (g_0, J_1, J_2, J_3) on $\mathbb{R}^4$. The radial vector field $v = x\partial_x + y\partial_y + z\partial_z + w\partial_w$ in $\mathbb{R}^4$ is Liouville for all symplectic structures in the family $\omega_t = \sum_{i=1}^3 t_i g_0(J_i\cdot, \cdot)$ parametrised by $t \in S^2 \subset \mathbb{R}^3$ (i.e. $\mathcal{L}_v\omega_u = \omega_u$) and v is transverse to $S^3 \subset \mathbb{R}^4$. Thus there is an S^2-family of contact forms α_t on S^3 given by $\alpha_t = \iota_V\omega_t$. This family of contact structures is SU(2)-invariant, and thus will descend to a family of contact structures on the quotients S^3/Γ by a finite subgroup $\Gamma \subset \mathrm{SU}(2)$. The manifolds S^3/Γ are precisely the links of the ADE singularities (which include e.g. the lens spaces $L(p, p-1)$ or the Poincaré sphere $\Sigma(2, 3, 5)$).

Let $Y = S^3/\Gamma$. A complex structure j_t on the contact distribution $\xi_t = \ker\alpha_t$ is obtained by restricting $J_t = \sum_i t_i J_i$. We thus have a family $(\xi_t, \alpha_t, j_t) \in \mathcal{CM}(S^3, \xi_0)$, and we take the base triple $(\xi_0, \alpha_0, j_0) := (\xi_t, \alpha_t, j_t)|_{t=(1,0,0)}$. The associated family of almost-Kähler structures on $K = [1, +\infty) \times Y$ agrees with the flat hyperkähler structure under the identification

$$K \cong (\mathbb{R}^4 \setminus \{x^2 + y^2 + z^2 + w^2 < 1\})/\Gamma\ ,\ \ (s, (x, y, z, w)) \mapsto (sx, sy, sz, sw).$$

We next calculate canonical configurations. The positive spinor bundle is $S^+ = \mathbb{C} \oplus \Lambda^{0,2}_{J_1}T^*K \cong \mathbb{C}^2$, and we have trivialised $\Lambda^{0,2}_{J_1}T^*K$ using $\frac{1}{2}d\overline{z}^1 \wedge d\overline{z}^2$. Likewise $S^- \cong \mathbb{C}^2$. The Clifford multiplications $\rho_t = \rho$ are independent of t. The symplectic forms $\omega_i = g_0(J_i,)$ have the following Clifford actions on $S^+ = \mathbb{C}^2$:

$$\rho(\omega_1) = \begin{pmatrix} -2i & 0 \\ 0 & 2i \end{pmatrix} \quad \rho(\omega_2) = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \quad \rho(\omega_3) = \begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix}.$$

and thus

$$\rho(\omega_t) = \begin{pmatrix} -2it_1 & -2(t_2 - it_3) \\ 2(t_2 + it_3) & 2it_1 \end{pmatrix}.$$

An S^2-family of sections of $-2i$-eigenspace of $\rho(\omega_t)$ is $(1+t_1, it_2 - t_3)$. Each has a transverse zero at $t = (-1,0,0)$ and is non-vanishing elsewhere. This means that E_- is not trivial over $K \times S^2$. Normalising we obtain a family of unit length sections of $E_-(t)$ over K for $t \in U_1 := S^2 \setminus (-1,0,0)$:

$$\Phi_t = \Big(\sqrt{\frac{1+t_1}{2}}, \frac{it_2 - t_3}{\sqrt{2(1+t_1)}}\Big) \in S^+ = \mathbb{C}^2.$$

The corresponding family of spin-c connections A_t is independent of t and is given by the trivial connection on $S = S^+ \oplus S^- = \mathbb{C}^2 \oplus \mathbb{C}^2$. The family of canonical configurations carried by U_1 that we just constructed is also in translation-invariant form.

4.2.2 Space of configurations

We now construct a suitable space of configurations (A, Φ) over the symplectic end K which has the structure of a Banach manifold.

4.2.2.1 Sobolev spaces on non-compact manifolds

To work in the convenient setting of Fredholm theory we make use of Sobolev spaces over the non-compact symplectic end. On a Riemannian manifold (M, g) of *bounded geometry*, the various possible definitions of Sobolev spaces of sections will agree. We refer the reader to [[18], Chapter 11], or to [[16], §3.2] for an exposition of these results. The cone over a closed Riemannian manifold, the case that concerns us, falls into this desirable category.

In the above setting, given an Euclidean vector bundle $E \to M$ with an orthogonal connection A, the space of Sobolev sections $L^2_{k,g,A}(M, E)$ can be defined as the space of measurable sections s of E with distributional derivatives up to order k and such that

$$||s||^2_{L^2_k} := \sum_{j \le k} \int_M |\nabla^{(j)}_A s|^2_h \mathrm{dvol}_g < +\infty.$$

In the above formula, $\nabla^{(j)}_A$ is the connection on $E \otimes (T^*M)^{\otimes(j-1)}$ induced from A and the Levi-Civita connection of g, and the symbol $|\cdot|_h$ denotes the metric induced from h and g_t on the bundle $E \otimes (T^*K)^{\otimes j}$. The vector space $L^2_{k,g,A}(M, E)$ equipped with the L^2_k inner product becomes a Hilbert space. When g, E, or A are understood we might drop them from the notation.

From now on, we will fix an integer $k \geq 4$, which ensures that L^2_k configurations over a 4-manifold of bounded geometry are in C^0 by the Sobolev embedding theorem. We recall that we have been working thus far with the space $\mathcal{CM}(Y, \xi_0)$ of triples (ξ, α, j), where the regularity of ξ and α is C^l, and j is C^{l-1}. Hence $g_{\xi,\alpha,j}$ is C^{l-1}. We fix the the integer l so that $l - k - 2 \geq 2$, because we will later need that $C^{l-k-2} \subset C^2$.

4.2.2.2 *Boundary conditions over the symplectic end*

We now set up the relevant configuration spaces over the symplectic end, with asymptotics to the canonical configurations provided by the contact geometry. Because canonical configurations only exist over sufficiently small neighbourhoods $U \subset \mathcal{CM}(Y, \xi_0)$, our construction of configuration spaces will involve taking suitable limits over such neighbourhoods.

In what follows, it is convenient to consider the slightly larger region containing the symplectic end: $K' = [0, 1] \times Y \cup K \subset Z^+$. Let $U \subset \mathcal{CM}(Y, \xi_0)$ be an open contractible subset, carrying a family of canonical configurations $\gamma := ((A_t, \Phi_t))_{t \in U}$ defined over K which are in translation-invariant form (Definition 4.10).

Definition 4.11. For (U, γ) as above, the *configuration space* for (U, γ), denoted $C_k(K', \gamma)_U$, is the space of triples (t, A, Φ), where $t \in U$, A is a locally L^2_k spin-c connection on the spinor bundle S (for the spin-c structure $\mathfrak{r}_t$) defined over K' and Φ is a locally L^2_k section of S^+ over K', subject to the following asymptotics:

$$\Phi - \Phi_t \in L^2_{k,g_t,A_t}(K, S^+) \tag{4.6}$$

$$A - A_t \in L^2_{k,g_t}(K, T^*K \otimes i\mathbb{R}). \tag{4.7}$$

The relevant *gauge group* in this setting is the group $\mathcal{G}_{k+1}(K')$ of locally L^2_{k+1} maps $v : K' \to$

U(1) which approach the identity, i.e.

$$1 - v \in L^2_{k+1,g_t}(K).$$

Again, the Sobolev space above does not depend on t. Observe that configurations in $C_k(K', \gamma)_U$ are necessarily *irreducible* (i.e. Φ doesn't vanish everywhere on K) due to the asymptotic condition (4.6). Hence $\mathcal{G}_{k+1}(K')$ acts freely on $C_k(K', \gamma_0)_U$.

Since for any two conical metrics g_0, g_1 over K the difference $g_1 g_0^{-1}$ is bounded over K and the configurations (A_t, Φ_t) were chosen in translation-invariant form (Definition 4.10), it follows that the Sobolev spaces $L^2_{k,g_t,A_t}(K, S^+)$ and $L^2_{k,g_t}(K, T^*K \otimes i\mathbb{R})$ are *independent* of $t \in U$. The configuration space for (U, γ) then forms a trivial bundle of affine Hilbert spaces

$$C_k(K', \gamma)_U \to U.$$

We make $C_k(K', \gamma)_U$ into a Banach manifold by identifying it with $L^2_k \times U$ via $(A, \Phi, t) \mapsto (A - A_t, \Phi - \Phi_t, t)$. In this "chart", the $\mathcal{G}_{k+1}(K')$-action $\mathcal{G}_{k+1}(K') \times (L^2_k \times U) \to (L^2_k \times U)$ acquires the rather odd-looking form: $v \cdot (a, \phi, t) = (a - v^{-1}dv, v\phi - (1 - v)\Phi_t, t)$. This action is only of class C^{l-k-2}. The reason is that Φ_t depends on first derivatives of the metric g_t (and also on α_t and j_t) which has regularity C^{l-1}; thus we may only differentiate $l - k - 2 = (l - 2) - k$ times the action $\mathcal{G}_{k+1}(K') \times C_k(K', \gamma)_U \to C_k(K', \gamma)_U \to$ in order to land inside L^2_k.

Most naturally, though, the tangent space at a given configuration (A, Φ, t) is identified with

$$T_{(A,\Phi,t)}C_k(K', \gamma)_U = \left\{(a, \phi, t) \mid \dot{t} \in T_t\mathcal{CM}(Y, \xi_0)\,,\ a - \frac{\partial}{\partial \dot{t}}A_t \in L^2_k(K)\,,\ \phi - \frac{\partial}{\partial \dot{t}}\Phi_t \in L^2_k(K)\right\}. \tag{4.8}$$

We omit the proof of the next result, which is done by carrying out the standard construction of slices for the gauge action (see [49] or [16]).

Lemma 4.5. *The gauge group $\mathcal{G}_{k+1}(K')$ is a Hilbert Lie group that acts freely in a C^{l-k-2} fashion* *on the Banach manifold $\mathcal{C}_k(K',\gamma)_U$ by*

$$v \cdot (t, A, \Phi) = (t, A - v^{-1}dv, v\Phi)$$

and the quotient $\mathcal{B}_k(K',\gamma)_U = \mathcal{C}_k(K',\gamma)_U/\mathcal{G}_{k+1}(K')$ is naturally a C^{l-k-2} Banach manifold.

Consider now a second open contractible subset $\tilde{U} \subset \mathcal{CM}(Y,\xi_0)$ together with a $\tilde{U}$-family of canonical configurations $\tilde{\gamma} = ((\tilde{A}_t, \tilde{\Phi}_t))_{t\in\tilde{U}}$ and with $U \subset \tilde{U}$. We also assume that the families of canonical configurations γ and $\tilde{\gamma}$ carried by U and $\tilde{U}$, respectively, are in *translation-invariant form* (Definition 4.10). Then we find a unique U-family of gauge transformations $v_t : K \to \mathrm{U}(1)$ ($t \in U$) such that $v_t \cdot (\Phi_t, A_t) = (\tilde{\Phi}_t, \tilde{A}_t)$. The translation-invariance of γ and $\tilde{\gamma}$ implies that the gauge-transformations v_t are translation-invariant over the symplectic end $K = [1,+\infty) \times Y$, namely $v_t(s,y) = v_t(1,y)$. In view of this, we may extend the v_t over to the larger region K' by translation. We warn the reader that the v_t need not satisfy the asymptotics $1 - v_t \in L^2_{k+1,g_t}(K)$. However, we do obtain an inclusion map

$$\begin{aligned} \mathcal{C}_k(K',\gamma)_U &\to \mathcal{C}_k(K',\tilde{\gamma})_{\tilde{U}} \\ (t, A, \Phi) &\mapsto (t, A - v_t^{-1}dv_t, v_t\Phi). \end{aligned} \tag{4.9}$$

Lemma 4.6. *The map (4.9) is a well-defined smooth $\mathcal{G}_{k+1}(K')$-equivariant map which is an open embedding.*

Proof. The only issue which requires checking is whether (4.9) is well-defined. That is, we must check that if (t,A,Φ) is in $\mathcal{C}_k(K',\gamma)_U$ then $(t,\tilde{A},\tilde{\Phi}) := v_t \cdot (t,A,\Phi) = (t, A - v_t^{-1}dv_t, v_t\Phi)$ satisfies the conditions of Definition 4.11:

- $\tilde{\Phi} - \tilde{\Phi}_t = v_t(\Phi - \Phi_t)$. Thus, $\tilde{\Phi} - \tilde{\Phi}_t$ is in $L^2(K)$, because $\Phi - \Phi_t \in L^2(K)$ and v_t has unit length

- $\nabla_{\tilde{A}_t}(\tilde{\Phi} - \tilde{\Phi}_t) = \nabla_{A_t - v_t^{-1}dv_t}(v_t(\Phi - \Phi_t)) = v_t \nabla_{A_t}(\Phi - \Phi_t)$. Since $|v_t| = 1$ and $\nabla_{A_t}(\Phi - \Phi_t) \in L^2(K)$ then $\nabla_{\tilde{A}_t}(\tilde{\Phi} - \tilde{\Phi}_t)$ is also in $L^2(K)$. Similarly, $\nabla^l_{\tilde{A}_t}(\tilde{\Phi} - \tilde{\Phi}_t) \in L^2(K)$ for all $l \geq 1$

- $\tilde{A} - \tilde{A}_t = A - A_t$ over K, and so $\tilde{A} - \tilde{A}_t \in L^2_k(K)$.

□

Thus, we have a *directed system* whose objects are the Banach manifolds $\mathcal{C}_k(K', \gamma)_U$, one for each tuples (U, γ) consisting of an open contractible set $U \subset C\mathcal{M}(Y, \xi_0)$ carrying the family of canonical configurations γ in translation-invariant form. A unique morphism (4.9), which is an open embedding of Banach manifolds, is associated with any two pairs $(U, \gamma), (\tilde{U}, \tilde{\gamma})$ such that $U \subset \tilde{U}$.

Definition 4.12. We define the *configuration space* $\mathcal{C}_k(K')$ as the direct limit of the above directed system

$$\mathcal{C}_k(K') = \varinjlim_{(U,\gamma)} \mathcal{C}_k(K', \gamma)_U.$$

$\mathcal{C}_k(K')$ is a Banach manifold. It is the total space of a bundle of affine Hilbert spaces

$$\mathcal{C}_k(K') \to C\mathcal{M}(Y, \xi_0)$$

equipped with a preferred connection i.e. a complementary (horizontal) subbundle to the vertical subbundle of $T\mathcal{C}_k(K')$. Over each $U \subset C\mathcal{M}(Y, \xi_0)$ carrying a family of canonical configurations γ this connection induces the trivial splitting of $T\mathcal{C}_k(K', \gamma)_U$ obtained from the fact that the Sobolev spaces $L^2_{k,g_t,A_t}(K')$ are independent of $t \in U$.

We also have the configuration space modulo gauge

$$\mathcal{B}_k(K') = \mathcal{C}_k(K')/\mathcal{G}_{k+1}(K') \cong \varinjlim_{(U,\gamma)} \mathcal{B}_k(K', \gamma)_U.$$

By Proposition 4.5, $\mathcal{B}_k(K')$ is a C^{l-k-2} Banach manifold, and it carries a natural projection to $C\mathcal{M}(Y, \xi_0)$.

4.2.2.3 Configuration space on Y

For future reference, we also introduce here the relevant configuration spaces for the 3-manifold Y. We refer the reader to [49] for further details. Given a spin-c structure $\mathfrak{s} = (g, S, \rho)$ on Y, we have the configuration space $C_{k-1/2}(Y, \mathfrak{s})$ of pairs (B, Ψ) consisting of a spin-c connection B and a section Ψ of S, both of regularity $L^2_{k-1/2}$. Those pairs with Ψ not identically vanishing are called irreducible, and the locus of such is denoted $C^*_{k-1/2}(Y, \mathfrak{s}) \subset C_{k-1/2}(Y, \mathfrak{s})$. The blown-up configuration space $C^\sigma_{k-1/2}(Y, \mathfrak{s})$ consists of triples (B, s, Ψ) where now $s \geq 0$ is a non-negative real number, and $||\Psi||_{L^2} = 1$. The respective quotients by the (free) action of the group of $L^2_{k+1/2}$ gauge transformations are denoted $\mathcal{B}^*_{k-1/2}(Y, \mathfrak{s})$ and $\mathcal{B}^\sigma_{k-1/2}(Y, \mathfrak{s})$. They are Hilbert manifolds in a natural way [[49], §9.3] (provided $k \geq 3$) and $\mathcal{B}^\sigma_{k-1/2}(Y, \mathfrak{s})$ has boundary given by configurations $(B, 0, \Psi)$ with $||\Psi||_{L^2} = 1$.

4.2.3 Moduli space and perturbations

We now construct the promised Seiberg–Witten moduli space $\mathfrak{M}([\mathfrak{a}], Z^+)$, which will be a Banach manifold equipped with a Fredholm map $\mathfrak{M}([\mathfrak{a}], Z^+) \xrightarrow{\pi} C\mathcal{M}(Y, \xi_0) \times \mathcal{P}$. This moduli is constructed by gluing together a moduli space over K' with a moduli space over the cylindrical end $Z = (-\infty, 0] \times Y$.

4.2.3.1 The moduli space over K′

The Seiberg–Witten equations define a $\mathcal{G}_{k+1}(K')$-equivariant section sw of a vector bundle $\Upsilon_{k-1} \to C_k(K')$, which we now describe. On configuration spaces over an open $U \subset C\mathcal{M}(Y, \xi_0)$ equipped with a family of canonical configurations, we have the Seiberg–Witten map

$$\begin{aligned} \mathrm{sw}_{\gamma,U} : C_k(K', \gamma)_U &\to \Upsilon_{k-1,\gamma,U} \\ (t, A, \Phi) &\mapsto (\frac{1}{2}\rho_t(F^{+,g_t}_{\hat{A}}) - (\Phi\Phi^*)_0, D^+_{A,g_t}\Phi). \end{aligned}$$

Remark 4.7. We explain the notation from the above formula. First $\Upsilon_{k-1,\gamma,U}$ is the bundle over $C_k(K',\gamma)_U$ with fibre over the point (t,A,Φ) given by $L^2_{k-1,g_t,A_t}(K', i\mathfrak{su}(S^+)\oplus S^-)$. Then $\rho_t(F^{+,g_t}_{\hat{A}})$ is the self-adjoint endomorphism S^+ arising from the Clifford action of the self-dual component of the curvature $F^{+,g_t}_{\hat{A}}$ of the $U(1)$ connection $\hat{A}$ on $\Lambda^2 S^+$. The quadratic term $(\Phi\Phi^*)_0$ is the endomorphism which acts on a given spinor $\phi \in S^+$ by

$$\phi \mapsto \langle \Phi, \phi \rangle \Phi - \frac{1}{2}|\Phi|^2 \phi.$$

As before, given two open contractible subsets $U \subset \tilde{U}$ carrying canonical configurations, there is also a transition map

$$\Upsilon_{k-1,\gamma,U} \xrightarrow{\iota} \Upsilon_{k-1,\tilde{\gamma},\tilde{U}}$$
$$((\sigma,\Psi),(t,A,\Phi)) \mapsto ((\sigma, v_t\Psi), \iota(t,A,\Phi))$$

compatible with projections to the base, which thus yields a limiting bundle $\Upsilon_{k-1} \to C_k(K')$. The Seiberg–Witten maps fit in to give a commutative diagram

$$\begin{array}{ccc} C_k(K',\gamma)_U & \xrightarrow{\iota} & C_k(K',\tilde{\gamma})_{\tilde{U}} \\ \Big\downarrow \mathrm{sw}_{\gamma,U} & & \Big\downarrow \mathrm{sw}_{\tilde{\gamma},\tilde{U}} \\ \Upsilon_{k-1,\gamma,U} & \xrightarrow{\iota} & \tilde{\Upsilon}_{k-1,\tilde{\gamma},\tilde{U}}. \end{array}$$

which provides a well-defined section sw of the bundle $\Upsilon_{k-1} \to C_k(K')$ that we call the *Seiberg–Witten map*.

In [[49], §11.6], a Banach space $\mathcal{P}$ of *tame* perturbations of the Chern-Simons-Dirac functional on a 3-manifold Y with a spin-c structure is constructed to achieve transversality for moduli spaces of gradient trajectories. In our context, a suitable perturbation scheme, following the approaches of [49] , [47] and [17], is introduced as follows. Let $\mathcal{P}$ be such a Banach space of tame perturbations of the Chern-Simons-Dirac functional of $(Y, g_{\xi_0,\alpha_0,j_0})$. We define a $\mathcal{G}_{k+1}(K')$-equivariant section

$\mu_{\gamma,U} : \mathcal{C}_k(K',\gamma)_U \times \mathcal{P} \to \Upsilon_{k-1,\gamma,U}$, of the form

$$\mu_{\gamma,U}(t, A, \Phi, \mathfrak{p}) = \varphi^1 \hat{\mathfrak{q}}(A, \Phi) + \varphi^2 \hat{\mathfrak{p}}(A, \Phi) + \varphi^3 \hat{\mathfrak{p}}_{K,t}. \tag{4.10}$$

We describe the items appearing in (4.10):

(i) we choose an admissible ([49], Definition 22.1.1) perturbation $\mathfrak{q}$ of the Chern-Simons-Dirac functional on $(Y, g_{\xi_0,\alpha_0,j_0})$. This induces a translation-invariant perturbation $\hat{\mathfrak{q}}(A, \Phi)$ over $\mathbb{R}\times Y$, as in [[49], §10.1]. Then φ^1 is a smooth cutoff function on $[0, +\infty)$, which is identically 1 on a neighbourhood of 0, and vanishes on a neighbourhood of $[1/2, +\infty)$

(ii) $\mathfrak{p} \in \mathcal{P}$ induces, as before, a translation-invariant perturbation $\hat{\mathfrak{p}}$ over $\mathbb{R} \times Y$. We choose φ^2 to be a bump function compactly supported in $(0, 1/2)$, and identically 1 at some interval in the interior

(iii) φ^3 is a cutoff function on $[0, +\infty)$ which is identically 1 over $[1, +\infty)$ and vanishing on a neighbourhood of $[0, 1/2]$. We take the family of sections of $\Upsilon_{k-1,\gamma_0,U}$ given by

$$\hat{\mathfrak{p}}_{K,t} = (-\frac{1}{2}\rho_t(F^{+,g_t}_{A_t}) + (\Phi_t\Phi_t^*)_0, 0).$$

The sections $\mu_{\gamma,U}$ glue to a section $\mu : \mathcal{C}_k(K') \times \mathcal{P} \to \Upsilon_{k-1}$, which we combine with $\mathrm{sw} : \mathcal{C}_k(K') \to \Upsilon_{k-1}$ to obtain the *perturbed Seiberg–Witten map*:

$$\mathrm{sw}_\mu = \mathrm{sw} + \mu : \mathcal{C}_k(K') \times \mathcal{P} \to \Upsilon_{k-1}. \tag{4.11}$$

The motivation for choosing the perturbation $\hat{\mathfrak{p}}_K$ comes from Taubes' work [78]. This perturbation term forces the canonical configurations to solve the equations $\mathrm{sw}_\mu = 0$ over the symplectic end $K \subset Z^+$. We include the perturbations $\hat{\mathfrak{q}}$, $\hat{\mathfrak{p}}$ to achieve the necessary transversality later on.

Definition 4.13. The *universal* moduli space of Seiberg–Witten monopoles over K' is

$$\mathfrak{M}_k(K') := \mathrm{sw}_\mu^{-1}(0)/\mathcal{G}_{k+1}(K') \cong \varinjlim_{(\gamma,U)} (\mathrm{sw}_{\gamma,U} + \mu_{\gamma,U})^{-1}(0)/\mathcal{G}_{k+1}(K').$$

The perturbed Seiberg–Witten map sw_μ descends to a section on the quotient bundle $\Upsilon_{k-1}/\mathcal{G}_{k+1}(K') \to \mathcal{B}_k(K') \times \mathcal{P}$.

In §A.1 we will show a general transversality result (based on those of [49] and [16]) which applies to the various moduli spaces that appear in this article. In particular it will give us:

Proposition 4.7. *The Seiberg–Witten map is a C^{l-k-2} section of $\Upsilon_{k-1}/\mathcal{G}_{k+1}(K') \to \mathcal{B}(K') \times \mathcal{P}$ which is transverse to the zero section. Thus $\mathfrak{M}_k(K')$ is a C^{l-k-2} Banach submanifold of $\mathcal{B}(K')\times\mathcal{P}$.*

4.2.3.2 The moduli space as a fibre product

Using the metric g_{ξ_0,α_0,j_0} on Y and the perturbation $\mathfrak{q} \in \mathcal{P}$, one can construct the moduli space of Seiberg–Witten monopoles over the half-infinite cylinder $((-\infty, 0]\times Y, dt^2+g_{\xi_0,\alpha_0,j_0})$ asymptotic to a critical point $[\mathfrak{a}]$ for the flow of the $\mathfrak{q}$-perturbed Chern-Simons-Dirac functional in the blowup. It follows that $[\mathfrak{a}]$ is either irreducible or unstable. This moduli is denoted $M_k([\mathfrak{a}], (-\infty, 0] \times Y)$ and it is a Hilbert manifold. We refer the reader to [49] for details.

There are restriction maps onto the blown-up configuration space of the slice $0 \times Y$

$$M_k([\mathfrak{a}], (-\infty, 0] \times Y) \xrightarrow{R_+} \mathcal{B}^\sigma_{k-1/2}(Y, \mathfrak{s}_{\xi_0,\alpha_0,j_0})$$
$$\mathfrak{M}_k(K') \xrightarrow{\mathfrak{R}_-} \mathcal{B}^\sigma_{k-1/2}(Y, \mathfrak{s}_{\xi_0,\alpha_0,j_0}).$$

That the restriction maps are indeed well-defined follows by a unique continuation principle for the Seiberg–Witten equations (Proposition 10.8.1 [49]). We will see in §A.1 that the sum of the derivatives of the restriction maps along the spinor and connection direction

$$dR_+ + d\mathfrak{R}_-(-, -, 0, 0)$$

is a Fredholm map and we will establish a transversality result:

Proposition 4.8. *The restriction maps R_+ and $\mathfrak{R}_-$ are transverse. Thus, the fibre product* $\mathrm{Fib}(R_+, \mathfrak{R}_+)$ *is a C^{l-k-2} Banach manifold together with a Fredholm map*

$$\mathrm{Fib}(R_+, \mathfrak{R}_+) \xrightarrow{\pi} C\mathcal{M}(Y, \xi_0) \times \mathcal{P}.$$

Definition 4.14. The *universal* moduli space of Seiberg–Witten monopoles over $Z^+ = Z \cup K'$ associated to the triple $(\xi_0, \alpha_0, j_0) \in C\mathcal{M}(Y, \xi_0)$ is the Banach manifold

$$\mathfrak{M}([\mathfrak{a}], Z^+) = \mathrm{Fib}(R_+, \mathfrak{R}_-).$$

By $\mathfrak{M}(Z^+)$ we denote the union over all critical points $[\mathfrak{a}]$ of the $\mathfrak{M}([\mathfrak{a}], Z^+)$.

Remark 4.8. By a standard argument (see [49], Lemma 24.2.2 and Lemma 19.1.1) one can see that any element in $\mathfrak{M}([\mathfrak{a}], Z^+) = \mathrm{Fib}(R_+, \mathfrak{R}_-)$ is represented by a solution $\gamma = (A, \Phi, t)$ to the Seiberg–Witten equations over the whole Z^+ (modulo gauge transformations v with $1 - v \in L^2_{k+1,g_t}$ on both ends of Z^+) such that

$$\begin{aligned} \gamma - (A_t, \Phi_t) &\in L^2_k(K) \\ \gamma - \gamma_{w\cdot\mathfrak{a}} &\in L^2_k(Z) \end{aligned}$$

where $w \in \mathcal{G}_{k+1/2}(Y)$, $\mathfrak{a}$ is a critical point of the $\mathfrak{q}$-perturbed Chern-Simons-Dirac functional, $\gamma_{w\cdot\mathfrak{a}}$ is the translation-invariant solution over the cylindrical end Z determined by $w \cdot \mathfrak{a}$, and (A_t, Φ_t) is a canonical configuration over K (in translation-invariant form).

4.2.3.3 Components of the moduli space of constant index

As with the moduli spaces that are studied in [49], the index of π will vary with the connected component of $\mathfrak{M}([\mathfrak{a}], Z^+)$. We give a more precise statement of this fact, following the ideas of

§24.4 in [49]. Denote by $\mathcal{B}_k([\mathfrak{a}], K')$ the preimage of $[\mathfrak{a}]$ under the partially defined restriction map to the slice $0 \times Y$

$$R_- : \mathcal{B}_k(K') \dashrightarrow \mathcal{B}^{\sigma}_{k-1/2}(Y, \mathfrak{s}_{\xi_0,\alpha_0,j_0}). \tag{4.12}$$

Any element of $\mathfrak{M}([\mathfrak{a}], Z^+)$ is, by definition, given by a quadruple $([\gamma_Z], [\gamma_{K'}], t, \mathfrak{p})$ with $[\gamma_Z|_Y] = [\gamma_{K'}|_Y]$. The cylinder configuration $[\gamma_Z]$ provides a path in $\mathcal{B}_k(K')$ (canonical up to homotopy) from $[\gamma_{K'}] \in \mathcal{B}_k([\mathfrak{a}], K')$ to the subspace $\mathcal{B}_k([\mathfrak{a}], K') \subset \mathcal{B}_k([\mathfrak{a}], K')$. Hence, each element of $\mathfrak{M}([\mathfrak{a}], Z^+)$ determines a connected component of $\mathcal{B}_k([\mathfrak{a}], K')$, giving a map

$$\pi_0\mathfrak{M}([\mathfrak{a}], Z^+) \to \pi_0\mathcal{B}_k([\mathfrak{a}], K'). \tag{4.13}$$

By the homotopy invariance of the index of a Fredholm operator we have:

Proposition 4.9. *The index of $\pi : \mathfrak{M}([\mathfrak{a}], Z^+) \to \mathcal{CM}(Y, \xi_0) \times \mathcal{P}$ is constant on the fibres of (4.13).*

Next we provide further information on $\pi_0\mathcal{B}_k([\mathfrak{a}]\ , K')$. Consider the natural projection $p : \mathcal{B}_k([\mathfrak{a}], K') \to \mathcal{CM}(K, \xi_0)$, and denote the fibre over a point t by $\mathcal{B}_k([\mathfrak{a}], Z^+)_t$.

Lemma 4.10. *(i) there is a bijection $\pi_0\mathcal{B}_k([\mathfrak{a}], K')_t \approx H^1(Y; \mathbb{Z})$*

(ii) p is a Serre fibration

(iii) the map $\pi_0\mathcal{B}_k([\mathfrak{a}], K')_t \to \pi_0\mathcal{B}_k([\mathfrak{a}], K')$ induced by inclusion is surjective.

Proof. For (i), we fix a canonical configuration $\gamma_t := (A_t, \Phi_t)$ at t, and fix a representative $\mathfrak{a}$ of $[\mathfrak{a}]$. We consider the space $C_k(\mathfrak{a}, K', \gamma_t)$ which is the fibre of the partially-defined restriction map $C_k(K', \gamma_t) \dashrightarrow C^{\sigma}_{k-1/2}(Y, \mathfrak{s}_{\xi_0,\alpha_0,j_0})$ over $\mathfrak{a}$. We choose a representative v_z from every connected component $z \in \pi_0\mathcal{G}_{k+1/2}(Y)$. Then we have a decomposition into disjoint closed subspaces

$$\mathcal{B}_k([\mathfrak{a}], K')_t = \bigcup_{z \in \pi_0\mathcal{G}_{k+1/2}(Y)} C_k(v_z \cdot \mathfrak{a}, K', \gamma_t)/\mathcal{G}_{k+1}(K', Y)$$

where $\mathcal{G}_{k+1}(K', Y)$ stands for the subgroup of $\mathcal{G}_{k+1}(K')$ consisting of gauge transformations which are the identity over $\{0\} \times Y = \partial K'$. Note that each of the disjoint subspaces above is connected (because $\mathcal{G}_{k+1}(K')$ is connected). This sets up a bijection $\pi_0 \mathcal{B}_k([\mathfrak{a}], K')_t \cong \pi_0 \mathcal{G}_{k+1/2}(Y)$ only depending on the representative $\mathfrak{a}$. Finally, the latter set is identified with the group $H^1(Y; \mathbb{Z}) = [Y, S^1]$.

Part (iii) follows from (ii) and the connectedness of the base of the Serre fibration p. For (ii) we must show: if $h : D \times [0, 1] \to \mathcal{CM}(Y, \xi_0)$ is any given homotopy, where D is a compact disc, and we are given a lift of h over $D \times 0$ to $\mathcal{B}_k([\mathfrak{a}], K')$, then there exists a lift of h over $D \times [0, 1]$ agreeing with the given one over $D \times \{0\}$. It suffices to consider the fibration p' obtained by pullback of p along h

$$p' : h^* \mathcal{B}_k([\mathfrak{a}], K') \to D \times [0, 1]$$

and construct a section of p' over $D \times [0, 1]$ which extends the given one over $D \times \{0\}$. This has the advantage that $D \times [0, 1]$ is contractible, hence we can find a family of canonical configurations $(A_{t,\tau}, \Phi_{t,\tau})$ carried by $(t, \tau) \in D \times [0, 1]$. The given section over $D \times \{0\}$ can be represented by a family of configurations of the form

$$\gamma_t = (A_{t,0}, \Phi_{t,0}) + (a_t, \phi_t)$$

such that $\gamma_t|_{\{0\} \times Y} = \mathfrak{a}$ for some representative $\mathfrak{a}$ of $[\mathfrak{a}]$, and $(a_t, \phi_t) \in L^2_k$. Let $\mathfrak{b}_{t,\tau}$ be the restriction to $\{0\} \times Y$ of $(A_{t,\tau}, \Phi_{t,\tau}) + (a_t, \phi_t)$. We extend the section γ_t over to $D \times [0, 1]$ by setting

$$\gamma_{t,\tau} = (A_{t,\tau}, \Phi_{t,\tau}) + (a_t, \phi_t) + \beta \cdot (\gamma_{\mathfrak{a}} - \gamma_{\mathfrak{b}_{t,\tau}})$$

where β is a smooth bump function $\beta : [0, +\infty) \to \mathbb{R}_{\geq 0}$ which is identically 1 on a small neighbourhood of 0 and vanishes outside of a compact set, and $\gamma_{\mathfrak{a}}$ stands for the translation-invariant configuration on the cylinder $\mathbb{R} \times Y$ associated to $\mathfrak{a}$ (and likewise for $\gamma_{\mathfrak{b}_{t,\tau}}$). The proof is now complete. □

Thus, by the previous results we can decompose $\mathfrak{M}([\mathfrak{a}], Z^+)$ into pieces where π has constant index

$$\mathfrak{M}([\mathfrak{a}], Z^+) = \bigcup_z \mathfrak{M}_z([\mathfrak{a}], Z^+).$$

which are parametrised by the connected components $z \in \pi_0 \mathcal{B}_k([\mathfrak{a}], K')$. Note that it does *not* hold necessarily that each $\mathfrak{M}_z([\mathfrak{a}], Z^+)$ is connected.

Remark 4.9. The map from Lemma 4.10(iii) is not injective, in general. More precisely, it is injective if and only if any loop (i.e. S^1-family) in $\mathcal{CM}(Y, \xi_0)$ has a corresponding S^1-family of canonical configurations over K.

4.2.3.4 Orientability

In order to define the families invariant when the coefficient ring R is not of characteristic 2, we will need to orient the Seiberg–Witten moduli spaces. In order to do so we need to orient the *determinant line bundle* of the Fredholm map

$$\mathfrak{M}(Z^+) = \bigcup_{[\mathfrak{a}]} \mathfrak{M}([\mathfrak{a}]\ , Z^+) \xrightarrow{\pi} \mathcal{CM}(Y, \xi_0) \times \mathcal{P}.$$

For the precise construction of this real line bundle $\det\pi \to \mathfrak{M}(Z^+)$ we refer to [[49] , §20.2]. Its fibre over a given $m \in \mathfrak{M}(Z^+)$ can be identified as

$$(\det\pi)_m = \Lambda^{\max}\ker(d\pi)_m \otimes \Lambda^{\max}(\operatorname{coker}(d\pi)_m)^*.$$

We now describe what goes into orienting the determinant line bundle.

The first ingredient is to orient the moduli spaces of trajectories in monopole Floer homology. This is formally analogous to the finite-dimensional Morse theory case. Given a critical point $[\mathfrak{a}] \in \mathcal{B}^\sigma(Y, \mathfrak{s})$ in the blowup, a 2-element set $\Lambda([\mathfrak{a}])$ is associated in [[49], §20.3], playing the role of the set of orientations for the unstable manifold of $[\mathfrak{a}]$ in the Morse theory picture.

The second ingredient is the following construction of a double covering Λ of $C(Y,\xi_0)$, in the spirit of [[49], §24.8]. Let $\gamma_0 = (t, A, \Phi) \in \mathcal{B}_k(K')$ be a configuration which restricts along $\{0\}\times Y$ onto a *reducible* configuration $[\mathfrak{a}_0] \in \mathcal{B}^\sigma_{k-1/2}(Y, \mathfrak{s}_{\xi_0,\alpha_0,j_0})$. In other words, the configuration γ_0 has image $[\mathfrak{a}_0]$ under the partially-defined map from (4.12)

$$R_- : \mathcal{B}_k(K') \dashrightarrow \mathcal{B}^\sigma_{k-1/2}(Y, \mathfrak{s}_{\xi_0,\alpha_0,j_0}).$$

The vector bundle bundle Υ_{k-1} on the right-hand side of (4.11) descends onto a vector bundle denoted $[\Upsilon_{k-1}]$ over $\mathcal{B}_k(K')$, and the (unperturbed) Seiberg-Witten map gives a section of this bundle

$$\mathrm{sw} : \mathcal{B}_k(K') \to [\Upsilon_{k-1}].$$

Consider also the map

$$\pi_{[\mathfrak{a}_0]} : T_{[\mathfrak{a}_0]}\mathcal{B}^\sigma_{k-1/2}(Y, \mathfrak{s}_{\xi_0,\alpha_0,j_0}) \to \mathcal{K}^+_{[\mathfrak{a}_0]}$$

given by orthogonal projection onto the subspace $\mathcal{K}^+_{[\mathfrak{a}]}$ which is defined as the closure of the span of the *non-negative* eigenvectors of the (unperturbed) Hessian $\mathrm{Hess}^\sigma : T_{[\mathfrak{a}_0]}\mathcal{B}^\sigma_k(Y, \mathfrak{s}_{\xi_0,\alpha_0,j_0}) \to T_{[\mathfrak{a}_0]}\mathcal{B}^\sigma_{k-1}(Y, \mathfrak{s}_{\xi_0,\alpha_0,j_0})$ of the Chern–Simons–Dirac functional. From these we assemble the Seiberg-Witten map with *Atiyah–Patodi–Singer boundary condition*

$$P_{\gamma_0} = (\mathcal{D}\mathrm{sw})_{\gamma_0} \oplus \pi_{[\mathfrak{a}_0]} \circ (dR_-)_{\gamma_0} : T_{\gamma_0}\mathcal{B}_k(K') \to [\Upsilon_{k-1}]_{\gamma_0} \times \mathcal{K}^+_{[\mathfrak{a}_0]}$$

where $(\mathcal{D}\mathrm{sw})_{\gamma_0}$ stands for the vertical component of the derivative of the section sw at γ_0 (taken with respect to the natural connection that the vector bundle $[\Upsilon_{k-1}]$ carries). As in [49], the Atiyah–Patodi–Singer theory establishes the Fredholm property of the linear operator P_{γ_0}.

Definition 4.15. We define a double covering Λ of $C(Y,\xi_0)$ with fibers $\Lambda(\xi)$ as follows. For a given $\xi \in C(Y,\xi_0)$ choose any configuration $\gamma_0 = (t, A, \Phi) \in \mathcal{B}_k(K')$ lying over ξ (i.e. $t = (\xi, \alpha, j)$ for some α and j) and which restricts onto a *reducible* configuration $[\mathfrak{a}_0] \in \mathcal{B}^\sigma_{k-1/2}(Y, \mathfrak{s}_{\xi_0,\alpha_0,j_0})$ along $\{0\}\times Y$. We define $\Lambda(\xi)$ to be the two-element set of orientations of $\det(P_{\gamma_0})$.

Remark 4.10. The two-element set $\Lambda(\xi)$ is independent of the choice of γ_0 or $[\mathfrak{a}_0]$, up to canonical bijection. Furthermore, it is also independent of our chosen base configuration (ξ_0, α_0, j_0), up to canonical bijection. These assertions all follow from [[49], Lemma 20.3.3] and standard homotopy arguments as those found in [[49], §20.3 and §24.8].

Associated to our double cover Λ there is a local system $\Lambda_{\mathbb{Z}}$ whose fibers are $\mathbb{Z}$-modules of rank 1. Explicitly, we can take the fiber $\Lambda_{\mathbb{Z}}(\xi)$ to be the quotient of the free $\mathbb{Z}$-module on the two-element set $\Lambda(\xi) = \{\mathfrak{o}, \mathfrak{o}'\}$ by the submodule generated by the element $\mathfrak{o} + \mathfrak{o}'$, and the monodromy action of paths is inherited from that on Λ. We write Λ_R for the local system of free R-modules of rank 1 obtained by taking the tensor product $\Lambda_{\mathbb{Z}} \otimes_{\mathbb{Z}} R$.

The proof of the next result follows the same argument as in §24.8 of [49]

Proposition 4.11. *Given a choice of an element in each orientation set $\Lambda([\mathfrak{a}])$ for each critical point $[\mathfrak{a}]$, there is a canonical homotopy class of isomorphism of real line bundles* $\det\pi \cong \pi^*\Lambda_{\mathbb{R}}$ *over $\mathfrak{M}(Z^+)$. Here* $\det\pi$ *is the determinant line bundle of $\pi : \mathfrak{M}(Z^+) \to \mathcal{CM}(Y, \xi_0) \times \mathcal{P}$, and $\pi^*\Lambda_{\mathbb{R}}$ is the pullback of $\Lambda_{\mathbb{R}}$ by $\pi : \mathfrak{M}(Z^+) \to \mathcal{CM}(Y, \xi_0) \times \mathcal{P} \simeq \mathcal{C}(Y, \xi_0)$.*

We explain how this orients the moduli spaces that will be relevant. Consider a C^2 map $\sigma : \Delta^n \to \mathcal{CM}(Y, \xi_0) \times \mathcal{P}$ from the standard n-simplex $\Delta^n = \{x \in (\mathbb{R}_{\geq 0})^{n+1} \mid \sum_{i=1}^{n+1} x_i = 1\}$. We equip Δ^n with its canonical orientation. Suppose σ is transverse to $\pi : \mathfrak{M}_z(Z^+) \to \mathcal{CM}(Y, \xi_0) \times \mathcal{P}$ along each stratum of Δ^n. Then we obtain a C^2 manifold $M_z([\mathfrak{a}], \sigma) = \mathrm{Fib}(\pi, \sigma)$ as the fibre product of $\pi : \mathfrak{M}_z([\mathfrak{a}], Z^+) \to \mathcal{CM}(Y, \xi_0) \times \mathcal{P}$ with σ, which is of dimension $\mathrm{ind}\pi + n$, where $\mathrm{ind}\pi$ is computed over the component $\mathfrak{M}_z([\mathfrak{a}], Z^+)$.

If choices in each $\Lambda([\mathfrak{a}])$ are made and we are given an orientation in $\Lambda(\sigma(b))$, where b stands for the barycenter of Δ^n, then Proposition 4.11 picks out preferred orientations of all the moduli spaces $M_z([\mathfrak{a}], T)$. This is is a matter of linear algebra:

Lemma 4.12. *Consider transverse linear maps $M \xrightarrow{\pi} C \xleftarrow{\sigma} \Delta$ of Banach spaces, with π Fredholm and Δ of finite dimension. Let $F = \pi - \sigma : M \oplus \Delta \to C$. Then F is Fredholm and there is a* *canonical isomorphism*

$$\det F \cong \det\pi \otimes \Lambda^{\max}\Delta.$$

Proof. Because of the transversality assumption, one has the canonical isomorphism (see the construction of [[49],§20.2], and put $J = \mathrm{Im}\sigma$)

$$\det\pi \cong \Lambda^{\max}\pi^{-1}(\mathrm{Im}\sigma) \otimes \left(\Lambda^{\max}\mathrm{Im}\sigma\right)^{*}.$$

Then the short exact sequences

$$0 \to \mathrm{Ker}\sigma \to \mathrm{Ker}F \to \pi^{-1}(\mathrm{Im}\sigma) \to 0$$
$$0 \to \mathrm{Ker}\sigma \to \Delta \to \mathrm{Im}\sigma \to 0$$

provide us with canonical isomorphisms

$$\begin{aligned}\Lambda^{\max}\pi^{-1}(\mathrm{Im}\sigma) &\cong \Lambda^{\max}\mathrm{Ker}F \otimes \left(\Lambda^{\max}\mathrm{Ker}\sigma\right)^{*}\\ &\cong \Lambda^{\max}\mathrm{Ker}F \otimes \left(\Lambda^{\max}\Delta\right)^{*} \otimes \Lambda^{\max}\mathrm{Im}\sigma.\end{aligned}$$

This says $\det\pi \cong \det F \otimes \left(\Lambda^{\max}N\right)^{*}$. □

More precisely, one orients $M_z([\mathfrak{a}], \sigma)$ by following the proof of Lemma 4.12 above using the *fibre-first convention* for orienting vector spaces in a short exact sequence. This agrees with orientation conventions in [49] (see p.525) for parametrised moduli spaces over an oriented manifold.

We refer to this as the *canonical orientation* of $M_z([\mathfrak{a}], \sigma)$ (depending on the choices of elements in $\Lambda([\mathfrak{a}])$ and $\Lambda(\sigma(b))$). Whenever these moduli are 0-dimensional and we use them to make counts of points, each point is counted with a sign corresponding to its canonical orientation (relative to the natural orientation of a point).

4.2.4 The families contact invariant

We describe now the construction of the homomorphism (1.4). We will write C for the Banach manifold $C\mathcal{M}(Y,\xi_0)\times\mathcal{P}$ for ease in notation. This space has the weak homotopy type of the space of contact structures $C(Y,\xi_0)$.

We fix orientations in $\Lambda([\mathfrak{a}])$ for all critical points $[\mathfrak{a}]$. We fix a ring R (commutative, unital).

4.2.4.1 Transverse singular chains

Let $M \xrightarrow{\pi} C$ be a C^r Fredholm map of C^r Banach manifolds. We assume that C is connected but allow M disconnected, with at most countably many components. The index of π, $\mathrm{ind}\pi \in \mathbb{Z}$, depends on the chosen connected component of M.

Below we view the standard n-simplex $\Delta^n = \{x \in (\mathbb{R}_{\geq 0})^{n+1} \mid \sum_{i=1}^{n+1} x_i = 1\}$ as a manifold with corners, and by a C^r map with domain Δ^n we mean a map which extends to a C^r map on an open neighbourhood of $\Delta^n \subset \mathbb{R}^{n+1}$.

Definition 4.16. A C^r singular n-simplex $\sigma : \Delta^n \to C$ is *transverse* to π if the restriction of σ to each stratum (i.e. face) of the n-simplex Δ^n is transverse to π. In particular, the image of each vertex of Δ^n under σ is a regular value of π.

For our purposes it suffices to take $r = 2$. Next we set up a version of the complex of singular chains on C with coefficients in the local system Λ_R, made up of transverse chains

Definition 4.17. Let $(S^{\pi}_{*}(C;\Lambda_R),\partial)$ be the chain complex over R given by finite formal sums

$$\sum a\cdot\sigma$$

where σ is a C^2 singular simplex $\sigma : \Delta^n \to C$ (with $n \geq 0$) which is *transverse* to π along components of M with $\mathrm{ind}\pi \leq 1-n$; and a is an element of the ring $\Lambda_R(\sigma(b))$, where $b \in \Delta^n$ is the barycenter of Δ^n. The differential ∂ is the singular differential coupled to the isomorphism

$\Lambda(\sigma(b)) \to \Lambda(\sigma_i(b_i))$ associated with the straight line segment from b to b_i, where σ_i denotes the restriction of σ to the ith codimension 1 face Δ^n_i of Δ^n and b_i the barycenter of Δ^n_i.

Remark 4.11. For ease in notation, whenever we refer to a singular n-simplex $\sigma : \Delta^n \to C$ we will assume it is equipped with an element in $\Lambda(\sigma(b))$, and regard instead the coefficient a as an element in the ring R.

The restriction to components of M with $\mathrm{ind}\pi \leq 1 - n$ is imposed on us by the Thom-Smale transversality theorem [74]. This result states that for C^r maps ($r \geq 1$) of C^r Banach manifolds $X \xrightarrow{f} Y \xleftarrow{g} Z$ with $\dim X = n < +\infty$ and g Fredholm, one can always C^r-approximate f by a map f' which is transverse to g, provided that $r > \max(\mathrm{ind} g + n, 0)$. Furthermore, if f was already transverse to g along a closed subset $X' \subset X$ then one can choose f' to agree with f along X'. Then, by the Thom-Smale transversality theorem we learn that the inclusion of $S^\pi_*(C; \Lambda_R)$ into the chain complex of (continuous) singular chains on C with coefficients in the local system Λ_R induces a quasi-isomorphism, so that $(S^\pi_*(C; \Lambda_R), \partial)$ computes the singular homology $H_*(C; \Lambda_R) \cong H_*(C(Y, \xi_0); \Lambda_R)$.

4.2.4.2 *Counting solutions to the Seiberg–Witten equations*

Consider a C^2 singular n-simplex $\sigma : \Delta^n \to C$ satisfying the transversality condition of Definition 4.17 with respect to the Fredholm map $\pi : \mathfrak{M}(Z^+) \to C$ (of regularity $C^{l-k-2} \subset C^2$). For such σ and each pair $([\mathfrak{a}], z)$ we have the space $M_z([\mathfrak{a}], \sigma)$ consisting of solutions of the Seiberg–Witten equations over the singular simplex σ. Namely, $M_z([\mathfrak{a}], \sigma) = \mathrm{Fib}(\pi, \sigma)$ is the fibre product of $\pi : \mathfrak{M}_z([\mathfrak{a}], Z^+) \to C$ and σ. Whenever the expected dimension of $M_z([\mathfrak{a}], \sigma)$ is ≤ 1, i.e. $\mathrm{ind}\pi \leq 1 - n$, we can guarantee that this fibre product is transverse, and hence that $M_z([\mathfrak{a}], \sigma)$ will be a C^2-manifold with corners. We denote by $\# M_z([\mathfrak{a}], \sigma)$ the count of points in the discrete (0-dimensional) moduli space $M_z([\mathfrak{a}], \sigma)$ when $\mathrm{ind}\pi = -n$, counted with the signs corresponding to their canonical orientation (see §4.2.3.4); and we set $\# M_z([\mathfrak{a}], \sigma) = 0$ if $\mathrm{ind}\pi \neq -n$. The possibility to make such count relies on the fact that the 0-dimensional moduli spaces $M_z([\mathfrak{a}], \sigma)$ are indeed finite, which we will address momentarily.

We can now assemble the counts of solutions to the Seiberg–Witten equations into a homomorphism of R-modules

$$\begin{aligned} \psi : S_*(C;\Lambda_R) &\to \widehat{C}^*(Y, \mathfrak{s}_{\xi_0,\alpha_0,j_0}; R) \\ \sigma &\mapsto \mathfrak{M}(Z^+)\cdot\sigma := \sum_{[\mathfrak{a}],z} \left(\# M_z([\mathfrak{a}],\sigma)\right)\cdot[\mathfrak{a}] \ . \end{aligned} \tag{4.14}$$

The right side of (4.14) is the monopole Floer *cochain* complex of Y (in the *from* version), obtained by taking the dual of the monopole Floer chain complex $\widehat{C}_*(Y, \mathfrak{s}_{\xi_0,\alpha_0,j_0}; R)$ with differential $\widehat{\partial}$. The latter complex is constructed from the spin-c structure $\mathfrak{s}_{\xi_0,\alpha_0,j_0}$ and admissible perturbation $\mathfrak{q}$. It is freely generated over R by the union of the sets $\mathfrak{C}^o$, $\mathfrak{C}^u$ of irreducible and unstable critical points, which gives a decomposition $\widehat{C}_*(Y, \mathfrak{s}_{\xi_0,\alpha_0,j_0}) = C^o_* \oplus C^u_*$. The Floer differential is given by the following matrix (see [49], Definition 22.1.3)

$$\widehat{\partial} = \begin{pmatrix} \partial^o_o & \partial^u_o \\ -\overline{\partial}^s_u\partial^o_s & -\overline{\partial}^u_u - \overline{\partial}^s_u\partial^u_s \end{pmatrix}. \tag{4.15}$$

Remark 4.12. For the expression (4.14) to be well-defined, we require the fact that there are only finitely many pairs $([\mathfrak{a}], z)$ for which $M_z([\mathfrak{a}],\sigma)$ is of dimension 0 and non-empty. This can be shown following the standard arguments in [49], and we defer a discussion of this fact to §A.2.

Proposition 4.13. *Up to signs, ψ is a chain map. Precisely, $\psi(\partial\sigma) = (-1)^n\widehat{\partial}^*\psi(\sigma)$, where σ is a singular n-simplex.*

To see this, we first make some remarks on the compactness properties of the moduli spaces $M_z([\mathfrak{a}],\sigma)$. We restrict to the case of the moduli spaces of expected dimension ≤ 1, since for the higher dimensional ones we cannot guarantee that they are transversely cut out. The $M_z([\mathfrak{a}],\sigma)$ are, in general, non-compact manifolds with corners. However, we have

Proposition 4.14. *The 0-dimensional moduli spaces $M_z([\mathfrak{a}],\sigma)$ consist of finitely-many points. The 1-dimensional moduli spaces $M_z([\mathfrak{a}],\sigma)$ admit a compactification into a space $M^+_z([\mathfrak{a}],\sigma)$*

stratified by manifolds. The top stratum consists of $M_z([\mathfrak{a}], \sigma)$ itself, and the boundary of the top stratum consists of "broken" configurations of the form

(a) $\breve{M}_{z_1}([\mathfrak{a}], [\mathfrak{b}]) \times M_{z_0}([\mathfrak{b}], \sigma)$

(b) $\breve{M}_{z_2}([\mathfrak{a}], [\mathfrak{b}]) \times \breve{M}_{z_1}([\mathfrak{b}], [\mathfrak{c}]) \times M_{z_0}([\mathfrak{c}], \sigma)$

where the middle factor in (b) is boundary-obstructed; together with configurations arising from the boundary stratum of Δ^n, which is the union of the $(n-1)$-simplices $\Delta_0^{n-1}, \Delta_1^{n-1}, \ldots, \Delta_n^{n-1}$ that are codimension-1 faces of Δ^n:

(c) $\bigcup_{i=0,n} M_z([\mathfrak{a}], \sigma_{|\Delta_i^{n-1}})$.

For each boundary stratum above, the homotopy classes must concatenate to z (e.g. for (a) we need $z_1 \circ z_0 = z$). Furthermore, the structure near each boundary stratum is: C^0 manifold-with-boundary structure at (a); a codimension-1 δ-structure (a more general structure than C^0 manifold-with-boundary, see [[49], Definition 19.5.3]) at (b); and a C^2 manifold-with-boundary structure at (c).

All the analysis required to deduce these results is provided by the techniques in [49], [47], [81]. We discuss in §A.2 some technical results that are involved.

Proof of Proposition 4.13. In general, for any singular simplex σ transverse to $\pi : \mathfrak{M}_z([\mathfrak{a}], Z^+) \to C$, one can construct a compactification $M_z^+([\mathfrak{a}], \sigma)$ of $M_z([\mathfrak{a}], \sigma)$ by adding broken configurations as in [49]. In the case when $M_z([\mathfrak{a}], \sigma)$ is transversely cut out and of dimension 0, it follows from index reasons that no broken configurations are added, and so the moduli consists of finitely-many points. In the case where the dimension of $M_z([\mathfrak{a}], \sigma)$ is 1, the corresponding compactification $M_z^+([\mathfrak{a}], \sigma)$ is a 1-dimensional stratified space with a codimension-1 δ-structure along its boundary. Such a space enjoys the nice property that the enumeration of its boundary points gives total count zero [[49], Corollary 21.3.2]. Thus, enumerating the boundary points, of types (a), (b) and (c) as above, yields corresponding identities

$$\langle\psi(\sigma)^o, \partial^o_o[\mathfrak{a}]\rangle - \langle\psi(\sigma)^u, \overline{\partial}^s_u\partial^o_s[\mathfrak{a}]\rangle + (-1)^{n-1}\langle\psi(\partial\sigma)^o, [\mathfrak{a}]\rangle = 0\,, \quad \forall[\mathfrak{a}] \in \mathfrak{C}^o$$
$$\langle\psi(\sigma)^o, \partial^u_o[\mathfrak{a}]\rangle + \langle\psi(\sigma)^u, \overline{\partial}^u_u[\mathfrak{a}]\rangle - \langle\psi(\sigma)^u, \overline{\partial}^s_u\partial^u[\mathfrak{a}]\rangle + (-1)^{n-1}\langle\psi(\partial\sigma)^u, [\mathfrak{a}]\rangle = 0\,, \quad \forall[\mathfrak{a}] \in \mathfrak{C}^u$$

which give the required equality $\psi(\partial\sigma) = (-1)^n\widehat{\partial}^*\psi(\sigma)$. For the origin of the signs see Lemma A.14 [2] in §A.2. □

Definition 4.18. The *families contact invariant* of (Y, ξ_0) is the homomorphism induced by the chain map ψ

$$\mathbf{Fc} := \psi_* : H_*(C(Y,\xi_0);\Lambda_R) \cong H_*(C;\Lambda_R) \to \widehat{HM}^*(Y, \mathfrak{s}_{\xi_0}; R). \tag{4.16}$$

Some observations are relevant now:

Remark 4.13. (i) *Invariant for a single contact structure*. Fixing an element of the 2-element set $\Lambda(\xi_0)$ fixes the sign of the contact invariant $\mathbf{c}(\xi_0)$ of Kronheimer–Mrowka-Ozsváth-Szabó [46]. In turn, this also picks out a canonical generator $1 \in H_0(C(Y,\xi_0);\Lambda_R)(= R$ or $R/2R$ accoding as to whether the local system Λ is trivial or not). It is clear from our construction that $\mathbf{c}(\xi_0)$ agrees with $\mathbf{Fc}(1)$. Part (A) of Theorem 1.5 is then proved.

(ii) *Gradings*. With respect to the natural grading of the Floer cohomology groups by the set of homotopy classes of oriented 2-plane fields, the map ψ defining (1.4) has the form

$$\psi : S^\pi_n(C(Y,\xi_0);\Lambda_R) \to \widehat{C}^{[\xi_0]-n}(Y, \mathfrak{s}_{\xi_0}; R), \quad n \geq 0.$$

For $n = 0$, i.e. for the contact invariant $\mathbf{c}(\xi_0)$, a proof of this fact can be found in [[16], §7.1]. For higher $n \geq 0$ the statement follows in a straightforward way from the $n = 0$ case and the

[2] A rather technical point is that the sign of $+\langle\psi(\sigma)^u, \overline{\partial}^u_u[\mathfrak{a}]\rangle$ written above should be flipped if one follows the *reducible convention* for orienting the moduli $M_{z_1}([\mathfrak{a}],[\mathfrak{b}])$ when both $[\mathfrak{a}],[\mathfrak{b}]$ are boundary-unstable (see §20.6 [49]). This reducible convention is meant when writing the term $-\overline{\partial}^u_u$ in the Floer differential (4.15). The signs listed in Lemma A.14 follow the usual convention. These two conventions differ by the sign $(-1)^{\dim M_{z_1}([\mathfrak{a}],[\mathfrak{b}])} = -1$.

identity of expected dimensions $\dim M_z([\mathfrak{a}], \sigma) = n + \dim M_z([\mathfrak{a}], *)$, with $\sigma : \Delta^n \to C$ an n-simplex and $* : \{*\} \to C$ the inclusion of a point.

(iii) *Criterion for triviality of* Λ. It is unclear to the author whether the double cover Λ can be non-trivial in general. However, under the assumption that the contact invariant $\mathbf{c}(\xi_0)$ with $R = \mathbb{Z}$ coefficients is *not* a 2-torsion element, then we can conclude that Λ is trivial (Corollary 1.6). This criterion applies in many cases of interest, e.g. whenever the contact structure admits a strong symplectic filling.

(iv) *Sign-ambiguity*. Even when the double cover Λ of $C(Y, \xi_0)$ is trivial, there is no canonical choice in the 2-element set $\Lambda(\xi_0)$. In fact, Lin–Ruberman–Saveliev [53] have shown that one *cannot* associate canonically an element in $\Lambda(\xi_0)$ to each isotopy class of a contact structure ξ_0 in such a way that the contact invariant $\mathbf{c}(\xi_0)$ is natural with respect to orientation-preserving diffeomorphisms of Y. This is done by showing that the unique tight contact structure on $-\Sigma(2,3,7)$ admits a contactomorphism which reverses the sign of $\mathbf{c}(\xi_0)$. We also note that the local system Λ is trivial, because this contact structure is strongly (and in fact Stein) fillable.

(v) *Invariance*. The construction of (4.16) involved choices. The main ones were the base contact structure ξ_0 together with a base triple $(\xi_0, \alpha_0, j_0) \in C\mathcal{M}(Y, \xi_0)$ and an admissible perturbation $\mathfrak{q} \in \mathcal{P}$. The remaining ones were rather inessential choices of cutoff functions (§4.2.1.1 , §4.2.3). Given two choices $(\xi_i, \alpha_i, j_i) \in C\mathcal{M}(Y, \xi_0)$, $i = 0, 1$, together with perturbations and cutoff functions that we omit from the notation, we obtain two corresponding chain maps

$$\psi_i : S_*^{\pi_i}(C; \Lambda_R) \to \widehat{C}^*(Y, \mathfrak{s}_{\xi_i, \alpha_i, j_i}; R), \quad i = 0, 1.$$

If we choose a generic path from (ξ_0, α_0, j_0) to (ξ_1, α_1, j_1) this yields a spin-c structure $\mathfrak{s}_W$ on $W = [0,1] \times Y$ and after a further choice of perturbations there is an associated cobordism map $\widehat{m}(W, \mathfrak{s}_W) : \widehat{C}^*(Y, \mathfrak{s}_{\xi_1, \alpha_1, j_1}; R) \to \widehat{C}^*(Y, \mathfrak{s}_{\xi_0, \alpha_0, j_0}; R)$ (see [49], p.518). We also define a subcomplex $S_* \subset S_*^{\pi_i}(C;\Lambda_R)$ of chains transverse to both π_0 and π_1 (in the same index range as before). The inclusion of this subcomplex is a quasi-isomorphism. Then one concludes by showing that the following diagram is homotopy-commutative, which is a standard argument

$$
\begin{array}{ccccc}
 & & S_*^{\pi_0}(C;\Lambda_R) & \xrightarrow{\psi_0} & \widehat{C}^*(Y,\mathfrak{s}_{\xi_0,\alpha_0,j_0};R) \\
 & \nearrow & & & \uparrow {\scriptstyle \widehat{m}(W,\mathfrak{s}_W)} \\
S_* & & & & \\
 & \searrow & & & \\
 & & S_*^{\pi_1}(C;\Lambda_R) & \xrightarrow{\psi_1} & \widehat{C}^*(Y,\mathfrak{s}_{\xi_1,\alpha_1,j_1};R)
\end{array}
$$

(vi) *Naturality*. The assertion on naturality from Theorem 1.5 (see the Remark after the aforementioned Theorem) readily follows from the construction from this section.

Chapter 5: The U map and families of contact structures

5.1 Module structures

In this section we define the module structures that Theorem 1.5 (B) refers to. We consider the graded ring

$$\mathbb{A}(Y;\mathbb{Z}) = \mathbb{Z}[U] \otimes_{\mathbb{Z}} \Lambda^*\big(H_1(Y;\mathbb{Z})/\text{torsion}\big)$$
$$|U| = 2, \quad |\gamma| = 1 \quad \gamma \in H_1(Y;\mathbb{Z})/\text{torsion}.$$

We write $\mathbb{A}^\dagger(Y;\mathbb{Z})$ for the opposite ring, with the opposite grading: $|U| = -2$, $|\gamma| = -1$ for $\gamma \in H_1(Y;\mathbb{Z})/\text{torsion}$. For a given (commutative, unital) ring R, we obtain graded R-algebras $\mathbb{A}(Y;R) := \mathbb{A}(Y;\mathbb{Z}) \otimes R$ and $\mathbb{A}^\dagger(Y;R) := \mathbb{A}^\dagger(Y;\mathbb{Z}) \otimes R$.

Remark 5.1. A different notation was used earlier, namely $\mathbb{A}(R) = \mathbb{A}^\dagger(Y;R)$ (see (1.2)).

The Floer cohomology groups $\widehat{HM}^*(Y,\mathfrak{s};R)$ carry a natural module structure over the graded R-algebra $\mathbb{A}(Y;R)$ [49]. In this section, we first give a chain level description of this module structure which is well-suited to our purposes. We make no claim of originality here, as the material presented here is surely known to the experts. Our approach is "dual" to that of [[49], §VII], and in a similar spirit to the construction of the U map given in [[46], §4.11]. Finally, we introduce the analogous $\mathbb{A}^\dagger(Y;R)$-module structure on $H_*(C(Y,\xi_0);\Lambda_R)$. The geometric interpretation of these algebraic structures that we provide in this section will be a key ingredient in the proof of Theorem 1.5 (B).

5.1.1 The module structure on $\widehat{HM}^*(Y, \mathfrak{s})$

Throughout this subsection, we fix a closed oriented 3-manifold Y, and a spin-c structure $\mathfrak{s} = (g, S, \rho)$ on Y. A construction reminiscent of the cup product pairing on the cohomology of $\mathcal{B}^\sigma(Y, \mathfrak{s})$ yields a pairing

$$H^k(\mathcal{B}^\sigma(Y, \mathfrak{s}); R) \otimes \widehat{HM}^*(Y, \mathfrak{s}; R) \xrightarrow{\cup} \widehat{HM}^{*+k}(Y, \mathfrak{s}; R). \tag{5.1}$$

The $\mathbb{A}(Y; R)$-module structure on $\widehat{HM}^*(Y, \mathfrak{s}; R)$ is then obtained from a canonical isomorphism $\mathbb{A}(Y; R) \cong H^*(\mathcal{B}^\sigma(Y, \mathfrak{s}); R)$. In what follows, our goal is to first describe this isomorphism (Proposition 5.6) and later define the pairing (5.1).

5.1.1.1 The cohomology ring of configuration space

We consider the blown-up configuration space $\mathcal{B}^\sigma(Y, \mathfrak{s})$ as in §4.2.2.3, where we have dropped the $k - 1/2$ subscript for ease in notation. Its homotopy type is that of $\mathbb{C}P^\infty \times T$, where T is a torus of dimension $b_1(Y) = \operatorname{rank} H_1(Y; \mathbb{Z})$. This fact is proved in [[49], §9.7]. Because we will use it later, we present here a short argument (in the same spirit) that proves a weaker statement.

Proposition 5.1 ([49]). *There is an isomorphism of graded algebras*

$$H^*(\mathcal{B}^\sigma(Y, \mathfrak{s}); \mathbb{Z}) \cong \mathbb{A}(Y; \mathbb{Z}).$$

Remark 5.2. The isomorphism given in the proof below is not canonical. We will obtain a canonical isomorphism in Proposition 5.6 using a different approach.

Proof. The inclusion of $\mathcal{B}^*(Y, \mathfrak{s})$ into the blown-up configuration space $\mathcal{B}^\sigma(Y, \mathfrak{s})$ induces a homotopy-equivalence, so we work with the former. We fix a spin-c connection B_0 on S. For another spin-c connection B we have the Hodge decomposition $B - B_0 = h + d\alpha + d^*\beta$ where h is harmonic. The projection $(B, \Psi) \mapsto h + d^*\beta$ induces a well-defined fibre bundle projection

$$\mathcal{B}^*(Y, \mathfrak{s}) \to \{a \in \Omega^1(Y; i\mathbb{R}) \mid d^* a = 0\}/\mathcal{G}^h(Y) \tag{5.2}$$

Here $\mathcal{G}^h(Y)$ stands for the group of harmonic maps $Y \to \mathrm{U}(1)$. The fiber of (5.2) is given by the projectivisation of the complex vector space of $L^2_{k-1/2}$ sections of S, which has the weak homotopy-type of $\mathbb{C}P^\infty$. By further projecting to the harmonic part, we obtain a homotopy equivalence of the base of (5.2) with the torus of harmonic 1-forms $\mathcal{H}^1(Y; i\mathbb{R})/\mathcal{H}^1(Y; 2\pi i\mathbb{Z})$, which is diffeomorphic to a torus T of rank $b_1(Y)$.

We next argue that this fibre bundle is cohomologically trivial, which completes the proof. The space $\mathcal{B}^*(Y, \mathfrak{s})$ is the base of a principal $\mathcal{G}(Y)$-bundle with weakly contractible total space $C^*(Y, \mathfrak{s}) \simeq *$ and so $\mathcal{B}^*(Y, \mathfrak{s})$ is a model for the classifying space $B\mathcal{G}(Y)$. The inclusion of the fibre agrees with the map on classifying spaces induced by the inclusion map $\mathrm{U}(1) \to \mathcal{G}(Y)$ by the constant gauge transformations. Now, the inclusion followed by evaluation at a fixed point $\mathrm{U}(1) \to \mathcal{G}(Y) \to \mathrm{U}(1)$ has degree 1, which shows that the inclusion of the fibre induces a surjective map on cohomology. This shows that the bundle is cohomologically trivial by the Theorem of Leray-Hirsch. □

5.1.1.2 The slant product construction

A standard construction [[15], §5] involving the *slant product*

$$\begin{aligned} \backslash : H_k(C_*) \otimes H^n((C_*)\check{} \otimes B^*) &\to H^{n-k}(B^*) \\ \alpha \otimes c &\mapsto \alpha \backslash c \end{aligned}$$

can be used to produce cohomology classes on $\mathcal{B}^\sigma(Y, \mathfrak{s})$ from homology classes in Y, by taking the slant product with characteristic classes of bundles over $Y \times \mathcal{B}^\sigma(Y, \mathfrak{s})$. We now describe this construction adapted to our setting.

Definition 5.1. The *canonical line bundle* $\mathcal{U}$ over $Y \times \mathcal{B}^\sigma(Y, \mathfrak{s})$ is constructed from the trivial complex line bundle $\mathbb{C} \times Y \times \mathcal{C}^\sigma(Y, \mathfrak{s})$ over $Y \times \mathcal{C}^\sigma(Y, \mathfrak{s})$ as follows: make this vector bundle into a $\mathcal{G}(Y)$-equivariant vector bundle by acting on the base in the standard way and on the total space by $v \cdot (\lambda, p, B, s, \Psi) := (v(p)\lambda, p, B - v^{-1}dv, s, v\Psi)$, where $v \in \mathcal{G}(Y)$. The bundle $\mathcal{U}$ is obtained by taking the quotient by the $\mathcal{G}(Y)$-action.

Definition 5.2. The *slant product map* is defined for $k = 0, 1$ by

$$\begin{aligned} \mu : H_k(Y; \mathbb{Z})/\text{torsion} &\to H^{2-k}(\mathcal{B}^\sigma(Y, \mathfrak{s}); \mathbb{Z}) \\ \alpha &\mapsto \alpha \backslash c_1(\mathcal{U}) \end{aligned} \tag{5.3}$$

Remark 5.3. Observe that the torsion in $H_k(Y; \mathbb{Z})$ is not in play in (5.3) because the cohomology of $\mathcal{B}^\sigma(Y, \mathfrak{s})$ has no torsion (Proposition 5.1).

Recall from §4.1.4 that there is a *universal* family of spin-c structures and irreducible configurations on Y parametrised by $\mathcal{B}^\sigma(Y, \mathfrak{s})$. We denote by $\mathbb{S} := \underline{S_\infty} \to Y \times \mathcal{B}^\sigma(Y, \ \mathfrak{s})$ the universal family of spinor bundles, which arises from the quotient by the natural action of $\mathcal{G}(Y)$ on the fibres and base of the bundle $\mathrm{pr}_1^* S \to Y \times \mathcal{C}^\sigma(Y, \mathfrak{s})$. We denote by $L := \Lambda^2 S \to Y$ the line bundle associated to the spin-c structure $\mathfrak{s}$ on Y. Of course, there is the splitting $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$, and we observe that

Lemma 5.2. *There is an isomorphism* $\Lambda^2 \mathbb{S}^+ \cong (\mathrm{pr}_1^* L) \otimes \mathcal{U}^{\otimes 2}$ *of* $\mathrm{U}(1)$*-bundles over* $Y \times \mathcal{B}^\sigma(Y)$.

Proof. We have

$$\mathbb{S}^+ = (\mathrm{pr}_1^* S^+)/\mathcal{G}(Y) \cong \big(\mathrm{pr}_1^*(S^+ \otimes \mathbb{C})\big)/\mathcal{G}(Y) \cong (\mathrm{pr}_1^* S^+) \otimes \big((\mathrm{pr}_1^* \mathbb{C})/\mathcal{G}(Y)\big) = (\mathrm{pr}_1^* S^+) \otimes \mathcal{U}$$

and thus $\Lambda^2 \mathbb{S}^+ \cong (\mathrm{pr}_1^* \Lambda^2 S^+) \otimes \mathcal{U}^{\otimes 2} = (\mathrm{pr}_1^* L) \otimes \mathcal{U}^{\otimes 2}$. □

Thus, since $\mathrm{pr}_1^* L$ is pulled back from Y, one could have defined μ in terms of the bundle $\mathbb{S}^+$ instead, as

$$\mu(\alpha) = \frac{1}{2}\big(\alpha \backslash c_1(\mathbb{S}^+)\big) = \frac{1}{2}\big(\alpha \backslash c_1(\Lambda^2 \mathbb{S}^+)\big). \tag{5.4}$$

Below we provide geometric interpretations for the maps $\mu : H_k(Y;\mathbb{Z})/\text{torsion} \to H^{2-k}(\mathcal{B}^\sigma(Y,\mathfrak{s});\mathbb{Z})$ for $k = 0, 1$. The ultimate goal in doing so is to describe the image of μ from a dual point of view.

5.1.1.3 The case $k = 0$

From (5.4) it is clear that $\mu(1)$ agrees with the first Chern class of the restriction of the bundle $\mathcal{U}$ to a slice $p \times \mathcal{B}^\sigma(Y,\mathfrak{s}) : \mu(1) = c_1(\mathcal{U}|_{p\times\mathcal{B}^\sigma(Y,\mathfrak{s})})$. This class can be understood from a dual point of view, which we do now.

Remark 5.4. Below, any inclusion $M \subset B$ of manifolds M and B with boundary is assumed to provide an inclusion of the boundaries $\partial M \subset \partial B$ as well.

Definition 5.3. Let B be a Banach manifold with boundary, and let $Z \subset B$ be a Banach submanifold with boundary $\partial Z \subset \partial B$ and which is of finite codimension and cooriented. We say that Z is *Poincaré dual* to a given cohomology class $c \in H^q(B;\mathbb{Z})$ if for any finite-dimensional compact oriented submanifold $M \subset B$ with boundary $\partial M \subset \partial B$ such that M is embedded transversely to Z (in the sense that $M \cap Z$, $\partial M \cap Z$ and $M \cap \partial Z$ are transverse intersections in the ambient B) then the oriented submanifold $M \cap Z \subset M$ is Poincaré dual to the cohomology class c restricted onto M. Namely,

$$\text{PD}(c_{|M}) = [M \cap Z, \partial(M \cap Z)] \in H_{\dim M - q}(M, \partial M;\mathbb{Z}).$$

Remark 5.5. Above, $M \cap Z$ is oriented in the standard way: by the exact sequence $0 \to TM \cap TZ \to TM \to TB/TZ \to 0$.

Going back to our case of interest, a section of $\mathcal{U}|_{p\times\mathcal{B}^\sigma(Y,\mathfrak{s})}$ is provided by a $\mathcal{G}(Y)$-equivariant map $f : \mathcal{C}^\sigma(Y,\mathfrak{s}) \to \mathbb{C}$, with $v \in \mathcal{G}(Y)$ acting on $\mathbb{C}$ by the element $v(p) \in \text{U}(1)$. A concrete example of such map can be obtained as follows: fix a unitary trivialisation of the fibre of S at the point $p \in Y$, denoted by $\tau = (\tau_1, \tau_2) : S_p \xrightarrow{\cong} \mathbb{C}^2$, and set $f_\tau(B, s, \Psi) = \tau_1\Psi(p)$. The section f_τ just constructed is transverse to the zero section of $\mathcal{U}|_{p\times\mathcal{B}^\sigma(Y,\mathfrak{s})}$. We obtain:

Lemma 5.3. *The oriented submanifold with boundary $Z_\tau := f_\tau^{-1}(0) \subset \mathcal{B}^\sigma(Y,\mathfrak{s})$ is Poincaré dual to $\mu(1) \in H^2(\mathcal{B}^\sigma(Y,\mathfrak{s});\mathbb{Z})$.*

5.1.1.4 The case $k = 1$

The dual interpretation of the classes $\mu([\gamma]) \in H^1(\mathcal{B}^\sigma(Y,\mathfrak{s});\mathbb{Z})$ for a homology class $[\gamma] \in H_1(Y;\mathbb{Z})$ brings in the holonomy of U(1)-connections, as follows. The universal family of spinor bundles $\mathbb{S} \to Y \times \mathcal{B}^\sigma(Y;\mathbb{Z})$ carries a tautological family of connections along the Y-slices. For a given closed curve $\gamma : S^1 \to Y$, we obtain a *half-holonomy evaluation map*

$$\mathcal{B}^\sigma(Y,\mathfrak{s}) \xrightarrow{h_\gamma} \mathrm{U}(1)$$
$$[B,s,\Psi] \mapsto \exp\Big(\frac{1}{2}\int_\gamma \hat{B}\Big).$$

We now explain this formula. As before, $\hat{B}$ stands for the U(1) connection induced by B on $L = \Lambda^2 S$. By the integral above we mean the following: choose a trivialisation of S along the closed curve γ, so as to identify $\hat{B}$ with a 1-form b on γ with values in $i\mathbb{R}$, and evaluate $\exp\frac{1}{2}\int_\gamma b$. This element of U(1) does not depend on the chosen trivialisation of S along γ and the chosen representative of the gauge-equivalence class, since for different choices the 1-form b changes by adding $-2v^{-1}dv$ for some smooth function $v : S^1 \to \mathrm{U}(1)$ (and note that $\int_{S^1} v^{-1}dv \in 2\pi i\mathbb{Z}$).

Thus, the half-holonomy evaluation map h_γ provides a square root of the "usual" holonomy map $\mathcal{B}^\sigma(Y,\mathfrak{s}) \to \mathrm{U}(1)$ given by $[B] \mapsto \exp\int_\gamma \hat{B}$. The geometric content of the slant map for $k = 1$ is described by the next result:

Proposition 5.4. *Let γ be an oriented closed curve in Y. The class $\mu([\gamma]) \in H^1(\mathcal{B}^\sigma(Y,\mathfrak{s});\mathbb{Z})$ is represented by the half-holonomy map $h_\gamma : \mathcal{B}^\sigma(Y,\mathfrak{s}) \to \mathrm{U}(1)$. Thus, $\mu(\gamma)$ is Poincaré dual to the fibres of the submersion h_γ .*

To show this, we consider a hermitian line bundle L over a finite-dimensional manifold X. Denote by $\mathcal{A}$ the affine space of unitary connections on L, and by $\mathcal{G} = C^\infty(X,\mathrm{U}(1))$ the gauge group of L. As before, there is a tautological unitary line bundle $\mathcal{L}$ over $(\mathcal{A}/\mathcal{G}) \times X$ carrying a tautological family of unitary connections on the X-slices.

Lemma 5.5. *For each $\gamma \in H_1(X;\mathbb{Z})$, the class $\gamma\backslash c_1(\mathcal{L}) \in H^1(\mathcal{A}/\mathcal{G};\mathbb{Z})$ is the cohomology class represented by the holonomy map* $\mathrm{hol}_\gamma : \mathcal{A}/\mathcal{G} \to \mathrm{U}(1)$ *given by* $\mathrm{hol}_\gamma([B]) = \exp\int_\gamma B$.

Proof. We view U(1) as $i\mathbb{R}/2\pi i\mathbb{Z}$, and denote by $\omega = [\frac{1}{2\pi}dx] \in H^1(\mathrm{U}(1);\mathbb{Z})$ the fundamental cohomology class. We must establish the identity $\mathrm{hol}_\gamma^*\omega = \gamma\backslash c_1(\mathcal{L})$, which is equivalent to the following: for any integral 1-cycle δ in $\mathcal{A}/\mathcal{G}$ we have

$$\langle \omega, (\mathrm{hol}_\gamma)_*\delta\rangle = \langle c_1(\mathcal{L}), \gamma \times \delta\rangle. \tag{5.5}$$

That it suffices to show (5.5) follows from the fact $\mathcal{A}/\mathcal{G}$ has no torsion in its cohomology.

We may suppose that δ is a smooth map $\delta : S^1 \to \mathcal{A}/\mathcal{G}$. This can be viewed as a path $t \mapsto B(t)$ of unitary connections on L with $B(0)$ gauge-equivalent to $B(1)$. We see that

$$\langle \omega, (\mathrm{hol}_\gamma)_*\delta\rangle = \frac{1}{2\pi i}\int_{t=0}^{1}\Big(\int_\gamma \frac{\partial B(t)}{\partial t}\Big)dt = -i\int_{\gamma\times\delta}\frac{\partial B(t)}{\partial t}\wedge\omega.$$

We now provide a representative for the class $c_1(\mathcal{L})|_{\gamma\times\delta}$. The bundle $\mathcal{L}|_{\gamma\times\delta}$ carries the family of connections $B(t)$ on the γ-slices, and these induce a well-defined connection $\mathbf{B}$ on $\mathcal{L}|_{\gamma\times\delta}$ by setting $\nabla_{\mathbf{B}} = \frac{d}{dt} + \nabla_{B(t)}$. The class $c_1(\mathcal{L})|_{\gamma\times\delta}$ is represented by the Chern-Weil form

$$\frac{i}{2\pi}F_{\mathbf{B}} = \frac{i}{2\pi}\Big(F_{B(t)} + dt\wedge\frac{\partial B(t)}{\partial t}\Big)$$

and hence

$$\langle c_1(\mathcal{L}), \gamma\times\delta\rangle = i\int_{\gamma\times\delta}\omega\wedge\frac{\partial B(t)}{\partial t} = -i\int_{\gamma\times\delta}\frac{\partial B(t)}{\partial t}\wedge\omega.$$

□

Proof of Proposition 5.4. We specialise our discussion from above to $X := Y$ and $L := \Lambda^2 S^+$. We have a natural map $p : \mathcal{B}^\sigma(Y,\mathfrak{s}) \to \mathcal{A}/\mathcal{G}$ given by $p([B,s,\Psi]) = [\hat{B}]$, and the identity

$\mathrm{hol}_\gamma \circ p = (h_\gamma)^2$. It follows from this and Lemma 5.5 that

$$2 \cdot h_\gamma^*[\mathrm{U}(1)]^\vee = p^*\mathrm{hol}_\gamma^*[\mathrm{U}(1)]^\vee = p^*(\gamma \backslash c_1(\mathcal{L})) = \gamma \backslash (\mathrm{id}_Y \times p)^* c_1(\mathcal{L}).$$

From Lemma 5.2 we have $\Lambda^2\mathbb{S}^+ \cong (\mathrm{pr}_1^*\Lambda^2 S^+) \otimes \mathcal{U}^{\otimes 2}$, and likewise one has $\mathcal{L} \cong (\mathrm{pr}_1^* L) \otimes \mathcal{U}$ where $\mathcal{U} = (\mathbb{C} \times Y \times \mathcal{A})/\mathcal{G}$ is the canonical line bundle over $Y \times \mathcal{A}/\mathcal{G}$. Because the fiberwise $\mathrm{U}(1)$-action on S^+ induces the *weight two* $\mathrm{U}(1)$-action on $L = \Lambda^2 S^+$ it follows that $(\mathrm{id}_Y \times p)^*\mathcal{U} \cong \mathcal{U}^2$, and hence

$$(\mathrm{id}_Y \times p)^*\mathcal{L} \cong (\mathrm{id}_Y \times p)^*\big((\mathrm{pr}_1^* L) \otimes \mathcal{U}\big) = (\mathrm{pr}_1^* L) \otimes \mathcal{U}^{\otimes 2} \cong \Lambda^2\mathbb{S}^+.$$

Putting everything together we have $2 \cdot h_\gamma^*[\mathrm{U}(1)]^\vee = \gamma \backslash c_1(\Lambda^2\mathbb{S}^+) = 2\mu(\gamma)$. This gives $h_\gamma^*[\mathrm{U}(1)]^\vee = \mu(\gamma)$ and from this identity the result follows. □

5.1.1.5 The cohomology ring of the configuration space, again

We can now upgrade the isomorphism in Proposition 5.1 to a canonical one:

Proposition 5.6. *The slant map μ induces an isomorphism of graded rings*

$$\mathbb{A}(Y;\mathbb{Z}) \xrightarrow{\cong} H^*(\mathcal{B}^\sigma(Y, \mathfrak{s});\mathbb{Z})$$

determined by sending $U \mapsto \mu(1)$, and $[\gamma] \mapsto \mu([\gamma])$ for $[\gamma] \in H_1(Y;\mathbb{Z})/\mathrm{torsion}$.

Proof. We consider the fibre bundle (5.2) from the proof of Proposition 5.1. Its fibre has the weak homotopy-type of $\mathbb{C}P^\infty$, and the line bundle $\mathcal{U}|_{p \times \mathcal{B}^*(Y,\mathfrak{s})}$ restricts to the canonical line bundle $\mathcal{O}(1)$ over $\mathbb{C}P^\infty$. Hence the class $\mu(1) = c_1(\mathcal{U}|_{p\times\mathcal{B}^*(Y,\mathfrak{s})}) \in H^2(\mathcal{B}^*(Y,\mathfrak{s});\mathbb{Z})$ restricts to a generator of the cohomology ring of the fibres.

On the other hand, the base of the fibre bundle has the homotopy type of the torus $\mathcal{H}^1(Y;i\mathbb{R})/\mathcal{H}^1(Y;2\pi i\mathbb{Z})$. Choosing a $\mathbb{Z}$-basis of oriented closed curves $(\gamma_i)_{i=1,\dots,b_1(Y)}$ for $H_1(Y;\mathbb{Z})/\mathrm{torsion}$ we obtain an explicit identification with the torus $T = \mathrm{U}(1)^{\times b_1(Y)}$

$$\mathcal{H}^1(Y; i\mathbb{R})/\mathcal{H}^1(Y; 2\pi i\mathbb{Z}) \xrightarrow{\cong} T\,, \qquad [b] \mapsto \left(\exp \int_{\gamma_i} b\right)_{i=1,\dots,b_1(Y)}$$

and the bundle projection $\mathcal{B}^*(Y, \mathfrak{s}) \to T$ is then identified with

$$[B, \Psi] \mapsto \left(\exp \int_{\gamma_i} (B - B_0)^{\mathcal{H}}\right)_{i=1,\dots,b_1(Y)}.$$

The latter map is easily seen to be homotopic to the product of the half-holonomy maps hol_{γ_i}, and hence a basis for the cohomology of the base of the fibre bundle pulls back to the classes $\mu(\gamma_i)$ (using Proposition 5.4). The fact that the fibre bundle is cohomologically trivial was shown in the proof of 5.1, so the result follows. □

5.1.1.6 The module structure in Floer cohomology

The cup product pairing (5.1) in Floer cohomology is obtained, roughly speaking, by integrating cohomology classes in $\mathcal{B}^\sigma(Y, \mathfrak{s})$ over the moduli spaces $M_z([\mathfrak{a}], [\mathfrak{b}])$. A general definition using Čech cohomology is given in [[49], §25]. Using our dual description of the generators of the cohomology of $\mathcal{B}^\sigma(Y, \mathfrak{s})$ we now give an equivalent description of this pairing which will serve better our purposes. We note that the construction given here for the U map is essentially identical to the construction found in [[46],§4.11], except that here we make use of the explicit sections f_τ of the canonical line bundle $\mathcal{U}$, rather than arbitrary ones.

After choosing a metric and admissible perturbation $(g, \mathfrak{q})$, there is a (universal) Seiberg–Witten moduli space $\mathfrak{M}'([\mathfrak{a}], [\mathfrak{b}]) \to \mathcal{P}$ over the cylinder $(\mathbb{R} \times Y, dt^2 + g)$. This is constructed in [[49] , §25] in the more general setting of cobordism maps, as a fibre product of moduli spaces over the cylinders $(-\infty, -1/2] \times Y$, $[-1/2, 1/2] \times Y$ and $[1/2, +\infty) \times Y$. Here, the moduli space over $[-1/2, 1/2] \times Y$ consists of configurations $(A, \Phi, \mathfrak{t})$ where $\mathfrak{t} \in \mathcal{P}$ is used to construct a perturbation term supported in a collar neighbourhood of the boundary, by taking $\eta\hat{\mathfrak{t}}$, with $\eta(t)$ a bump function compactly supported in $(-1/2, 0) \cup (0, 1/2)$.

By the unique continuation principle, the moduli $\mathfrak{M}'([\mathfrak{a}],[\mathfrak{b}])$ can be regarded as a subset of the configuration space $\mathcal{B}^\sigma([-1/2,1/2]\times Y,\mathfrak{s})\times\mathcal{P}$. On the latter space we have two maps defined. First, there is the section f_τ of the canonical line bundle $\mathcal{U}$, $f_\tau(A,s,\Phi)=\tau_1\Phi(0,p)\in\mathbb{C}$, defined by a choice of unitary splitting $\tau=(\tau_1,\tau_2):S^+_{(0,p)}\xrightarrow{\cong}\mathbb{C}^2$ at $(0,p)\in\mathbb{R}\times Y$. On the other hand, we have the half-holonomy map $h_\gamma(A,s,\Phi)=\exp\frac{1}{2}\int_{0\times\gamma}\hat{A}\in\mathrm{U}(1)$ obtained from an oriented closed curve $0\times\gamma$ in the slice $0\times Y\subset\mathbb{R}\times Y$.

Proposition 5.7. *Fix oriented closed curves $\gamma_i\subset Y, i=1,\dots,b_1(Y)$ providing a basis of $H_1(Y;\mathbb{Z})/\mathrm{torsion}$. Then*

(i) the fibre product defining the moduli spaces $\mathfrak{M}'([\mathfrak{a}]\,,[\mathfrak{b}])$ is transverse

(ii) $Z_\tau=f_\tau^{-1}(0)$ is transverse to the submanifold $\mathfrak{M}'([\mathfrak{a}],[\mathfrak{b}])\subset\mathcal{B}^\sigma([-1/2,1/2]\times Y,\mathfrak{s})\times\mathcal{P}$

(iii) for each i, $Z_{\gamma_i,\kappa}=h_{\gamma_i}^{-1}(\kappa)$ is transverse to the submanifold $\mathfrak{M}'([\mathfrak{a}],[\mathfrak{b}])\subset\mathcal{B}^\sigma([-1/2,1/2]\times Y,\mathfrak{s})\times\mathcal{P}$, where $\kappa\in\mathrm{U}(1)$ is any given value.

Part (i) is proved in [[49], §25] in a more general setting, and (ii)-(iii) follow in a similar way as the transversality results presented in §A.1. To define the module structure on the monopole Floer cohomology group $\widehat{HM}^*(Y,\mathfrak{s};\mathbb{Z})$ one chooses a perturbation $\mathfrak{t}\in\mathcal{P}$ that is a regular value of the Fredholm maps

$$Z_\tau\cap\mathfrak{M}'([\mathfrak{a}],[\mathfrak{b}])\to\mathcal{P}$$

$$Z_{\gamma_i,\kappa}\cap\mathfrak{M}'([\mathfrak{a}],[\mathfrak{b}])\to\mathcal{P}$$

for all i and all pairs of critical points $[\mathfrak{a}],[\mathfrak{b}]$, and we denote by $M([\mathfrak{a}],U,[\mathfrak{b}];\tau)$ and $M([\mathfrak{a}],\gamma_i,[\mathfrak{b}];\kappa)$ the corresponding fibres over $\mathfrak{t}$, which are smooth manifolds of finite dimension.

The $\mathbb{A}(Y;\mathbb{Z})$-module structure on $\widehat{HM}^*(Y,\mathfrak{s};\mathbb{Z})$ is now constructed by writing down maps $\widehat{m}_\tau(U)^*,\widehat{m}_\kappa(\gamma_i)^*:\widehat{C}^*(Y,\mathfrak{s})\to\widehat{C}^*(Y,\mathfrak{s})$ as follows. Each enumerates trajectories between joining critical points of the three kinds, e.g.

$$m(U)^u_o : C^u \to C^o, \quad [\mathfrak{a}] \mapsto \sum_{[\mathfrak{b}] \in \mathfrak{C}^o} \# M([\mathfrak{a}], U, [\mathfrak{b}]; \tau) \cdot [\mathfrak{b}]$$

and similarly for maps $m(U)^o_s, m(U)^o_o, m(U)^s_u, m(U)^u_s$ together with similar maps $\overline{m}(U)^s_u, \overline{m}(U)^u_u, \overline{m}(U)^s_u$ for the reducible loci in the moduli spaces. These assemble into a chain map $\widehat{m}_\tau(U) : \widehat{C}_*(Y, \mathfrak{s}; \mathbb{Z}) \to \widehat{C}_*(Y, \mathfrak{s}; \mathbb{Z})$ given by

$$\begin{pmatrix} m(U)^o_o & m(U)^u_o \\ \overline{m}(U)^s_u \partial^o_s - \overline{\partial}^s_u m(U)^o_s & \overline{m}(U)^u_u + \overline{m}(U)^s_u \partial^u_s - \overline{\partial}^s_u m(U)^u_s \end{pmatrix}$$

and dualising yields the desired cochain map $\widehat{m}(U; \tau)^*$. Similarly one obtains the cochain map $\widehat{m}_\kappa(\gamma_i, \kappa)^*$. Up to chain homotopy, these maps do not depend on τ or κ, and passing to cohomology defines the action of $U, \gamma_i \in \mathbb{A}(Y, \mathbb{Z})$ on $\widehat{HM}^*(Y, \mathfrak{s}; \mathbb{Z})$, which gives the pairing (5.1) when $R = \mathbb{Z}$. For a general ring R, we tensor the cochain maps $\widehat{m}_\tau(U)^*, \widehat{m}_\kappa(\gamma_i)^*$ with R, and this induces the action of $\mathbb{A}(Y; R) = \mathbb{A}(Y; \mathbb{Z}) \otimes R$ on the monopole Floer cohomology $\widehat{HM}^*(Y, \mathfrak{s}; R)$. This completes our description of the module structure (5.1) in monopole Floer cohomology.

Remark 5.6. We briefly mention why the above description of the module structure on Floer homology agrees with the Čech cohomology construction given in [[49],§25]. As in [[49],§21] we consider a compact d-dimensional space N^d stratified by oriented smooth manifolds, where we denote by $M^d \subset N^d$ the top dimensional stratum, and let $\mathfrak{U}$ be an open cover of N^d transverse to the strata (in the sense of [[49], §21.2]). Given a connected component $M^d_\alpha \subset M^d$ of the top dimensional stratum there is an associated integration map (see [49], formula (21.3)) $\langle -, [M^d_\alpha] \rangle : \check{C}^d(\mathfrak{U}; \mathbb{Z}) \to \mathbb{Z}$, where $\check{C}^d(\mathfrak{U}; \mathbb{Z})$ is the abelian group of degree d Čech cochains (which are automatically closed and vanish when restricted to $C^{d-1}(\mathfrak{U}|_{N^{d-1}}; \mathbb{Z})$ because of the transversality assumption). The essential point is the following: if $u \in \check{C}^d(\mathfrak{U}; \mathbb{Z})$ is a degree d cochain representing a cohomology class in $H^d_c(M^d; \mathbb{Z}) \cong H^d(N^d, N^{d-1}; \mathbb{Z})$ which is Poincaré dual to an oriented submanifold $Z \subset M$ (necessarily of dimension zero) which is transverse to the strata of N^d, then one has that $\langle u, [M_\alpha^d]\rangle$ agrees with the signed count of points in $Z \cap M_\alpha^d$. Applying this observation when N^d is the compactification by broken trajectories of a d-dimensional Seiberg-Witten moduli space $M([\mathfrak{a}],[\mathfrak{b}])$ on the cylinder $\mathbb{R} \times Y$ (here we mean the "smaller" version of the compactification, as in Definition 24.6.9 in [49]) it follows that our definition of the module structure agrees with the one given in [49].

5.1.2 The module structure on $H_*(C(Y,\xi_0);\Lambda_R)$

We now fix a closed oriented contact 3-manifold (Y,ξ_0). We will define a graded $\mathbb{A}^\dagger(Y;R)$-module structure

$$\mathbb{A}^\dagger(Y;R) \otimes H_*(C(Y,\xi_0);\Lambda_R) \to H_{*-k}(C(Y,\xi_0);\Lambda_R)$$

and describe its geometric meaning.

5.1.2.1 The slant construction

We do a similar construction as before, using the slant product

$$H_k(Y;\mathbb{Z}) \otimes H^n(Y \times C(Y,\xi_0);\mathbb{Z}) \to H^{n-k}(C(Y,\xi_0);\mathbb{Z}).$$

There is a tautological family of contact structures on Y parametrised by $C(Y,\xi_0)$, which provide us with a real oriented rank 2 vector bundle $\boldsymbol{\xi} \to Y \times C(Y,\xi_0)$. The bundle $\boldsymbol{\xi}$ is a subbundle of a trivial rank 3 bundle (since TY is trivial for any closed oriented 3-manifold), so its second Stiefel-Whitney class $w_2(\boldsymbol{\xi})$ vanishes. Consequently, the Euler class $e(\boldsymbol{\xi}) \in H^2(Y \times C(Y,\xi_0);\mathbb{Z})$ is divisible by 2.

Definition 5.4. For $k = 0, 1$ we define

$$\begin{aligned} \overline{\mu} : H_k(Y;\mathbb{Z})/\text{torsion} &\to H^{2-k}(C(Y,\xi_0);\mathbb{Z}) \\ \alpha &\mapsto \frac{1}{2}\alpha \backslash e(\boldsymbol{\xi}). \end{aligned} \tag{5.6}$$

Remark 5.7. Observe that for $k = 0$, the slant product map (5.6) is, a priori, only well-defined as a map into $H^2(C(Y,\xi_0);\mathbb{Z})$ *modulo* the 2-torsion subgroup. This ambiguity arises from dividing by 2 in (5.6). However, we now explain that there is a canonical lift, which we take as the definition of (5.6). Observe that taking $\alpha = 1 \in H_0(Y;\mathbb{Z}) = \mathbb{Z}$ we have $\alpha\backslash e(\boldsymbol{\xi}) = e(\boldsymbol{\xi}|_{p\times C(Y,\xi_0)})$, so the matter reduces to having a preferred square root of $\boldsymbol{\xi}|_{p\times C(Y,\xi_0)}$, up to isomorphism. This rank 2 bundle comes with a preferred homotopy class of embeddings into the trivial rank 3 bundle, simply obtained by fixing a positive framing $T_pY \cong \mathbb{R}^3$. In other words, there is a canonical homotopy class of maps $C(Y,\xi_0) \to \widetilde{\mathrm{Gr}}_2(\mathbb{R}^3)$ into the Grassmannian of oriented 2-planes in $\mathbb{R}^3$, which by pullling back the tautological 2-plane bundle over $\widetilde{\mathrm{Gr}}_2(\mathbb{R}^3)$ yield $\boldsymbol{\xi}|_{p\times C(Y,\xi_0)}$. It is now elementary to observe that there is a unique square root (i.e. spin structure) for the tautological 2-plane bundle over $\widetilde{\mathrm{Gr}}_2(\mathbb{R}^3)$.

Definition 5.5. We endow $H_*(C(Y,\xi_0);\Lambda_R)$ with a graded $\mathbb{A}^\dagger(Y;R)$-module structure

$$\mathbb{A}^\dagger(Y;R)\otimes H_*(C(Y,\xi_0);\Lambda_R) \to H_*(C(Y,\xi_0);\Lambda_R) \tag{5.7}$$

by setting: for $T \in H_n(C(Y,\xi_0);\Lambda_R)$

$$U\cdot T := \overline{\mu}(1)\cap T \in H_{n-2}(C(Y,\xi_0);\Lambda_R)$$
$$\gamma\cdot T := \overline{\mu}(\gamma)\cap T \in H_{n-1}(C(Y,\xi_0);\Lambda_R)\,,\quad \gamma\in H_1(Y;\mathbb{Z})/\text{torsion}.$$

Here $\cap$ denotes the cap product with *coefficients in the local system* Λ_R:

$$H^k(C(Y,\xi_0);\mathbb{Z})\otimes H_n(C(Y,\xi_0);\Lambda_R)\to H_{n-k}(C(Y,\xi_0);\Lambda_R).$$

We now relate the slant product maps μ and $\overline{\mu}$. The space $C\mathcal{M}(Y,\xi_0)$ of triples (ξ,α,j) parametrises a family of spin-c structures and irreducible configurations on Y (see §4.1.5.2), and to this it corresponds a classifying map $f: C\mathcal{M}(Y,\xi_0)\to\mathcal{B}^*(Y,\mathfrak{s}_{\xi_0,\alpha_0,j_0})$ (see Lemma 4.1). Here, (ξ_0,α_0,j_0) is a fixed triple.

Lemma 5.8. *We have the identity $\overline{\mu} = f^*\mu$.*

Proof. Indeed, under the map $\mathrm{id}_Y \times f : Y \times C(Y, \xi_0) \to Y \times \mathcal{B}^*(Y, \mathfrak{s}_{\xi_0,\alpha_0,j_0})$ the bundle $\Lambda^2\mathbb{S}^+$ pulls back to the bundle $\boldsymbol{\xi}$, and hence $e(\boldsymbol{\xi}) = (\mathrm{id}_Y \times f)^* c_1(\Lambda^2\mathbb{S}^+)$. □

5.1.2.2 Geometric interpretations

We conclude this section by interpreting the module action (5.1) in geometric terms. We start with the U map

$$H_*(C(Y, \xi_0); \Lambda_R) \xrightarrow{U} H_{*-2}(C(Y, \xi_0; \Lambda_R)\,, \quad T \mapsto \overline{\mu}(1) \cap T.$$

Fixing a point $p \in Y$, there is a natural evaluation map to the Grassmanian of oriented planes in T_pY

$$C(Y, \xi_0) \xrightarrow{ev} \widetilde{\mathrm{Gr}}_2(T_pY) \cong S^2\,, \quad \xi \mapsto \xi(p).$$

The main geometric content is:

Proposition 5.9. *The class $\overline{\mu}(1) \in H^2(C(Y, \xi_0); \mathbb{Z})$ is represented by the map ev, i.e. $\overline{\mu}(1) = ev^*[S^2]^\vee$.*

From this it follows that the more geometric description of the U action on $H_*(C(Y, \xi_0))$ given in §1 agrees with the one just given in Definition 5.5.

Proof. For each unitary framing $(\tau_1, \tau_2) : S_p \xrightarrow{\cong} \mathbb{C}^2$ of the fibre over $p \in Y$ of the spinor bundle $S := S_{\xi_0,\alpha_0,j_0}$ we have the section $f_\tau(B, \Psi) = \tau_1\Psi(p)$ of the canonical line bundle $\mathcal{U}$ restricted to $p \times \mathcal{B}^*(Y, \mathfrak{s}_{\xi_0,\alpha_0,j_0})$. This "pencil" of sections induces a map e to the projectivisation of S_p, away from the base locus $B = \{[B, \Psi] \,:\, \Psi(p) = 0\}$

$$\mathcal{B}^*(Y, \mathfrak{s}_{\xi_0,\alpha_0,j_0}) \setminus B \xrightarrow{e} \mathbb{P}(S_p) \cong \mathbb{P}^1\,, \quad [B, \Psi] \mapsto \mathbb{C} \cdot \Psi(p).$$

Note that if $Z_\tau := f_\tau^{-1}(0)$ then $Z_\tau \setminus B \subset \mathcal{B}^*(Y, \mathfrak{s}_{\xi_0,\alpha_0,j_0}) \setminus B$ is a fibre of e.

Observe now that the image of the classifying map $f : C\mathcal{M}(Y, \xi_0) \to \mathcal{B}^*(Y, \mathfrak{s}_{\xi_0,\alpha_0,j_0})$ does not meet B. From this, together with the fact that Z_τ is Poincaré dual to $\mu(1)$ (Lemma 5.3) and $f^*\mu = \overline{\mu}$ (Lemma 5.8), it follows that the cohomology class $\overline{\mu}(1) \in H^2(C\mathcal{M}(Y, \xi_0))$ is represented by the map $e \circ f : C\mathcal{M}(Y, \xi_0) \to \mathbb{P}(S_p)$.

The key observation is now the following

Lemma 5.10. *Let (V, g) be a 3-dimensional real oriented inner product vector space, and let $S \cong \mathbb{C}^2$ be its fundamental spin-c representation. Then there exists a canonical diffeomorphism $\widetilde{\mathrm{Gr}}_2(V) \cong \mathbb{P}(S)$.*

Proof. The Grassmanian $\widetilde{\mathrm{Gr}}_2(V)$ is diffeomorphic to the unit sphere in V^* via

$$\mathrm{Sph}(V^*, g) \xrightarrow{\cong} \widetilde{\mathrm{Gr}}_2(V)\,, \quad \alpha \mapsto \ker \alpha.$$

Recall that the fundamental spin-c representation provides an isomorphism $V^* \cong \mathfrak{su}(S)$ as $\mathrm{Spin}^{\mathbb{C}}(3)$ modules. Under this isomorphism, the Clifford multiplication, a given $\alpha \in \mathrm{Sph}(V^*, g)$ acts on S decomposing it into $\pm i$ eigenspaces $S = l^+ \oplus l^-$. Each $l^\pm$ is a complex line in S, and the assignment

$$\mathrm{Sph}(V^*, g) \to \mathbb{P}(S)\,, \quad \alpha \mapsto l^+$$

provides a diffeomophism, concluding the proof. □

To conclude, apply Lemma 5.10 for each $t \in C\mathcal{M}(Y, \xi_0)$ to the inner product spaces $(T_pY, g_{\xi_t,\alpha_t,j_t})$ and spin-c structures $(g_{\xi_t,\alpha_t,j_t}, S_{\xi_0,\alpha_0,j_0}, \rho_{\xi_0,\alpha_0,j_0} \circ b^*_{g,g_{\xi_0,\alpha_0,j_0}})$. Under the diffeomorphism described in the proof of Lemma 5.10, one identifies the maps $e \circ f$ and ev as the same. □

Finally, we briefly comment on the action of $\gamma \in H_1(Y, \mathbb{Z})$

$$H_*(C(Y, \xi_0); \Lambda_R) \xrightarrow{\gamma} H_{*-1}(C(Y, \xi_0); \Lambda_R).$$

The geometric interpretation that we will need in the subsequent sections is already provided by Lemma 5.5: upon choosing a reduction of the structure group of $\boldsymbol{\xi} \to Y \times C(Y, \xi_0)$ to $\mathrm{U}(1)$, and a family of unitary connections $\{B_\xi\}$ over the Y-slices, one obtains a holonomy map

$$C(Y, \xi_0) \to \mathrm{U}(1)\,, \quad \xi \mapsto \exp \int_\gamma B_\xi$$

whose regular fibres are Poincare dual to $2\overline{\mu}(\gamma) \in H^1(C(Y, \xi_0); \mathbb{Z})$. In particular, the canonical spin-c connections $\hat{B}_{\xi,\alpha,j}$ on Y parametrised by $(\xi, \alpha, j) \in C\mathcal{M}(Y, \xi_0) \simeq C(Y, \xi_0)$ provide such a family of connections.

5.2 The neck-stretching argument

In this section we establish Theorem 1.5 (B). This asserts that the families contact invariant $\mathbf{Fc}$: $H_*(C(Y, \xi_0); \Lambda_R) \to \widehat{HM}^*(Y, \mathfrak{s}_{\xi_0}; R)$ intertwines the module structures, which were introduced in §5.1. We must show: for $T \in H_*(C(Y, \xi_0); \Lambda_R)$ and a homology class $\gamma \in H_1(Y; \mathbb{Z})$

$$U \cdot \mathbf{Fc}(T) = \mathbf{Fc}(U \cdot T) \tag{5.8}$$

$$\gamma \cdot \mathbf{Fc}(T) = \mathbf{Fc}(\gamma \cdot T). \tag{5.9}$$

We sketch now the main ideas in the case of U. The key is to consider, for a given simplex σ : $\Delta^n \to C\mathcal{M}(Y, \xi_0)$, a moduli space $\mathcal{M}([\mathfrak{a}], U, \sigma; \tau) \to \Delta^n \times \mathbb{R} \ni (t, s)$ of solutions to the Seiberg–Witten equations on Z^+ that meet certain evaluation constraint at the point $(s, p) \in \mathbb{R} \times Y \cong Z^+$. Here $p \in Y$ is fixed, whereas $s \in \mathbb{R}$ is not, and hence the evaluation constraint is thought of as travelling through Z^+ from the cylindrical to the symplectic end. The evaluation constraint itself is that the spinor Φ satisfies $\tau_1 \Phi = 0$ at the point (s, p), for a suitably chosen trivialisation $\tau = (\tau_1, \tau_2)$ of the bundle S^+ along the line $\mathbb{R} \times p \subset Z^+$. Such moduli spaces will be referred to as *parametrised evaluation moduli spaces*. The main part of the argument is to analyse the ends of the (non-compact) moduli $\mathcal{M}([\mathfrak{a}], U, \sigma; \tau)$ as $s \to \pm\infty$. As $s \to -\infty$ we will see that the solutions to the equations degenerate into *broken* configurations, which in the simplest case consist of pairs of configurations (γ_1, γ_0), the first of which solves the Seiberg–Witten equations over an infinite cylinder $\mathbb{R} \times Y$ with an evaluation constraint, and the second is an unconstrained solution over Z^+. The interesting part of the moduli space, however, shows up as $s \to +\infty$. Here we will show that $\mathcal{M}([\mathfrak{a}], U, \sigma; \tau)$ looks like the product $\mathbb{R} \times M$ where M is, in a sense, the intersection of the moduli $M([\mathfrak{a}], \sigma)$ over the simplex (constructed in §4.2.4) with a fibre of the map $\mathcal{B}^*(Y, \mathfrak{s}_{\xi_0, \alpha_0, j_0}) \dashrightarrow \mathbb{P}^1$ from §5.1.

This will allow us to construct compactifications of the parametrised evaluation moduli spaces, and the identities (5.8)-(5.9) arise from counting the boundary points of the compactified 1-dimensional parametrised evaluation moduli.

5.2.1 Parametrised evaluation moduli spaces over Z^+

5.2.1.1 A family of perturbations

As a starting point for the construction of the parametrised evaluation moduli spaces we introduce an intermediate moduli space

$$\mathcal{M}([\mathfrak{a}], Z^+) \to C\mathcal{M}(Y, \xi_0) \times \mathcal{P} \times \mathbb{R} \tag{5.10}$$

analogous to $\mathfrak{M}([\mathfrak{a}], Z^+)$ from §4.2.3. The only new feature is that the $\mathbb{R}$ factor in the base will parametrise various perturbations of the Seiberg–Witten equations. The parametrised evaluation moduli space will result from imposing constraints on the configurations in $\mathcal{M}([\mathfrak{a}]\ , Z^+)$. Following the same scheme as in §4.2.3, we construct $\mathcal{M}([\mathfrak{a}], Z^+)$ as a fibre product of moduli.

The first step is constructing a moduli space

$$\mathcal{M}_k \to C\mathcal{M}(Y, \xi_0) \times \mathcal{P} \times \mathbb{R} \tag{5.11}$$

in the same flavour of $\mathfrak{M}_k(K') \to C\mathcal{M}(Y, \xi_0) \times \mathcal{P}$. The moduli space $\mathcal{M}_k$ consists of gauge-equivalence classes of quintuples $(A, \Phi, t, \mathfrak{p}, s)$, where the variables $(t, \mathfrak{p}, s) \in \mathcal{CM}(Y, \xi_0) \times \mathcal{P} \times \mathbb{R}$ provide the map in (5.11), and (A, Φ) are configurations over the region

$$K(s) = [\mathrm{m}(s), +\infty) \times Y \subset Z^+ \tag{5.12}$$

Here $\mathrm{m}(s)$ stands for the function $\min(s-1, 0)$, or rather, a suitable smooth approximation of it. Such $(A, \Phi, t, \mathfrak{p}, s)$ must be asymptotic to canonical configurations as before, and satisfy the Seiberg–Witten equations

$$\mathrm{sw}(A, \Phi, u) + \lambda(A, \Phi, t, \mathfrak{p}, s) = 0$$

perturbed by a certain quantity $\lambda(A, \Phi, t, \mathfrak{p}, s) \in \Upsilon_{k-1}$ which we now describe. It is given by the section

$$\begin{aligned} \lambda : \mathcal{C}_k(K(s)) \times \mathcal{P} \times \mathbb{R} &\to \Upsilon_{k-1} \\ (A, \Phi, t, \mathfrak{p}, s) &\mapsto \varphi^1_s \hat{\mathfrak{q}}(A, \Phi) + \varphi^2_s \hat{\mathfrak{p}}(A, \Phi) + \eta_s \hat{\mathfrak{t}}(A, \Phi) + \varphi^3 \hat{\mathfrak{p}}_{K,u}. \end{aligned} \tag{5.13}$$

Here, $\mathfrak{q}$, $\mathfrak{t}$ are fixed generic perturbations chosen as in §4.2.3 and §5.1.1.6. Also, φ^1_s, φ^2_s, η_s are fixed $\mathbb{R}$-families of non-negative functions on $\mathbb{R}$, and φ^3 is the function we chose in §4.2.3. We require that they relate to the functions φ^1, φ^2 of §4.2.3, and η of §5.1.1.6 as follows. Let $(\tau_s f)(t) := f(t+s)$. Then

(i) $\varphi^1_s = \tau_{-\mathrm{m}(s)} \varphi^1$ for all $s \in \mathbb{R}$

(ii) $\varphi^2_s = \tau_{-\mathrm{m}(s)} \varphi^2$ for all $s \in \mathbb{R}$

(iii) $\eta_s = \tau_{-s} \eta$ for $s < 0$ very negative, and identically vanishing for $s \geq 0$.

The choice of such perturbation data will ultimately ensure the behaviour of the parametrised evaluation moduli spaces that we have described at the beginning of §5.2.

We want to make $\mathcal{M}_k$ into a Banach manifold. By applying the $\mathbb{R}$-family of translations $t \mapsto t - s$ we can view $\mathcal{M}_k$ as a subset of a suitable configuration space $\mathcal{B}_k = \mathcal{C}_k / \mathcal{G}_{k+1}$ over $[0, +\infty) \times Y$.

As before, the latter is a C^{l-k-2} Banach manifold and $\mathcal{M}_k$ is the transverse zero set of a section of a bundle over $\mathcal{B}_k$ given by the perturbed Seiberg–Witten map; hence a Banach manifold of the same regularity. The claimed transversality follows, once more, from the results in §A.1.

We then have restriction maps

$$\mathcal{M}_k \xrightarrow{\mathcal{R}_-} \mathcal{B}^{\sigma}_{k-1/2}(Y, \mathfrak{s}_{\xi_0,\alpha_0,j_0})$$
$$M_k([\mathfrak{a}], (-\infty, 0] \times Y) \xrightarrow{R_+} \mathcal{B}^{\sigma}_{k-1/2}(Y, \mathfrak{s}_{\xi_0,\alpha_0,j_0})$$

onto the left-most and right-most end, respectively. From §A.1 it will follow that the fibre product

$$\mathcal{M}([\mathfrak{a}], Z^+) = \mathrm{Fib}(R_+, \mathcal{R}_-)$$

is transverse, and that the projection to $C\mathcal{M}(Y, \xi_0) \times \mathcal{P} \times \mathbb{R}$ is Fredholm. This completes the construction of (5.10).

5.2.1.2 *The parametrised U-moduli space*

We fix a point $p \in Y$ throughout. We denote by τ an arbitrary unitary splitting of the fibre of the spinor bundle $S_{\xi_0,\alpha_0,j_0} \to Y$ over the point $p \in Y$, i.e. a unitary isomorphism $\tau = (\tau_1, \tau_2) : (S_{\xi_0,\alpha_0,j_0})_p \xrightarrow{\cong} \mathbb{C}^2$. Given such τ, which we may view as an element in the unitary group U(2), we obtain an extension to a unitary splitting of the positive spinor bundle $S^+ \to Z^+$ as follows. First, over the cylindrical end $Z = (-\infty, 0] \times Y \subset Z^+$ by translation. For the symplectic end $K = [1, +\infty) \times Y \subset Z^+$ we proceed as follows. Recall that in §4.2.1.3 that we introduced a rescaling operator $\mathcal{R}_0$, which upon acting on canonical configurations yields them translation-invariant in some gauge. We have the translation-invariant bundle over K given by

$$\overline{S^+} = \mathcal{R}_0^* S^+ = \mathbb{C} \oplus \Lambda^{0,2}_{J_0} T^* K.$$

Usinf the identification $(S_{\xi_0,\alpha_0,j_0})_p = S^+_{(1,p)} \cong \mathcal{R}_0^* S^+_{(1,p)}$ we may simply translate the splitting τ along K. In the transition region $[0,1] \times Y \subset Z^+$ we extend τ in an arbitrary manner.

We have the *canonical line bundle* $\mathcal{U} \to \mathcal{B}_k \times \mathbb{R}$, which arises from the $\mathbb{R}$-family of representations of the group of gauge transformations $\mathcal{G}_{k+1} \to \mathrm{U}(1)$ given by $v \mapsto v(s,p)$, where s varies within $\mathbb{R}$. We pullback this bundle over to $\mathcal{M}([\mathfrak{a}], Z^+)$, which can be identified naturally as a Banach submanifold of $\mathcal{B}(Z^+) \times \mathbb{R}$. We consider the section of this pullback bundle given by

$$f_\tau(A, \Phi, t, \mathfrak{p}, s) = \tau_1 \overline{\Phi}(s,p)$$

where $\overline{\Phi} = \mathcal{R}_0^* \Phi$ is the rescaled version of Φ. Note that we only defined $\mathcal{R}_0$ over the region K; we extend it here over the whole Z^+ as the identity over the cylindrical end Z.

The following will follow from §A.1:

Proposition 5.11. *The section f_τ is transverse to the zero section of $\mathcal{U} \to \mathcal{M}([\mathfrak{a}], Z^+)$.*

Definition 5.6. The (universal) *parametrised U-moduli space* is the Banach submanifold

$$\mathcal{M}([\mathfrak{a}]\,, U, Z^+; \tau) \subset \mathcal{M}([\mathfrak{a}], Z^+)$$

given by the zero set of the section f_τ.

Remark 5.8. Allowing for arbitrary splittings $\tau \in \mathrm{U}(2)$ might seem strange at this point. The main case to have in mind is the basic splitting $S_{\xi_0,\alpha_0,j_0} = \mathbb{C} \oplus \xi_0$, which over the symplectic end corresponds to the splitting $S^+ = \mathbb{C} \oplus \Lambda^{0,2}_{J_0} T^* K$. The section of $\mathcal{U}$ that we would want to take in this case is simply given by projecting $\Phi(s,p)$ to the trivial $\mathbb{C}$ factor. However, it will soon become apparent that, in order for the ends of the relevant moduli spaces in the neck-stretching argument to have a nice structure, we have to pass to a generic splitting τ.

5.2.1.3 The parametrised γ-moduli space

In a similar fashion, we fix a smooth oriented closed curve $\gamma \subset Y$ and consider the map

$$h_\gamma : \mathcal{M}([\mathfrak{a}], Z^+) \to \mathrm{U}(1) \tag{5.14}$$

obtained by associating to $(A, \Phi, t, \mathfrak{p}, s)$ the half-holonomy of the induced connection $\hat{A}$ on $\Lambda^2 S^+$ around the loop $s \times \gamma \subset Z^+$

$$h_\gamma(A, \Phi, t, \mathfrak{p}, s) = \exp \frac{1}{2} \int_{s\times\gamma} \hat{A}.$$

In §A.1 we show:

Proposition 5.12. *The map (5.14) is a submersion.*

Definition 5.7. The (universal) *parametrised γ-moduli space* is the Banach submanifold

$$\mathcal{M}([\mathfrak{a}], \gamma, Z^+; \kappa) \subset \mathcal{M}([\mathfrak{a}], Z^+)$$

given by the preimage of $\kappa \in \mathrm{U}(1)$ under (5.14).

5.2.2 Compactifications

5.2.2.1 The setup

We first introduce the moduli spaces that will be the main players in the neck-stretching argument that will follow. These are associated to a singular chain $\sigma : \Delta^n \to C := C\mathcal{M}(Y, \xi_0) \times \mathcal{P}$ equipped with a unitary splitting $\tau \in \mathrm{U}(2)$ and a value $\kappa \in \mathrm{U}(1)$.

First, by taking the fibre product of $\mathcal{M}_z([\mathfrak{a}], Z^+) \xrightarrow{\pi} C \times \mathbb{R}$ and $\sigma \times \mathrm{id}_\mathbb{R} : \Delta^n \times \mathbb{R} \to C \times \mathbb{R}$ we obtain the space $\mathcal{M}_z([\mathfrak{a}], \sigma)$ which is a C^2 manifold with corners provided the fibre product is transverse. Similarly, taking the fibre product of the each of the two maps $\mathcal{M}_z([\mathfrak{a}], U, Z^+), \mathcal{M}_z([\mathfrak{a}], \gamma, Z^+) \xrightarrow{\pi} C \times \mathbb{R}$ with $\sigma \times \mathrm{id}_\mathbb{R}$ we obtain C^2 manifolds with corners $\mathcal{M}_z([\mathfrak{a}], U, \sigma; \tau)$, $\mathcal{M}_z([\mathfrak{a}], \gamma, \sigma; \kappa)$ if transversality holds. In both cases the required transversality can be achieved by a C^2 perturbation of σ whenever the index of π is $\leq 1-n$, by the Thom-Smale transversality theorem (see §4.2.4.1).

The task that we take up for the remainder of this section is to analyze the ends of the 1-dimensional non-compact moduli spaces $\mathcal{M}_z([\mathfrak{a}]\ , U, \sigma; \tau)$, $\mathcal{M}_z([\mathfrak{a}], \gamma, \sigma; \kappa)$ and construct suitable compactifications of them with a nice boundary structure.

5.2.2.2 Exponential decay

Consider a configuration (A, Φ, t, s) in the moduli $\mathcal{M}_z([\mathfrak{a}], \sigma)$. Over the symplectic end $K \subset Z^+$ the positive spinor bundle $S^+ \to Z^+$ decomposes into the $\pm 2i$ eigenspaces of the Clifford action of ω_t. The canonical spinor Φ_t provides a framing of the $-2i$ eigenspace, and we decompose Φ accordingly

$$\Phi = \alpha\Phi_t + \beta \tag{5.15}$$

where α is a function, and β is a section of the $+2i$ eigenspace. Similarly, using the canonical connection A_t we obtain a decomposition

$$A = A_t + a \tag{5.16}$$

for an $i\mathbb{R}$-valued 1-form a. We regard a as a unitary connection $\nabla_a = d + a$ on the trivial line bundle, with curvature given by $F_a = da$. There is a also a unitary connection $\tilde{\nabla}_A$ on the $+2i$ eigenspace $E_+(t)$ obtained from A by orthogonal projection.

The main ingredient for the various compactness results needed in this article is the following *exponential decay estimate*, which follows from the work of Kronheimer–Mrowka [47] and Zhang [81].

Theorem 5.13. *There exists constants $C, \epsilon > 0$ depending on σ, with the following significance:*

if $(A, \Phi, t, s) \in \mathcal{M}_z([\mathfrak{a}], \sigma)$ *for some* $[\mathfrak{a}], z$*, we have the following estimate over* $K \subset Z^+$

$$|1 - |\alpha|^2 + |\beta|^2|^2 + |\beta|^2 + |\nabla_a \alpha|^2 + |\tilde{\nabla}_A \beta|^2 + |F_a|^2 \leq Ce^{-\epsilon s}.$$

Corollary 5.14. *For any element in* $\mathcal{M}_z([\mathfrak{a}], \sigma)$ *there is a gauge representative* (A, Φ, t, s) *of it such that* $A - A_t$ *and* $\Phi - \Phi_t$ *decay exponentially over* K *with first derivatives (with constants* $C, \epsilon > 0$ *only depending on* σ*).*

Proof. The only part which doesn't follow directly from Theorem 5.13 is that $|A - A_t|^2 \leq Ce^{-\epsilon s}$. This is proved exactly as in Corollary 3.16 of [47]. □

5.2.2.3 The boundary of $\mathcal{M}_z([\mathfrak{a}], U, \sigma; \tau)$ *at* $s = +\infty$

We now describe the behaviour of configurations $(A, \Phi, t, s) \in \mathcal{M}_z([\mathfrak{a}], U, \sigma; \tau)$ when s approaches $+\infty$.

Denote by $e_\infty : \Delta^n \to \mathbb{P}(S^+_{(1,p)})$ the map that associates to $t \in \Delta^n$ the fibre over the point $(1, p) \in Z^+$ of the $-2i$-eigenspace for $\rho_t(\omega_t)$, namely the line

$$\mathbb{C} \cdot \Phi_t(1, p) \subset S^+_{(1,p)}.$$

We encountered this map in the proof of Proposition 5.9. Recall that τ provides a translation-invariant unitary splitting $\tau = (\tau_1, \tau_2) : \overline{S^+}|_{[1,+\infty)\times p} \cong \mathbb{C}^2$ as in §5.2.1.2. This provides us with a preferred line $l_\tau \in \mathbb{P}(S^+_p)$, namely that line which corresponds with $0 : 1$ under the identification $\mathbb{P}(S^+_p) \cong \mathbb{P}(\mathbb{C}^2)$ given by τ.

Definition 5.8. The *U-limiting locus* at $s = +\infty$ of σ is the subset $Z_{\infty,\tau}(\sigma) := e_\infty^{-1}(l_\tau) \subset \Delta^n$.

The limiting set at infinity is a compact subset of Δ^n. Later we will require that l_τ is a regular value (by varying τ), so that $Z_{\infty,\tau}(\sigma)$ will be a submanifold (with corners) of Δ^n. The terminology we chose is justified by the next observation:

Lemma 5.15. *Suppose $(A_n, \Phi_n, t_n, s_n) \in \mathcal{M}_z([\mathfrak{a}], U, \sigma; \tau)$ is a sequence of configurations such that $\lim_{n\to+\infty} s_n = +\infty$ and $\lim_{n\to+\infty} t_n = t^*$ for some $t^* \in \Delta^n$. Then t^* lies in $Z_{\infty,\tau}(\sigma) \subset \Delta^n$*

Proof. We choose a family of canonical configurations (A_t, Φ_t) defined for $t \in \Delta^n$, since Δ^n is contractible. By Lemma 4.4 we may assume, after passing to a different gauge, that $\overline{\Phi_t}$ are translation-invariant spinors over the symplectic end K.

By Theorem 5.13, there exist constants $C > 0$ and $\epsilon > 0$ independent of n, such that for any $s \in \mathbb{R}$ and $y \in Y$

$$|\Phi_n(s, y) - \Phi_{t_n}(s, y)| \leq Ce^{-\epsilon s}.$$

Thus

$$|\overline{\Phi}_n(s, y) - \overline{\Phi}_{0,t_n}(s, y)| \leq |\mathcal{R}_0^*(s)||\Phi_n(s, y) - \Phi_{t_n}(s, y)| \leq C|\mathcal{R}_0^*(s)|e^{-\epsilon s}\ .$$

where $|\mathcal{R}_0^*(s)|$ denotes the pointwise norm of the rescaling operator, which for $s \geq 1$ equals 1.

On the other hand, by the definition of $\mathcal{M}_z([\mathfrak{a}], U, \sigma; \tau)$, at the point (s_n, p) the evaluation constraint $\tau_1\overline{\Phi}_n(s_n, p) = 0$ holds. By the above bound, $|\overline{\Phi}_n(s_n, p) - \overline{\Phi}_{t_n}(s_n, p)|$ converges to zero, and hence $\lim_{n\to\infty} \tau_1\overline{\Phi}_{t_n}(s_n, p) = 0$. By translation-invariance we have $\tau_1\overline{\Phi}_{t_n}(1, p) = \tau_1\overline{\Phi}_{t_n}(s_n, p) = 0$. Hence we obtain $\tau_1\overline{\Phi}_{t^*}(1, p) = \lim_{n\to\infty} \tau_1\overline{\Phi}_{t_n}(1, p) = 0$, which means that $e_\infty(t^*) = l_\tau$, as required. □

Definition 5.9. The *U-limiting moduli space* at $s = +\infty$ is the preimage of the U-limiting locus $Z_{\infty,\tau}(\sigma) = e_\infty^{-1}(l_\tau) \subset \Delta^n$ under the map $M_z([\mathfrak{a}], \sigma) \to \Delta^n$. We denote it by $M_z([\mathfrak{a}], Z_{\infty,\tau}(\sigma))$.

The next is the main result of this section. It describes the shape of $\mathcal{M}_z([\mathfrak{a}], U,\ \sigma; \tau)$ as the evaluation constraint goes to $+\infty$.

Theorem 5.16. *Let σ be a C^2 singular chain in $C = C\mathcal{M}(Y, \xi_0) \times \mathcal{P}$. After a C^2 pertubation of σ and a residual choice of splitting $\tau \in \mathrm{U}(2)$, there exists a constant $s_0 > 0$ such that the following holds for all $[\mathfrak{a}], z$ for which the moduli space $\mathcal{M}_z([\mathfrak{a}], U, \sigma; \tau)$ has expected dimension 1:*

- *the moduli spaces $\mathcal{M}_z([\mathfrak{a}], U, \sigma; \tau)$ are transversely cut out and the moduli spaces $M_z([\mathfrak{a}], Z_{\infty,\tau}(\sigma))$ consist of a finite set of transversely cut out points*

- *there is a homeomorphism of the open subset $\{s > s_0\} \cap \mathcal{M}_z([\mathfrak{a}], U, \sigma; \tau)$ with the product $M_z([\mathfrak{a}], Z_{\infty,\tau}(\sigma)) \times (s_0, +\infty)$, compatible with the projection to $(s_0, +\infty)$.*

Proof. We start with some preliminary observations. First, note that the transversality assertion for the moduli spaces $\mathcal{M}_z([\mathfrak{a}], U, \sigma; \tau)$ of dimension 1 follows by an application of the Thom-Smale transversality theorem, in the same way as for the moduli spaces $M_z([\mathfrak{a}], \sigma)$. In this case, again by standard finiteness results (see §A.2) we only have finitely many non-empty $\mathcal{M}_z([\mathfrak{a}], U, \sigma; \tau)$ with dimension 1. Also, the moduli space $M_z([\mathfrak{a}], Z_{\infty,\tau}(\sigma))$ is compact, since its expected dimension is 0. Thus, if it is transversely cut out then it will consist of finitely-many points.

We choose a family of canonical configurations (A_t, Φ_t) parametrised by $t \in \Delta^n$ in translation-invariant form (see Proposition 4.4, Definition 4.10). The open subset $\{s > 0\} \cap \mathcal{M}_z([\mathfrak{a}], \sigma)$ is canonically identified with the product $M_z([\mathfrak{a}], \sigma) \times (0, +\infty)$, compatibly with the projection to $(0, +\infty)$. For this product structure, the canonical line bundle $\mathcal{U} \to \mathcal{M}_z([\mathfrak{a}], \sigma)$ is identified with a pullback to the first factor in the product. Next, we extend the section f_τ of $\mathcal{U}$ (whose zeros give $\mathcal{M}_z([\mathfrak{a}], U, \sigma; \tau) \subset \mathcal{M}_z([\mathfrak{a}], \sigma; \tau))$ to a section F defined over $s = +\infty$ as follows

$$
\begin{aligned}
F : M_z([\mathfrak{a}], \sigma) \times (0, +\infty] &\to \mathcal{U} \\
(A, \Phi, t, s) &\mapsto \tau_1 \overline{\Phi}(s, p), \text{ if } s \neq +\infty \\
(A, \Phi, t, +\infty) &\mapsto \tau_1 \overline{\Phi}_t(1, p).
\end{aligned}
$$

We write F_s for the smooth section given by restriction of F to the slice $M_z([\mathfrak{a}], \sigma) \times \{s\}$. The required result is thus of Implicit Function Theorem type. Namely, from the version of this given in [[49], Lemma 19.3.3] it will follow that the map $F^{-1}(0) \to (0, +\infty]$ (given by the projection $(A, \Phi, t, s) \mapsto s$) defines a *topological* submersion over $(F_\infty)^{-1}(0) = M_z([\mathfrak{a}], Z_{\infty,\tau}(\sigma))$ (and therefore the required homeomorphism $\mathcal{M}_z([\mathfrak{a}], U, \sigma; \tau) \cong M_z([\mathfrak{a}], Z_{\infty,\tau}(\sigma)) \times (s_0, +\infty)$ for some large $s_0 > 0$) provided we can show

(i) F is continuous

(ii) $F_s \longrightarrow F_\infty$ in C^1_{loc}

(iii) F_∞ is transverse to the zero section.

Item (i) follows from the exponential decay estimates (Theorem 5.13). For item (ii) we proceed as follows. Recall that the configuration space $\mathcal{C}_k(K')$ over the symplectic end is a Banach manifold with tangent space

$$T_{(A,\Phi,t)}\mathcal{C}_k(K') \cong \left\{(a,\phi,\dot{t}) \mid a - \frac{\partial}{\partial \dot{t}}A_t \in L^2_k(K,g_t)\,,\ \phi - \frac{\partial}{\partial \dot{t}}\Phi_t \in L^2_k(K,g_t)\,,\ \dot{t} \in T_t\mathcal{CM}(Y,\xi_0)\right\}$$

(see (4.8)) and a Banach space norm induced from a local chart is given by

$$||(a,\phi,\dot{t})|| = ||a - \frac{\partial}{\partial \dot{t}}A_t||_{L^2_k(K,g_t)} + ||\phi - \frac{\partial}{\partial \dot{t}}\Phi_t||_{L^2_k(K,g_t)} + ||\dot{t}||.$$

The vertical components (taken with respect to the obvious connection on $\mathcal{U}$) of the derivatives of the sections F_s and F_∞ are

$$\begin{aligned}(\mathcal{D}f_s)_{(A,\Phi,t)}(a,\phi,\dot{t}) &= \overline{\phi}(s,p)\\ (\mathcal{D}f_\infty)_{(A,\Phi,t)}(a,\phi,\dot{t}) &= \frac{\partial}{\partial \dot{t}}\overline{\Phi}_t(1,p).\end{aligned}$$

We then use the continuous embedding $L^2_{k,\overline{g}_t,\overline{A}_t}(K,\overline{S}^+) \hookrightarrow C^0(K,\overline{S}^+)$ (recall that the *cylindrical* metric is $\overline{g}_t = ds^2 + g_{\xi_t,\alpha_t,j_t}$ over K) together with the identity of Riemannian volume forms $d\mathrm{vol}_{g_t} = s^3 d\mathrm{vol}_{\overline{g}_t}$ to obtain the estimate

$$\begin{aligned}s^{3/2}\cdot|(\mathcal{D}(F_s - F_\infty))_{(A,\Phi,t)}(a,\phi,\dot{t})| &\leq ||s^{3/2}(\overline{\phi} - \frac{\partial}{\partial \dot{t}}\overline{\Phi}_t)||_{C^0(K,\overline{S}^+)}\\ &\leq C\cdot||s^{3/2}(\overline{\phi} - \frac{\partial}{\partial \dot{t}}\overline{\Phi}_t)||_{L^2_{k,\overline{g}_t,\overline{A}_t}(K,\overline{S}^+)}\\ &\leq C||\phi - \frac{\partial}{\partial \dot{t}}\Phi_t||_{L^2_{k,g_t,A_t}(K,S^+)}.\end{aligned}$$

From this we deduce that $||F_s - F_\infty||_{C^1(M_z([\mathfrak{a}],\sigma))} \leq C/s^{3/2}$, and in particular we have C^1_{loc} convergence $F_s \longrightarrow F_\infty$ as $s \longrightarrow +\infty$ follows.

For (iii) recall that $Z_{\infty,\tau} = e_\infty^{-1}(l_\tau)$. The space of unitary splittings τ is identified with the unitary group $\mathrm{U}(2)$. There exists, by Sard's theorem, a residual subset of the space unitary splittings $\tau \in \mathrm{U}(2)$ for which $l_\tau \in \mathbb{P}(S^+_{(1,p)})$ is a regular value of e_∞. It is straightforward to see then that

Lemma 5.17. *If l_τ is a regular value of e_∞, then the map from the configuration space*

$$C_k([0,+\infty)\times Y)|_{\Delta^n} \xrightarrow{F_\infty} \mathbb{C}$$
$$(A,\Phi,t) \mapsto \tau_1\overline{\Phi}_t(1,p)$$

has a regular value at 0.

In §A.1 we establish a general transversality result for moduli spaces with evaluation constraints, which applies to certain evaluation maps that fall into an suitable class (Definition A.1). Lemma 5.17 shows that F_∞ falls into this class. This general result implies in this instance that $M_z([\mathfrak{a}],\sigma) \xrightarrow{F_\infty} \mathcal{U}$ is transverse to the zero section of $\mathcal{U}$. This concludes the proof of Theorem 5.16. □

5.2.2.4 The boundary of $\mathcal{M}_z([\mathfrak{a}],\gamma,\sigma;\kappa)$ at $s = +\infty$

We now carry out an analogous study of the shape as s approaches $+\infty$ of the second kind of parametrised evaluation moduli spaces $\mathcal{M}_z([\mathfrak{a}],\gamma,\sigma;\kappa)$ where $\gamma \subset Y$ is a smooth oriented closed curve. First, have the analogue of Lemma 5.15. This time it involves the map $e^\gamma_\infty : \Delta^n \longrightarrow \mathrm{U}(1)$ which associates to $t \in \Delta^n$ the half-holonomy $\exp\frac{1}{2}\int_{1\times\gamma}\hat{A}_t$.

Definition 5.10. The *γ-limiting locus* at $s = +\infty$ of σ is the subset $Z^\gamma_{\infty,\kappa}(\sigma) = (e^\gamma_\infty)^{-1}(\kappa) \subset \Delta^n$.

Lemma 5.18. *Suppose $(A_n,\Phi_n,t_n,s_n) \in \mathcal{M}_z([\mathfrak{a}],\gamma,\sigma;\kappa)$ is a sequence of configurations such that* $\lim_{n\to+\infty} s_n = +\infty$ *and* $\lim_{n\to+\infty} t_n = t^*$ *for some $t^* \in \Delta^n$. Then t^* lies in $Z^\gamma_{\infty,\kappa} \subset \Delta^n$.*

Proof. Let $a_n = A_n - A_{t_n}$. By Corollary 5.14 we may assume $|a_n|^2 \leq Ce^{-\epsilon s}$ over K. For convenience, regard $\mathrm{U}(1)$ as $i\mathbb{R}/2\pi i\mathbb{Z}$. There we have the identity

$$\frac{1}{2}\int_{s_n\times\gamma}\hat{A}_n - \frac{1}{2}\int_{1\times\gamma}\hat{A}_{t_n} = \int_{s_n\times\gamma} a_n + \frac{1}{2}\int_{s_n\times\gamma-1\times\gamma}\hat{A}_{t_n}.$$

The second term on the right-hand side vanishes (mod $2\pi i\mathbb{Z}$) by the translation-invariance property of the canonical connection A_t. From the exponential decay estimate on $|a_n|$ it follows that the first term goes to zero as $n \to \infty$. The result follows. □

Definition 5.11. The *γ-limiting moduli space* at $s = +\infty$ is the preimage of the γ-limiting locus $Z^{\gamma}_{\infty,\kappa} \subset \Delta^n$ under the map $M_z([\mathfrak{a}],\sigma) \to \Delta^n$. We denote it by $M_z([\mathfrak{a}], Z^{\gamma}_{\infty,\kappa}(\sigma))$.

We have the following analogue of Theorem 5.16, describing the shape of $\mathcal{M}_z([\mathfrak{a}],\gamma,\sigma;\kappa)$ as the evaluation constraint goes to $+\infty$.

Theorem 5.19. *Let σ be a C^2 singular chain in $C = C\mathcal{M}(Y,\xi_0)\times\mathcal{P}$. After a C^2 pertubation of σ and a residual choice of $\kappa \in \mathrm{U}(1)$, there exists a constant $s_0 > 0$ such that the following holds for all $[\mathfrak{a}], z$ such that $\mathcal{M}_z([\mathfrak{a}],\gamma,\sigma;\kappa)$ has expected dimension 1:*

- *all the moduli spaces $\mathcal{M}_z([\mathfrak{a}],U,\sigma;\kappa)$ are transversely cut out and the moduli spaces $M_z([\mathfrak{a}], Z^{\gamma}_{\infty,\kappa}(\sigma))$ consist of a finite set of transversely cut out points*
- *there is a homeomorphism of the open subset $\{s > s_0\} \subset \mathcal{M}_z([\mathfrak{a}],\gamma,\sigma;\kappa)$ with the product $M_z([\mathfrak{a}], Z^{\gamma}_{\infty,\kappa}(\sigma)) \times (s_0,+\infty)$, compatible with the projection to $(s_0,+\infty)$.*

Proof. The strategy is the same as in the proof of Theorem 5.16. Rather than working with the half-holonomy map $h_\gamma(A,\Phi,t,s) = \exp\frac{1}{2}\int_{t\times\gamma}\hat{A}$ we view $\mathrm{U}(1)$ as $i\mathbb{R}/2\pi i\mathbb{Z}$ and work with $f_\gamma(A,\Phi,t,s) = \frac{1}{2}\int_{\{s\}\times\gamma}\hat{A}$. We extend this to a map F defined over $s = +\infty$ in a similar fashion as before:

$$
\begin{aligned}
F : M_z([\mathfrak{a}],\sigma)\times(0,+\infty] &\longrightarrow i\mathbb{R}/2\pi i\mathbb{Z}\cong \mathrm{U}(1)\\
(A,\Phi,t,s) &\mapsto \frac{1}{2}\int_{\{s\}\times\gamma}\hat{A} \text{ if } s\neq+\infty\\
(A,\Phi,t,+\infty) &\mapsto \frac{1}{2}\int_{\{1\}\times\gamma}\hat{A}_t.
\end{aligned}
$$

As in the proof of Theorem 5.16 we need to show that the restrictions to the slices F_s satisfy (i)-(iii). For (iii) we have the statement analogous to Lemma 5.17: for residual $\kappa\in\mathrm{U}(1)$ the map

$$
\begin{aligned}
C_k([0,+\infty)\times Y)|_{\Delta^n} &\xrightarrow{f_\infty} \mathrm{U}(1)\\
(A,\Phi,t) &\mapsto \frac{1}{2}\int_{1\times\gamma}\hat{A}_t
\end{aligned}
$$

has a regular value at κ. Indeed, κ has this property whenever the map $e^{\gamma}_{\infty}:\Delta^n\to\mathrm{U}(1)$ has a regular value at κ. Then the general transversality results of §A.1 imply (iii). This concludes the proof of Theorem 5.19.

□

5.2.2.5 *The compactification of $\mathcal{M}_z([\mathfrak{a}],U,\sigma;\tau)$ and $\mathcal{M}_z([\mathfrak{a}],[\mathfrak{a}],\gamma,\sigma;\kappa)$*

We are now set to describe the compactification of the parametrised evaluation moduli spaces over a simplex σ. This brings together the various moduli spaces that we have studied: $M_z([\mathfrak{a}],\sigma)$ (§4.2.4), $\mathcal{M}_z([\mathfrak{a}],U,\sigma;\tau)$ and $\mathcal{M}_z([\mathfrak{a}],\gamma,\sigma;\kappa)$ (§5.2.2.1). In addition, we also have the usual moduli spaces of Floer trajectories $\breve{M}_z([\mathfrak{a}],[\mathfrak{b}])$ (where we quotient by the reparametrization action of $\mathbb{R}$, as usual), and the U and γ-moduli spaces over cylinders $M_z([\mathfrak{a}],U,[\mathfrak{b}];\tau)$, $M_z([\mathfrak{a}],\gamma,[\mathfrak{b}];\kappa)$ introduced in §5.1.1.6.

First we introduce some definitions. Recall our notation $C := C\mathcal{M}(Y,\xi_0)\times\mathcal{P}$.

Definition 5.12. Given a unitary splitting $\tau\in U(2)$, a C^2 singular simplex $\sigma:\Delta^n\to C$ will be called *τ-transverse* provided it satisfies the following transversality requirements:

(i) σ is transverse to the Fredholm map $\mathfrak{M}(Z^+) \to C$ (see §4.2.3) along components with index $\leq 1 - n$

(ii) the map $\Delta^n \times \mathbb{R} \ni (t, s) \mapsto (\sigma(t), s) \in C \times \mathbb{R}$ is transverse to the Fredholm map $\mathcal{M}(U, Z^+; \tau) \to C \times \mathbb{R}$ (see §5.2.1) along the components of index $\leq 1 - n$.

(iii) the map $e_\infty : \Delta^n \to \mathbb{P}(S^+_{(1,p)})$ has a regular value at $l_\tau \in \mathbb{P}(S^+_{(1,p)})$.

Likewise, given $\kappa \in U(1)$ a C^2 singular simplex σ will be called κ-transverse if it satisfies (i) together with the following analogues of (ii) and (iii):

(i)' the map $\Delta^n \times \mathbb{R} \ni (t, s) \mapsto (\sigma(t), s) \in C \times \mathbb{R}$ is transverse to the Fredholm map $\mathcal{M}(\gamma, Z^+; \kappa) \to C \times \mathbb{R}$ along the components of index $\leq 1 - n$.

(ii)' the map $e^\gamma_\infty : \Delta^n \to \mathrm{U}(1)$, defined in terms of σ, has a regular value at $\kappa \in U(1)$.

By the Thom-Smale transversality Theorem, after a small perturbation one can ensure that any σ becomes τ-transverse for a residual choice of τ (likewise for κ-transverse).

The following result describes the structure of the compactifications of our moduli spaces:

Proposition 5.20. *Let σ be τ-transverse C^2 singular simplex $\sigma : \Delta^n \to C$. If the moduli space $\mathcal{M}_z([\mathfrak{a}], U, \sigma; \tau)$ has expected dimension 0, then it consists of finitely-many transversely cut-out points. If it has expected dimension 1, then it is a C^2 manifold with boundary which admits a compactification $\mathcal{M}^+_z([\mathfrak{a}], U, \sigma; \tau)$ with the structure of a space stratified by manifolds. Its top stratum is given by $\mathcal{M}_z([\mathfrak{a}], U, \sigma; \tau)$ itself, and the boundary of the top stratum consists of configurations of the following types:*

(a) $M_z([\mathfrak{a}], Z_{\infty,\tau}(\sigma))$

(b) the moduli $\mathcal{M}_z([\mathfrak{a}], U, \sigma|_{\Delta^{n-1}_i}; \tau)$ *over the codimension 1 faces* $\Delta^{n-1}_i \subset \Delta^n$ *of* σ

(c) $\breve{M}_{z_1}([\mathfrak{a}], [\mathfrak{b}]) \times \mathcal{M}_{z_0}([\mathfrak{b}], U, \sigma; \tau)$

(d) $\breve{M}_{z_2}([\mathfrak{a}], [\mathfrak{b}]) \times \breve{M}_{z_1}([\mathfrak{b}], [\mathfrak{c}]) \times \mathcal{M}_{z_0}([\mathfrak{c}], U, \sigma; \tau)$

(e) $M_{z_1}([\mathfrak{a}], U, [\mathfrak{b}]; \tau) \times M_{z_0}([\mathfrak{b}], \sigma)$

(f) $M_{z_2}([\mathfrak{a}], U, [\mathfrak{b}]; \tau) \times \breve{M}_{z_1}([\mathfrak{b}], [\mathfrak{c}]) \times M_{z_0}([\mathfrak{c}], \sigma)$

(g) $\breve{M}_{z_2}([\mathfrak{a}], [\mathfrak{b}]) \times M_{z_1}([\mathfrak{b}], U, [\mathfrak{c}]; \tau) \times M_{z_0}([\mathfrak{c}], \sigma)$.

(Here, the middle factor in the triple products must be boundary-obstructed. The concatenation of the homotopy classes z_i in every product must equal z.)

Furthermore, the structure near the boundary strata of type (a),(b),(c),(e) is that of a C^0 manifold with boundary, and the structure near (d), (f),(g) is that of a codimension 1 δ-structure (see [49], Definition 19.5.3.)

The analogous result holds for the γ-moduli spaces.

More generally, a compactification by broken trajectories of the moduli $\mathcal{M}_z([\mathfrak{a}], U, \sigma; \tau)$ of any dimension can be constructed, provided transversality holds. However, we will only use those of dimension 0 or 1. We refer to §A.2 for an outline of the standard technical results that enable us to establish the compactness. We have carried out in this section the analysis of the structure of the compactification near the boundary stratum of type (a). This component of the boundary stratum is the most interesting, and will be the key to the proof of Theorem 1.5 (B). For the strata of type (c)-(g) the required gluing analysis follows similar techniques as those in [49].

5.2.3 The proof of Theorem 1.5 (B)

We are now ready to complete the proof of Theorem 1.5 (B). This follows from chain level identities arising from enumeration of boundary points of $\mathcal{M}_z([\mathfrak{a}], U, \sigma; \tau)$ and $\mathcal{M}_z([\mathfrak{a}], \gamma; \sigma; \kappa)$.

5.2.3.1 Orientations

We explained in §4.2.3.4 how to orient the moduli $M_z([\mathfrak{a}], \sigma)$. We now want to orient the parametrised moduli $\mathcal{M}_z([\mathfrak{a}], U, \sigma; \tau)$ and $\mathcal{M}_z([\mathfrak{a}], \gamma, \sigma; \kappa)$, and for this we first orient the bigger moduli $\mathcal{M}_z([\mathfrak{a}], \sigma)$ that contains them. The latter moduli is defined as the fibre product $\mathrm{Fib}(\pi, \sigma)$

of the natural map $\pi : \mathcal{M}_z([\mathfrak{a}], Z^+) \to C := C\mathcal{M}(Y, \xi_0) \times \mathcal{P}$ (as in (5.10) but projecting the $\mathbb{R}$ factor away) and σ.

To orient $\mathcal{M}_z([\mathfrak{a}], \sigma)$ we need to orient the determinant line $\det\pi$ of the Fredholm map $\pi : \mathcal{M}_z([\mathfrak{a}], Z^+) \to C$. Once that is done $\mathcal{M}_z([\mathfrak{a}], \sigma)$ becomes oriented by Lemma 4.12. Since the moduli $\mathfrak{M}_z([\mathfrak{a}], Z^+)$ is the fibre over $s = 1$ of the natural map

$$\mathcal{M}_z([\mathfrak{a}], Z^+) \to \mathbb{R}$$

then an orientation of $\det\pi$ is determined by an orientation of the determinant line of $\mathfrak{M}_z([\mathfrak{a}], \sigma) \to C$ and the convention that the $\mathbb{R}$ factor *goes first*. This is contrary to the usual fibre-first convention, but agrees with standard conventions in [49].

The remaining moduli spaces are oriented as follows:

- $\mathcal{M}_z([\mathfrak{a}], U, \sigma; \tau)$ is the zero set of a section of a complex line bundle over $\mathcal{M}_z([\mathfrak{a}], \sigma)$, so we orient it as such.

- $\mathcal{M}_z([\mathfrak{a}], \gamma, \sigma; \kappa)$ is the fibre of a map $\mathcal{M}_z([\mathfrak{a}], \sigma) \to \mathrm{U}(1)$, so we orient it using the fibre-first convention.

- $M_z([\mathfrak{a}], Z_{\infty,\tau}(\sigma))$ is the fibre of a map $M_z([\mathfrak{a}], \sigma) \to \mathbb{P}^1$, so we orient it by the fibre-first convention.

- $M_z([\mathfrak{a}], U, [\mathfrak{b}]; \tau)$ and $M_z([\mathfrak{a}], \gamma, [\mathfrak{b}]; \kappa)$ are analogous to the first two bullets.

We refer to the above as the *canonical orientations* of the moduli spaces.

5.2.3.2 Counting solutions to the Seiberg–Witten equations

If σ is a transverse C^2 singular simplex in $C = C\mathcal{M}(Y, \xi_0) \times \mathcal{P}$ (together with an element from the two-element set $\Lambda(\sigma(b))$, where $b \in \Delta^n$ is the barycenter, which we will omit from notation) then we have defined an associated monopole cochain in §4.2.4

$$\psi(\sigma) = \sum_{[\mathfrak{a}],z} \#M_z([\mathfrak{a}],\sigma)\cdot[\mathfrak{a}] \in \widehat{C}^*(Y,\mathfrak{s}_{\xi_0,\alpha_0,j_0}).$$

which induces the family contact invariant **Fc**. Given a τ-transverse C^2 singular simplex $\sigma : \Delta^n \to C$ for some $\tau \in U(2)$, we will now define new associated monopole cochains

$$\theta_\tau(\sigma)\,,\psi_{\infty,\tau}(\sigma) \in \widehat{C}^*(Y,\mathfrak{s}_{\xi_0,\alpha_0,j_0})$$

and if, in addition, σ is κ-transverse for some $\kappa \in U(1)$, we will define new monopole cochains

$$\theta_{\gamma,\kappa}(\sigma),\psi_{\infty,\gamma,\kappa}(\sigma) \in \widehat{C}^*(Y,\mathfrak{s}_{\xi_0,\alpha_0,j_0}).$$

These are given by

$$\begin{aligned}
\theta_\tau(\sigma) &= \sum_{[\mathfrak{a}],z} \big(\#\mathcal{M}_z([\mathfrak{a}],U,\sigma;\tau)\big)\cdot[\mathfrak{a}] \\
\theta_{\gamma,\kappa}(\sigma)(\alpha) &= \sum_{[\mathfrak{a}],z} \big(\#\mathcal{M}_z([\mathfrak{a}],\gamma,\sigma;\kappa)\big)\cdot[\mathfrak{a}] \\
\psi_\infty(\sigma) &= \sum_{[\mathfrak{a}],z} \big(\#M_z([\mathfrak{a}],Z_{\infty,\tau}(\sigma))\big)\cdot[\mathfrak{a}] \\
\psi^\gamma_{\infty,\gamma,\kappa}(\sigma) &= \sum_{[\mathfrak{a}],z} \big(\#M_z([\mathfrak{a}],Z_{\infty,\kappa}(\sigma))\big)\cdot[\mathfrak{a}].
\end{aligned}$$

That all these sums are indeed finite follows from standard compactness arguments as in [49] that we will review in §A.2.

Similar arguments as for Proposition 4.13 show that $\psi_{\infty,\tau}(\sigma)$ and $\psi_{\infty,\gamma}(\sigma)$ satisfy chain map relations (up to signs): $\widehat{\partial}^*\psi_{\infty,\tau}(\sigma) = (-1)^n\psi_{\infty,\tau}(\partial\sigma)$ on simplices of dimension n, and similarly for $\psi_{\infty,\gamma,\kappa}(\sigma)$. In addition, given a σ which is transverse for both τ and τ', the difference $\psi_{\infty,\tau}(\sigma)-\psi_{\infty,\tau'}(\sigma)$ is easily seen to be *exact* (likewise for two κ and κ'), so we obtain well-defined homomorphisms at the level of homology

$$(\psi_\infty)_* ,\ (\psi_{\infty,\gamma})_* : H_*(C(Y,\xi_0);\Lambda_R) \to \widehat{HM}^*(Y,\mathfrak{s}_{\xi_0})$$

defined as follows: any homology class $[T]$ in $C(Y,\xi_0)$ with coefficients twisted by Λ_R can be represented by a cycle T whose simplices are τ-transverse for a fixed τ. Then one sets $(\psi_\infty)_*[T] := [\psi_{\infty,\tau}(T)]$, and likewise for $(\psi_{\infty,\kappa})_*$. We have the following crucial identities:

Proposition 5.21. *For any homology class* $[T] \in H_*(C(Y,\xi_0);\Lambda_R)$ *we have*

$$(\psi_\infty)_*([T]) = \mathbf{Fc}(U \cdot [T])$$
$$(\psi_{\infty,\gamma})_*([T]) = \mathbf{Fc}([\gamma]\cdot[T]).$$

Proof. We explain the first identity, and the second follows identically. Recall from Lemma 5.3 that the cohomology class $\overline{\mu}(1) \in H^2(C(Y,\xi_0);R)$ is Poincaré dual to the zero set of the section $f_\tau : \mathcal{B}^\sigma(Y,\mathfrak{s}_{\xi_0,\alpha_0,j_0}) \to \mathcal{U}$ restricted to $C\mathcal{M}(Y,\xi_0) \subset \mathcal{B}^\sigma(Y,\mathfrak{s}_{\xi_0,\alpha_0,j_0})$. For residual τ the section f_τ will be transverse to the zero section along $C\mathcal{M}(Y,\xi_0)$. Any given homology class $[T]$ can be represented by a C^2 cycle T in $C(Y,\xi_0)$ that intersects transversely the zero set $f_\tau^{-1}(0)$. This intersection is given by restricting T to the union over the limiting loci $Z_{\infty,\tau}(\sigma)$, where σ runs over subfaces $\sigma \subset T$ of all dimensions. This intersection can be given the structure of a cycle T_∞, and it follows that in homology $[T_\infty] = \overline{\mu}(1) \cap [T] =: U \cdot [T]$. The result now follows from applying **Fc** to both sides. □

5.2.3.3 The chain homotopy relation

Theorem 1.5 (B) now follows from combining Proposition 5.21 and

Proposition 5.22. *Let* $\sigma : \Delta^n \to C$ *be a singular* C^2 *chain which is* τ*-transverse. Then the* *following chain homotopy relation (up to signs) holds*

$$\widehat{m}_\tau(U)^*\psi(\sigma) - \psi_{\infty,\tau}(\sigma) = (\widehat{\partial})^*\theta_\tau(\sigma) + (-1)^n\theta_\tau(\partial\sigma).$$

Likewise, if σ is κ-transverse we have

$$\widehat{m}_\kappa(\gamma)^*\psi(\alpha) - \psi_{\infty,\gamma,\kappa}(\sigma) = (\widehat{\partial})^*\theta_{\gamma,\kappa}(\sigma) + (-1)^{n-1}\theta_{\gamma,\kappa}(\partial\sigma).$$

Proof. We show how the first identity is obtained. For the second we proceed identically. We write down for reference the two operators involved (see [[49], Definition 22.1.3, Definition 25.3.3]), namely the differential $\widehat{\partial} : \widehat{C}_*(Y, \mathfrak{s}_{\xi_0}) \to \widehat{C}_{*-1}(Y, \mathfrak{s}_{\xi_0,\alpha_0,j_0})$ and the chain map $\widehat{m}(U) : \widehat{C}_*(Y, \mathfrak{s}_{\xi_0,\alpha_0,j_0}) \to \widehat{C}_{*-2}(Y, \mathfrak{s}_{\xi_0,\alpha_0,j_0})$,

$$\widehat{\partial} = \begin{pmatrix} \partial^o_o & \partial^u_o \\ -\overline{\partial}^s_u\partial^o_s & -\overline{\partial}^u_u - \overline{\partial}^s_u\partial^u_s \end{pmatrix} \tag{5.17}$$

$$\widehat{m}_\tau(U) = \begin{pmatrix} m^o_o(U) & m^u_o(U) \\ \overline{m}^s_u(U)\partial^o_s - \overline{\partial}^s_u m^o_s(U) & \overline{m}^u_u(U) + \overline{m}^s_u(U)\partial^u_s - \overline{\partial}^s_u m^u_s(U) \end{pmatrix}. \tag{5.18}$$

Recall that we are interested in the duals $(\hat{\partial})^*$, $\hat{m}_\tau(U)^*$ of the above operators, acting on cochains.

We first let $[\mathfrak{a}]$ be an irreducible critical point, and z a component for which $\dim \mathcal{M}_z([\mathfrak{a}], U, \sigma; \tau) = 1$. By Proposition 5.20 its compactification $\mathcal{M}_z([\mathfrak{a}], U, \sigma; \tau)$ has a codimension 1 δ-structure near the boundary stratum. This has the desirable property that the total count of boundary points (with orientations) vanishes [[49] , Corollary 21.3.2]. Then, enumerating the points on the boundary strata yields the identity

$$
\begin{aligned}
0 = &+\#M_z([\mathfrak{a}], Z_{\infty,\tau}(\sigma)) \\
&+(-1)^n \cdot \sum_{\text{subfaces } \Delta_i^{n-1} \subset \Delta^n} (-1)^i \#\mathcal{M}_z([\mathfrak{a}], U, \sigma|_{\Delta_i^{n-1}}; \tau) \\
&+ \sum_{[\mathfrak{b}] \in \mathfrak{C}^o, z_1, z_0} \#\breve{M}_{z_1}([\mathfrak{a}], [\mathfrak{b}]) \#\mathcal{M}_{z_0}([\mathfrak{b}], U, \sigma; \tau) \\
&- \sum_{[\mathfrak{b}] \in \mathfrak{C}^s, [\mathfrak{c}] \in \mathfrak{C}^u, z_2, z_1, z_0} \#\breve{M}_{z_2}([\mathfrak{a}], [\mathfrak{b}]) \#\breve{M}_{z_1}([\mathfrak{b}], [\mathfrak{c}]) \#\mathcal{M}_{z_0}([\mathfrak{b}], U, \sigma; \tau) \\
&- \sum_{[\mathfrak{b}] \in \mathfrak{C}^o, z_1, z_0} \#M_{z_1}([\mathfrak{a}], U, [\mathfrak{b}]; \tau) \#M_{z_0}([\mathfrak{b}], \sigma) \\
&+ \sum_{[\mathfrak{b}] \in \mathfrak{C}^s, [\mathfrak{c}] \in \mathfrak{C}^u, z_2, z_1, z_0} \#M_{z_2}([\mathfrak{a}], U, [\mathfrak{b}]; \tau) \#\breve{M}_{z_1}([\mathfrak{b}], [\mathfrak{c}]) \#M_{z_0}([\mathfrak{c}]\ , \sigma) \\
&- \sum_{[\mathfrak{b}] \in \mathfrak{C}^s, [\mathfrak{c}] \in \mathfrak{C}^u, z_2, z_1, z_0} \#\breve{M}_{z_2}([\mathfrak{a}], [\mathfrak{b}]) \#M_{z_1}([\mathfrak{b}], U, [\mathfrak{c}]; \tau) \#M_{z_0}([\mathfrak{c}]\ , \sigma)
\end{aligned}
$$

Let $[\mathfrak{a}]$ be boundary-unstable now. The corresponding enumeration yields the identity

$$\begin{aligned}
0 = &+\#M_z([\mathfrak{a}], Z_{\infty,\tau}(\sigma)). \\
&+(-1)^n \cdot \sum_{\text{subfaces } \Delta_i^{n-1} \subset \Delta^n} (-1)^i \#\mathcal{M}_z([\mathfrak{a}], U, \sigma|_{\Delta_i^{n-1}}; \tau) && (5.19) \\
&+ \sum_{[\mathfrak{b}] \in \mathfrak{C}^o, z_1, z_0} \#\breve{M}_{z_1}([\mathfrak{a}], [\mathfrak{b}]) \#\mathcal{M}_{z_0}([\mathfrak{b}], U, \sigma; \tau) \\
&- \sum_{[\mathfrak{b}] \in \mathfrak{C}^u, z_1, z_0} \#\breve{M}_{z_1}([\mathfrak{a}], [\mathfrak{b}]) \#\mathcal{M}_{z_0}([\mathfrak{b}], U, \sigma; \tau) && (5.20) \\
&- \sum_{[\mathfrak{b}] \in \mathfrak{C}^s, [\mathfrak{c}] \in \mathfrak{C}^u, z_2, z_1, z_0} \#\breve{M}_{z_2}([\mathfrak{a}], [\mathfrak{b}]) \#\breve{M}_{z_1}([\mathfrak{b}], [\mathfrak{c}]) \#\mathcal{M}_{z_0}([\mathfrak{b}], U, \sigma; \tau) \\
&- \sum_{[\mathfrak{b}] \in \mathfrak{C}^o, z_1, z_0} \#M_{z_1}([\mathfrak{a}], U, [\mathfrak{b}]; \tau) \#M_{z_0}([\mathfrak{b}], \sigma) \\
&- \sum_{[\mathfrak{b}] \in \mathfrak{C}^u, z_1, z_0} \#M_{z_1}([\mathfrak{a}], U, [\mathfrak{b}]; \tau) \#M_{z_0}([\mathfrak{b}], \sigma) && (5.21) \\
&+ \sum_{[\mathfrak{b}] \in \mathfrak{C}^s, [\mathfrak{c}] \in \mathfrak{C}^u, z_2, z_1, z_0} \#M_{z_2}([\mathfrak{a}], U, [\mathfrak{b}]; \tau) \#\breve{M}_{z_1}([\mathfrak{b}], [\mathfrak{c}]) \#M_{z_0}([\mathfrak{c}], \sigma) \\
&- \sum_{[\mathfrak{b}] \in \mathfrak{C}^s, [\mathfrak{c}] \in \mathfrak{C}^u, z_2, z_1, z_0} \#\breve{M}_{z_2}([\mathfrak{a}], [\mathfrak{b}]) \#M_{z_1}([\mathfrak{b}], U, [\mathfrak{c}]; \tau) \#M_{z_0}([\mathfrak{c}], \sigma)
\end{aligned}$$

For the origin of the signs above we refer to Lemma A.15[1].

For each of the two cases considered above, the corresponding identity can be written in terms of the natural pairing $\langle \cdot, \cdot \rangle : \widehat{C}^*(Y) \otimes_R \widehat{C}_*(Y) \to R$ as

$$\langle \theta(U)(\sigma), \widehat{\partial}[\mathfrak{a}] \rangle + \langle \theta(U)(\partial\sigma), [\mathfrak{a}] \rangle - \langle \psi(\sigma), \widehat{m}(U)[\mathfrak{a}] \rangle + \langle \psi_\infty(\sigma), [\mathfrak{a}] \rangle = 0.$$

This concludes the proof of the desired identity. □

[1] Again, we encounter the technical point that we must change some signs if we follow the *reducible convention* for orienting the moduli $M_{z_1}([\mathfrak{a}], [\mathfrak{b}])$ or $M_{z_1}([\mathfrak{a}], U, [\mathfrak{b}])$ when both $[\mathfrak{a}], [\mathfrak{b}]$ are boundary-unstable (see §20.6 [49]). This reducible convention is meant when writing the term $-\overline{\partial}_u^u$ in the Floer differential (5.17) and the term $\overline{m}_u^u(U)$ in (5.18). The signs listed in Lemma A.15 follow the usual convention. The only sign that one must add is $(-1)^{\dim_{z_1} M([\mathfrak{a}],[\mathfrak{b}])} = -1$ for line (5.20). In line (5.21) the sign is correct, since the difference between the two conventions is given by the sign $(-1)^{\dim M_{z_1}([\mathfrak{a}],U,[\mathfrak{b}];\tau)} = +1$.

5.3 Exact triangles

For the whole of this section we assume that Λ is a trivial double cover of $C(Y, \xi_0)$ and fix a trivialization. See Corollary 1.6 for a criterion that ensures this and which applies in particular if ξ_0 is strongly fillable. We work throughout with homology and cohomology with coefficients in a ring R.

We recall that $C(Y, \xi_0, B) \subset C(Y, \xi_0)$ denotes the subspace of contact structures ξ which agree with ξ_0 over a Darboux ball B (for ξ_0) around the point $p \in Y$. The goal of this section is to establish Theorem 1.7. We rewrite this result in cohomological terms:

Theorem 5.23. *Associated to any closed contact 3-manifold (Y, ξ_0) for which the local system Λ is trivial, there is a natural diagram which is commutative up to signs*

$$
\begin{array}{ccccccccccc}
\cdots \longrightarrow & \widehat{HM}^*(Y,\mathfrak{s}_{\xi_0}) & \xrightarrow{U} & \widehat{HM}^{*+2}(Y,\mathfrak{s}_{\xi_0}) & \longrightarrow & \widetilde{HM}^*(Y,\mathfrak{s}_{\xi_0}) & \longrightarrow & \widehat{HM}^{*+1}(Y,\mathfrak{s}_{\xi_0}) & \xrightarrow{U} \cdots \\
& \mathbf{Fc}\big\uparrow & & \mathbf{Fc}\big\uparrow & & \widetilde{\mathbf{Fc}}\big\uparrow & & \mathbf{Fc}\big\uparrow & \\
\cdots \longrightarrow & H_*(C(Y,\xi_0)) & \longrightarrow & H_{*-2}(C(Y,\xi_0,B)) & \longrightarrow & H_{*-1}(C(Y,\xi_0,B)) & \longrightarrow & H_{*-1}(C(Y,\xi_0)) & \longrightarrow \cdots
\end{array}
$$

where the top row is the long exact sequence of the mapping cone of U in Floer cohomology, the bottom row is Wang's long exact sequence associated to the Serre fibration $C(Y, \xi_0, B) \to C(Y, \xi_0) \xrightarrow{ev} S^2$, the vertical arrows **Fc** *denote the families contact invariant, and* $\widetilde{\mathbf{Fc}}$ *is another families invariant which is to be defined.*

We recall that the "tilde" group $\widetilde{HM}^*(Y, \mathfrak{s})$ is defined as the cohomology of the algebraic mapping cone of the chain map $\widehat{m}_\tau(U)^* : \widehat{C}^*(Y, \mathfrak{s}) \to \widehat{C}^{*+2}(Y, \mathfrak{s})$, for any choice of $\tau \in U(2)$. We denote $\widetilde{C}^*(Y, \mathfrak{s}) := \mathrm{cone}(\widehat{m}_\tau(U)^*)$, and thus $\widetilde{HM}(Y, \mathfrak{s}) = H^*(\widetilde{C}^*(Y, \mathfrak{s}))$.

Remark 5.9. The conventions we use for the algebraic mapping cone $\mathrm{cone}(f)$ of a chain map $f : A_* \to B_*$ (and similarly for a cochain map) are the following: as a module $\mathrm{cone}(f)_* = A_{*-1} \oplus B_*$, and the differential in the cone is given by

$$d_{\text{cone}} = \begin{pmatrix} -d_A & 0 \\ -f & d_B \end{pmatrix}.$$

Associated to a chain map $f : A_* \to B_*$ there is a sequence of chain maps

$$A_* \xrightarrow{f} B_* \xrightarrow{i} \text{cone}(f) \xrightarrow{\delta} A_{*-1}$$

where $i(\beta) = (0, \beta)$ and $\delta(\alpha, \beta) = \alpha$. We recall that the sequence above becomes exact upon taking homology.

5.3.1 Achieving transversality with constant families of perturbations

To establish the above result, it is convenient to work with transverse singular chains in $C = C\mathcal{M}(Y, \xi_0) \times \mathcal{P}$ a stronger property: that the perturbation term $\mathfrak{p} \in \mathcal{P}$ is constant for each simplex. The main result is the following

Proposition 5.24. *Let $\sigma : \Delta^n \to C\mathcal{M}(Y, \xi_0)$ be any C^2 singular simplex. Then there exists a residual subset of perturbations $\mathfrak{p} \in \mathcal{P}$ and unitary splittings $\tau \in U(2)$ for which the singular chain $\sigma_{\mathfrak{p}} : \Delta^n \to C = C\mathcal{M}(Y, \xi_0) \times \mathcal{P}$ defined by $\sigma_{\mathfrak{p}}(u) = (\sigma(u), \mathfrak{p})$ is τ-transverse.*

We explain now how to establish this result. Let $\mathcal{M}$ stand for either of the moduli spaces $\mathfrak{M}(Z^+)$, $\mathcal{M}(Z^+)$ or $\mathcal{M}(U, Z^+; \tau)$. Recall that there is a natural Fredholm map $\mathcal{M} \xrightarrow{\pi} A \times \mathcal{P}$ where $A = C\mathcal{M}(Y, \xi_0)$ in the first case, and $A = C\mathcal{M}(Y, \xi_0) \times \mathbb{R}$ in the other two. We write $\text{pr} : A \times \mathcal{P} \to C\mathcal{M}(Y, \xi_0) \times \mathcal{P}$ for the natural projection in all cases above. In §A.1 we deal with establishing the various transversality statements used in this paper. From the arguments there, we can deduce a finer transversality property than those stated thus far: that in order to achieve transversality for $\mathcal{M}$ one does not need to consider variations along the A direction. Essentially, this is a consequence of the fact that the fibre product construction of the moduli space involved a restriction map to the slice $0 \times Y$, over which the family of spin-c structures was constant, independent of A. We have the following result, which will follow from §A.1:

Lemma 5.25. *The map* $\mathcal{M} \xrightarrow{\pi} A \times \mathcal{P} \xrightarrow{\mathrm{pr}_1} A$ *is a submersion.*

Proof of Proposition 5.24. Fix a residual τ (such that Lemma 5.17 holds). Then we need to show that $\sigma_{\mathfrak{p}}$ is transverse to $\pi' := \mathrm{pr} \circ \pi : \mathcal{M} \xrightarrow{\pi} A \times \mathcal{P} \xrightarrow{\mathrm{pr}} \mathcal{CM}(Y, \xi_0) \times \mathcal{P}$ along components of $\mathcal{M}$ with $\mathrm{ind}\pi' \leq 1 - n$.

By Lemma 5.25, the product map $\sigma \times \mathrm{id}_{\mathcal{P}} : \Delta^n \times \mathcal{P} \to \mathcal{CM}(Y, \xi_0) \times \mathcal{P}$ is transverse to π', and so their fibre product is transverse:

$$\begin{array}{ccc} \mathcal{M}(\sigma) := \mathrm{Fib}(\sigma \times \mathrm{id}_{\mathcal{P}}, \pi') & \longrightarrow & \mathcal{M} \\ \downarrow{\scriptstyle \pi'_\sigma} & & \downarrow{\scriptstyle \pi'} \\ \Delta^n \times \mathcal{P} & \xrightarrow{\sigma \times \mathrm{id}_{\mathcal{P}}} & = \mathcal{CM}(Y, \xi_0) \times \mathcal{P}. \end{array}$$

Now, the C^2 map $\mathrm{pr}_2 \circ \pi'_\sigma : \mathcal{M}(\sigma) \to \mathcal{P}$ is Fredholm and has index $\mathrm{ind}(\pi'_\sigma) = \mathrm{ind}\pi' + n$, where $\mathrm{ind}\pi'$ depends on the component of $\mathcal{M}$. The Sard-Smale theorem [74] gives us a residual subset of perturbations $\mathfrak{p} \in \mathcal{P}$ which are regular values for the map $\mathrm{pr}_2 \circ \pi'_\sigma$, provided that $\mathrm{ind}(\mathrm{pr}_2 \circ \pi'_\sigma) \leq 1$ (because $\mathrm{pr}_2 \circ \pi'_\sigma$ is C^2). For those $\mathfrak{p}$, the map $\iota_{\mathfrak{p}} : \Delta^n \to \Delta^n \times \mathcal{P}$ given by $u \mapsto (u, \mathfrak{p})$ is transverse to π'_σ, and we obtain a transverse fibre product:

$$\begin{array}{ccc} M(\sigma) := \mathrm{Fib}(\iota_{\mathfrak{p}}, \pi'_\sigma) & \longrightarrow & \mathcal{M}(\sigma) \\ \downarrow & & \downarrow{\scriptstyle \pi'_\sigma} \\ \Delta^n & \xrightarrow{\iota_{\mathfrak{p}}} & \Delta^n \times \mathcal{P} \end{array}$$

A simple diagram chasing argument involving the two diagrams above shows now that $\sigma_{\mathfrak{p}} = (\sigma \times \mathrm{id}_{\mathcal{P}}) \circ \iota_{\mathfrak{p}}$ is transverse to π'. □

5.3.2 The map between triangles

Remark 5.10. The assignments $\sigma \mapsto \psi(\sigma)$, $\psi_{\infty,\tau}(\sigma)$, $\theta_\tau(U)(\sigma)$ satisfied the chain map or chain homotopy up to signs (see Proposition 4.14, Proposition 5.22). To avoid dealing with this plethora of signs, we find it convenient to resolve this issue now, by redefining $\psi_\tau, \psi_{\infty,\tau}, \theta_\tau(U)$ simply by placing the sign $(-1)^{\frac{n(n+1)}{2}}$ whenever they act on a singular simplex σ of dimension n. It is straightforward to verify that these redefined operations now satisfy the chain map or chain homotopy relations strictly:

$$\psi\partial = \widehat{\partial}^*\psi$$
$$\psi_{\infty,\tau}\partial = \widehat{\partial}^*\psi_{\infty,\tau}$$
$$\widehat{m}_\tau(U)^*\psi - \psi_{\infty,\tau} = \widehat{\partial}^*\theta_\tau(U) + \theta_\tau(U)\partial.$$

Consider the subspace $C\mathcal{M}(Y,\xi_0,B) \subset C\mathcal{M}(Y,\xi_0)$ of triples (ξ,α,j) which over B agree with the fixed triple (ξ_0,α_0,j_0). Of course, the forgetful map gives a homotopy equivalence $C\mathcal{M}(Y,\xi_0,B) \xrightarrow{\simeq} C(Y,\xi_0,B)$. The next result uses the τ-transverse singular simplices $\sigma_\mathfrak{p} : \Delta^n \to C = C\mathcal{M}(Y,\xi_0,B)\times\mathcal{P}$ from Proposition 5.24, and it shows that the U action annihilates the image of $H_*(C(Y,\xi_0,B))$ in $H_*(C(Y,\xi_0))$:

Lemma 5.26. *Let $\sigma : \Delta^n \to C\mathcal{M}(Y,\xi_0,B)$ be a singular simplex such that $\sigma_\mathfrak{p}$ is τ-transverse for the perturbation $\mathfrak{p} \in \mathcal{P}$ (see Proposition 5.24). Then $\psi_{\infty,\tau}(\sigma_\mathfrak{p}) = 0$. Thus, by Proposition 5.22 we have*

$$\widehat{m}_\tau(U)^*\psi(\sigma_\mathfrak{p}) = \widehat{\partial}^*\theta_\tau(U)(\sigma_\mathfrak{p}) + \theta_\tau(U)(\partial\sigma_\mathfrak{p}).$$

Proof. Observe that the moduli spaces $M_z([\mathfrak{a}], Z_{\infty,\tau}(\sigma_\mathfrak{p}))$ of dimension 0 are empty. The point is that $\sigma : \Delta^n \to C\mathcal{M}(Y,\xi_0,B)$ parametrises triples that agree with (ξ_0,α_0,j_0) on a neighbourhood of $p \in Y$. Thus the limiting set $Z_{\infty,\tau}(\sigma_\mathfrak{p}) \subset \Delta^n$ must be either empty, or equal to Δ^n. But $Z_{\infty,\tau}(\sigma_\mathfrak{p}) \subset \Delta^n$ is a codimension 2 submanifold with corners that is cut out transversely by the τ-transversality assumption. Thus $Z_{\infty,\tau}(\sigma_\mathfrak{p})$ must be empty, and hence the moduli $M_z([\mathfrak{a}], Z_{\infty,\tau}(\sigma_\mathfrak{p}))$ of dimension 0 are empty. □

With this in place we now start building the map between the exact triangles from Theorem

5.23. First, we define a homomorphism

$$\Psi_\infty : H_*\big(C(Y,\xi_0), C(Y,\xi_0,B)\big) \to \widehat{HM}^{*+2}(Y,\mathfrak{s}_{\xi_0})$$

as follows. Choose a relative homology class in $H_*\big(C(Y,\xi_0), C(Y,\xi_0,B)\big)$ together with a chain representing it, and lift this to a chain T in $C\mathcal{M}(Y,\xi_0)$ with boundary in $C\mathcal{M}(Y,\xi_0,B)$. Choosing a generic τ and $\mathfrak{p}$, we can ensure that for each simplex σ of T the simplex $\sigma_{\mathfrak{p}}$ is τ-transverse. Let $T_{\mathfrak{p}}$ stand for the cycle in $C\mathcal{M}(Y,\xi_0,B)\times\mathcal{P}$ obtained by using the perturbation $\mathfrak{p}$ on each simplex of T. We then have that $\psi_{\infty,\tau}(T_{\mathfrak{p}}) = \psi_{\infty,\tau}(\partial T_{\mathfrak{p}}) = 0$, where the vanishing occurs due to Lemma 5.26. We can thus define $\Psi_\infty([T]) := [\psi_{\infty,\tau}(T_{\mathfrak{p}})]$.

Next, we define a homomorphism

$$\widetilde{\mathbf{Fc}} : H_{*-1}\big(C(Y,\xi_0,B)\big) \to \widetilde{HM}^{*}(Y,\mathfrak{s}_{\xi_0})$$

as follows. We choose a class in $H_*\big(C(Y,\xi_0,B)\big)[1]$ and a cycle in $C(Y,\xi_0,B)$ representing it. Then lift this to a cycle T in $C\mathcal{M}(Y,\xi_0,B)$. By choosing τ and $\mathfrak{p}$ generic we can again ensure that T is made up of simplices σ such that $\sigma_{\mathfrak{p}}$ is τ-transverse. It follows from Lemma 5.26 that the cochain $(\psi(T_{\mathfrak{p}}), -\theta_\tau(U)(T_{\mathfrak{p}})) \in \widetilde{C}^*(Y,\mathfrak{s})$ is closed for the mapping cone differential. Thus, we can define

$$\widetilde{\mathbf{Fc}}([T]) := [(\psi(T_{\mathfrak{p}}), \theta_\tau(U)(T_{\mathfrak{p}}))].$$

Lemma 5.27. *The maps* $\mathbf{Fc}$*,* Ψ_∞ *and* $\widetilde{\mathbf{Fc}}$ *fit into a commutative diagram (up to signs)*

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \widehat{HM}^{*}(Y,\mathfrak{s}_{\xi_0}) & \xrightarrow{U} & \widehat{HM}^{*+2}(Y,\mathfrak{s}_{\xi_0}) & \longrightarrow & \widetilde{HM}^{*}(Y,\mathfrak{s}_{\xi_0}) & \longrightarrow & \widehat{HM}^{*+1}(Y,\mathfrak{s}_{\xi_0}) & \xrightarrow{U} \cdots \\
& \uparrow{\scriptstyle \mathbf{Fc}} & & \uparrow{\scriptstyle \Psi_\infty} & & \uparrow{\scriptstyle \widetilde{\mathbf{Fc}}} & & \uparrow{\scriptstyle \mathbf{Fc}} & \\
\cdots \longrightarrow & H_*(C(Y,\xi_0)) & \longrightarrow & H_*\big(C(Y,\xi_0), C(Y,\xi_0,B)\big) & \longrightarrow & H_{*-1}(C(Y,\xi_0,B)) & \longrightarrow & H_{*-1}(C(Y,\xi_0)) & \longrightarrow \cdots
\end{array}$$

where the top row is the Gysin long exact sequence associated to the mapping cone of $\widehat{m}_\tau(U)$*, and the bottom row is the long exact sequence in homology of the pair* $\big(C(Y,\xi_0), C(Y,\xi_0,B)\big)$*.*

Proof. The third square is immediate. The first square commutes because of the identity $\widehat{m}_\tau(U)\psi - \psi_{\infty,\tau} = \widehat{\partial}^*\theta_\tau(U) + \theta_\tau(U)\partial$, and the second square anti-commutes because of the identity

$$\begin{pmatrix} 0 \\ -\psi_{\infty,\tau} \end{pmatrix} = \begin{pmatrix} \psi\partial \\ \theta_\tau(U)\partial \end{pmatrix} + \begin{pmatrix} -\widehat{\partial}^* & 0 \\ -\widehat{m}_\tau(U) & \widehat{\partial}^* \end{pmatrix} \cdot \begin{pmatrix} \psi \\ \theta_\tau(U) \end{pmatrix}.$$

□

The final step is to identify the bottom row in Lemma 5.27 as Wang's long exact sequence, and this is done by using a standard excision isomorphism

$$H_{n-2}(C(Y,\xi_0,B)) \xrightarrow{\cong} H_n(C(Y,\xi_0), C(Y,\xi_0,B)). \tag{5.22}$$

which follows from the fibration property of $C(Y,\xi_0,B) \to C(Y,\xi_0) \xrightarrow{ev} S^2$ (recall that $ev(\xi) = \xi(p)$). Let us recall how this isomorphism is constructed since we will need it. Let $x_0 \in S^2$ be the point corresponding to the plane $\xi_0(p)$, and $-x_0 \in S^2$ its antipodal. We take the standard CW structure of S^2, where x_0 is the 0-cell, and the 2-cell D^2 is centered at $-x_0$. The map $1 \times \mathrm{pr} : ev^{-1}(-x_0) \times D^2 \to S^2$ which collapses ∂D^2 to the point $x_0 \in S^2$ can be lifted through the fibration $ev : C(Y,\xi_0) \to S^2$ to produce a map of pairs

$$f : (ev^{-1}(-x_0) \times D^2, ev^{-1}(-x_0) \times \partial D^2) \to (C(Y,\xi_0), ev^{-1}(x_0))$$

which at the center $-x_0 \in D^2$ agrees with the fibre inclusion $ev^{-1}(-x_0) \hookrightarrow C(Y,\xi_0)$. The map f is a homotopy equivalence of pairs. The pair $(C(Y,\xi_0), ev^{-1}(x_0))$ is weakly homotopy equivalent to the pair $(C(Y,\xi_0), C(Y,\xi_0,B))$, so their homology is identified. The fibre transport along a path joining x_0 to $-x_0$ combined with the Künneth isomorphism yields an isomorphism

$$t_* : H_{n-2}(C(Y,\xi_0,B)) \xrightarrow{\cong} H_{n-2}(ev^{-1}(-x_0)) \xrightarrow{\cong} H_n(ev^{-1}(-x_0) \times D^2, ev^{-1}(-x_0) \times \partial D^2).$$

The map t_* is independent of the chosen path joining $x_0, -x_0$ by $\pi_1 S^2 = 0$. Then the excision isomorphism (5.22) is concretely described as the map $f_* \circ t_*$. Equipped with this description, we re-identify the map Ψ_*:

Lemma 5.28. *Under the excision isomorphism (5.22), the map Ψ_∞ is identified with the restriction to $H_{*-2}\big(C(Y,\xi_0,B)\big)$ of the families contact invariant* $\mathbf{Fc} : H_{*-2}\big(C(Y,\xi_0,B)\big) \to \widehat{HM}^{*+2}(Y,\mathfrak{s}_{\xi_0})$.

Proof. Choose an $(n-2)$-cycle T in $C(Y,\xi_0,B)$. The homology class $[T]$ corresponds on the right-hand side of (5.22) to the class of the chain $\widetilde{T} = f(T' \times D^2)$, where T' is the cycle in $ev^{-1}(-x_0)$ obtained by transporting T along a path from x_0 to $-x_0$. Thus, we need to compute the class $\Psi_\infty([\widetilde{T}])$.

The chain $\widetilde{T}$ can be lifted to a chain in $C\mathcal{M}(Y,\xi_0)$ with boundary in $C\mathcal{M}(Y,\xi_0,B)$, which we denote by $\widetilde{T}$ as well. As usual, we can assume that all simplices that make up $\widetilde{T}$ are τ-transverse when equipped with a fixed perturbation $\mathfrak{p}$, for a generic choice of τ and $\mathfrak{p}$. We have $\Psi_\infty([\widetilde{T}]) = [\psi_{\infty,\tau}(\widetilde{T})]$, and by construction the chain $\psi_{\infty,\tau}(\widetilde{T})$ agrees with the chain $\psi(T_\infty)$ where T_∞ is obtained by intersecting $\widetilde{T}$ with the union over the limiting loci $Z_{\infty,\tau}(\sigma)$ with σ running over the subfaces of $\tilde{T}$ of all dimensions. By transversality of these intersections, T_∞ can be given the structure of a chain. Now, as in the proof of Proposition 5.9, we know that T_∞ agrees with the intersection of $\widetilde{T}$ and a fibre of $ev : C(Y,\xi_0) \to S^2$. By the description of $\widetilde{T}$ as $f(T' \times D^2)$ we can now see that T_∞ is, in fact, a cycle in $C(Y,\xi_0)$ which is homologous to T. Thus, we have $\Psi_\infty([\widetilde{T}]) = \psi_*([T])$, as required. □

Combining Lemma 5.27 and Lemma 5.28, the proof of Theorem 5.23 is now complete.

Appendix A: Transversality, compactness and orientations

A.1 Transversality

We now take up the task of establishing the transversality results claimed in the previous sections. The arguments used follow quite closely those of [49] and [16], and we will focus on describing the differences. This section has the nature of an appendix.

We recall that we have chosen integers $k \geq 4$ and l with $l - k - 2 \geq 1$.

A.1.1 Main results

We consider the following setup, in the spirit of the one considering thus far. We consider a P-family of Riemannian metrics $\{g_p\}$ on $Z^+ = \mathbb{R} \times Y$. As before, we consider metrics of regularity C^{l-1}. The parametrising space P is a Banach manifold, possibly just finite-dimensional. The cases we have in mind are mainly $P = C\mathcal{M}(Y, \xi_0)$ and $P = C\mathcal{M}(Y, \xi_0) \times \mathbb{R}$. We assume that the metrics g_p coincide with a fixed cylindrical metric $g_0 = dt^2 + g_{0,Y}$ over the region $(-\infty, 1/2] \times Y$. We assume that $K = [1, +\infty) \times Y$ is equipped with a family of almost-Kähler structures $\{(\omega_p, J_p, g_p)\}$ such that $g_p = \omega_p(\cdot, J_p \cdot)$. We assume that each (ω_p, J_p, g_p) makes K an asymptotically flat almost-Kähler end (for the definition see [47] , §3(i)). We also assume that the differences $g_p g_0^{-1}$ are bounded over Z^+ (though not necessarily uniformly in P). There is then a P-family of spin-c structures on Z^+ constructed as in §4.2.1.2 using the triple (ω_0, J_0, g_0) and the P-family of metrics.

We remark at this point that if we consider a compact end or a cylindrical end, rather than an asymptotically flat almost-Kähler end, then the results of this section will still apply.

The corresponding space of configurations (A, Φ, p) over $K' = [0, +\infty) \times Y$, and its quotient by the group $\mathcal{G}_{k+1}(K')$ of L^2_{k+1} gauge transformations asymptotic to the identity, are denoted by $C_k(K')$ and $\mathcal{B}_k(K')$ respectively. $\mathcal{B}_k(K')$ is a C^{l-k-2} Banach manifold away from the reducible locus. The moduli space $\mathfrak{M}(K') \subset \mathcal{B}_k(K') \times \mathcal{P}$ is defined as the zero set of the perturbed Seiberg–Witten map sw_η, which is naturallly a section of a Hilbert bundle Υ_{k-1} over $\mathcal{B}_k(K') \times \mathcal{P}$. The perturbation η is taken of the form

$$\eta(A, \Phi, p, \mathfrak{p}) = \varphi_p^1 \hat{\mathfrak{q}}(A, \Phi) + \varphi_p^2 \hat{\mathfrak{p}}(A, \Phi) + \varphi_p^3 \hat{\mathfrak{p}}_{K,p}.$$

Here $\mathfrak{q}$ is a fixed admissible perturbation, and $\mathfrak{p}_{K,p}$ is the Taubes perturbation used earlier. We consider P-families of functions satisfying similar constraints as before. Namely, (i) cutoff functions φ_p^1 which are identically 1 on a neighbourhood of $(-\infty, 0]$ and vanishing on a neighbourhood of $[1/2, +\infty)$; (ii) bump functions φ_p^2 with compact support inside $(0, 1/2)$; (iii) and cutoff functions φ_p^3 which are identically 1 over $[1, +\infty)$ and vanish on a neighbourhood of $(-\infty, 1/2]$. One can include more perturbation terms in η to adjust to each particular situation, and as long as they don't depend on $\mathcal{P}$ the results of this section apply.

Below we introduce a suitable class of maps $\mathrm{ev} : C_k(K') \to V$ that we call *good* (Definition A.1 below). These are equivariant sections of a $\mathcal{G}_{k+1}(K')$-equivariant fibre bundle V over $C_k(K')$, and we wish to impose the constraint that $\mathrm{ev}(A, \Phi, p, \mathfrak{p}) = \sigma(A, \Phi, p, \mathfrak{p})$ on the moduli $\mathfrak{M}(K')$. Here σ is a fixed equivariant section of V (Definition A.1). We prove:

Proposition A.1. *The Seiberg–Witten map* $\mathrm{sw}_\eta : \mathcal{B}_k(K') \times \mathcal{P} \to \Upsilon_{k-1}$ *is transverse to the zero section. If* ev *is a good evaluation map, then* ev *and* σ *are transverse sections of* $V \to \mathfrak{M}(K')$.

Thus, the space $\mathfrak{M}_k(K', \mathrm{ev}) := \mathfrak{M}(K') \cap \mathrm{ev}^{-1}(\mathrm{Im}\sigma)$ is a Banach manifold, of class C^{l-k-2}. We have two restriction maps to the configuration space of the boundary

$$R_+ : M^*([\mathfrak{a}], (-\infty, 0] \times Y) \to \mathcal{B}^*_{k-1/2}(Y) \tag{A.1}$$

$$\mathfrak{R}_- : \mathfrak{M}(K', \mathrm{ev}) \to \mathcal{B}^*_{k-1/2}(Y). \tag{A.2}$$

and their fibre product $\mathfrak{M}([\mathfrak{a}], Z^+, \mathrm{ev}) = \mathrm{Fib}(R_+, \mathfrak{R}_-)$ is the moduli space we are interested in. The main result of this section is:

Proposition A.2. *For a good evaluation map, the maps R_+ and $\mathfrak{R}_-$ are transverse. Thus, the moduli space $\mathfrak{M}([\mathfrak{a}], Z^+, \mathrm{ev})_P$ is a C^{l-k-2} Banach manifold. The map $\mathfrak{M}([\mathfrak{a}], Z^+, \mathrm{ev}) \to P \times \mathcal{P}$ is C^{l-k-2} and Fredholm.*

We now describe the class of evaluation maps for which our transversality results apply.

Definition A.1. Fix a smooth $\mathcal{G}_{k+1}(K')$-equivariant fibre bundle $V \to \mathcal{C}_k(K')$ with finite-dimensional fibre, together with a preferred equivariant section σ and a connection on V along σ (that is, a connection on the pullback fibre bundle $\sigma^* V$). A *good* evaluation map compatible with such data is a section $\mathrm{ev} : \mathcal{C}_k(K') \to V$ subject to the following conditions:

(i) ev is a $\mathcal{G}_{k+1}(K')$-equivariant section

(ii) ev is transverse to σ as sections of $V \to \mathcal{C}_k(K')$

(iii) There exists a compact set $E \subset (1/2, 1) \times Y$ with $([0, +\infty) \times Y) \setminus E$ *connected* such that, for any $(A, \Phi, p) \in \mathrm{ev}^{-1}(\mathrm{Im}\sigma)$, all the smooth configurations tangent to (A, Φ, p) of the form

$$(a, \phi, 0) \in T_{(A,\Phi,p)}\mathcal{C}_k([0, +\infty) \times Y)$$

which are compactly supported away from E are contained in

$$T_{(A,\Phi,p)}\big(\mathrm{ev}^{-1}(\mathrm{Im}\nu)\big) = \ker(\mathcal{D}\mathrm{ev} - \mathcal{D}\sigma)_{(A,\Phi,p)}.$$

In (iii), $\mathcal{D}$ denotes the differential of a section projected onto the the vertical direction using the connection V defined along σ.

The evaluation constraints we have considered thus far in the article fall into the above category. These are:

Example A.1. One of the main examples (see §5.2.1) is the evaluation map $\mathrm{ev} : (A, \Phi, p) \mapsto \tau_1 \overline{\Phi}(x_0)$ induced by an unitary splitting τ of the fibre of the spinor bundle S^+ at a point $x_0 \in$

$(1/2, 1) \times Y$. Here V is the trivial bundle with fibre $\mathbb{C}$ carrying the $\mathcal{G}_{k+1}(K')$-action $v \cdot \lambda = v(x_0)\lambda$, and σ is the zero section. The subset E can be taken to be the point x_0.

In §5.2.1 we considered the moduli $\mathcal{M}([\mathfrak{a}], Z^+, U; \tau)$ with an evaluation constraint that travelled along the $\mathbb{R}$ direction: $\tau_1\overline{\Phi}(s, y_0) = 0$, $s \in \mathbb{R}$. By applying an $\mathbb{R}$-family of diffeomorphisms taking the point (s, y_0) to $(1/2, 1) \times \{y_0\}$ we see that the situation considered in §5.2.1 fall into our current setup.

Example A.2. Another example (see §5.2.1) is the half-holonomy evaluation map, associated to a smooth oriented closed curve $\gamma \subset Y$, given by $h_\gamma(A, \Phi, p) = \exp\frac{1}{2}\int_{s_0 \times \gamma} \hat{A}$, where s_0 is a fixed number in $(1/2, 1)$. Here, the fibre bundle V is equivariantly trivial, with fibre $\mathrm{U}(1)$, and we take σ constant. E can be taken to be $s_0 \times \gamma \subset (1/2, 1) \times Y$.

Example A.3. In §5.2.2 we considered a map that evaluates the canonical spinors at a point x_0. The zero set of this map are the limiting moduli space $M([\mathfrak{a}], Z_{\infty,\tau}(\sigma))$ (Definition 5.9). This was defined, after choosing an unitary splitting, by $\mathrm{ev} : (A, \Phi, p) \mapsto \tau_1\overline{\Phi}_p(x_0)$. The bundle V of which ev is a section is a vector bundle with trivial $\mathcal{G}_{k+1}(K')$ action, and σ is the zero section. This defines a good evaluation map for generic unitary splittings (Lemma 5.17). An analogous map was considered for the half-holonomy of the canonical connections.

Recall from §4.2.2.2 that $C(K') \to P$ is a bundle of affine Hilbert spaces equipped with a preferred connection on $C(K') \to P$ i.e. a complementary (horizontal) subbundle for the vertical subbundle of $TC_k(K')$.

Definition A.2. An admissible evaluation map $\mathrm{ev} : C_k(K') \to V$ is *very good* for the data V, σ if the transversality condition (ii) from Definition A.1 can be achieved without variation along the horizontal direction of $TC(K')$.

Remark A.1. Examples A.1 and A.2 are very good, while A.3 is not.

Finally, we will show:

Proposition A.3. *For a very good evaluation map* ev*, the map* $\mathfrak{M}([\mathfrak{a}], Z^+, \mathrm{ev}) \to P$ *is a submersion.*

A.1.2 Proof of Proposition A.1

Let $\gamma = (A, \Phi, p, \mathfrak{p})$ be a configuration in $C_k([0, +\infty) \times Y)$ solving the equations $\mathrm{sw}_\eta(\gamma) = 0$, and denote by $\mathbf{d}_\gamma : L^2_{k+1}([0, +\infty) \times Y, i\mathbb{R}) \to T_p P \times L^2_k([0, +\infty) \times Y, i\Lambda^1 \oplus S^+)$ the linearisation of the gauge action at γ. To establish the first statement in Proposition A.1 it suffices to show the stronger result that the operator $Q_\gamma = (\mathcal{D}\mathrm{sw}_\eta)_\gamma + \mathbf{d}^*_\gamma$ is surjective. This operator takes the form

$$L^2_k(K', i\Lambda^1 \oplus S^+) \times T_p P \times \mathcal{P} \to \\ L^2_{k-1}(K', i\mathfrak{su}(S^+) \oplus S^- \oplus i\mathbb{R})$$

The desired surjectivity is established in [[16], p.51] using similar ideas to [49]. We explain how to adapt these ideas to the case of the moduli space with evaluation constraint $\mathfrak{M}(K', \mathrm{ev})$. For this suppose that $\gamma = (A, \Phi, p, \mathfrak{p})$ satisfies, in addition, the constraint $\mathrm{ev} = \sigma$. We have the vertical derivative at γ of $\mathrm{ev} : C_k([0, +\infty) \times Y)_P \to V$, which is a linear map of the form

$$\mathcal{D}\mathrm{ev}_\gamma : L^2_k(K', i\Lambda^1 \oplus S^+) \times T_p P \times \mathcal{P} \to V_0$$

where V_0 denotes the fibre of V at γ. We wish to establish the surjectivity of the operator $Q_\gamma + (\mathcal{D}\mathrm{ev} - \mathcal{D}\sigma)_\gamma$, which takes the form

$$L^2_k(K', i\Lambda^1 \oplus S^+) \times T_p P \times \mathcal{P} \to \\ L^2_{k-1}(K', i\mathfrak{su}(S^+) \oplus S^- \oplus i\mathbb{R}) \oplus V_0.$$

Equivalently, because $(\mathcal{D}\mathrm{ev} - \mathcal{D}\sigma)_\gamma$ is surjective (condition (ii) of Definition A.1), it suffices to prove the surjectivity of

$$\ker(\mathcal{D}\mathrm{ev} - \mathcal{D}\sigma)_\gamma \xrightarrow{Q_\gamma} L^2_{k-1}(K', i\mathfrak{su}(S^+) \oplus S^- \oplus i\mathbb{R}). \tag{A.3}$$

Lemma A.4. *For all γ with $\mathrm{sw}_\eta(\gamma) = 0$ and $\mathrm{ev}(\gamma) = \sigma(\gamma)$, the operator (A.3) is surjective.*

Proof. We follow the argument in the proof of [[49], Proposition 24.3.1], and explain the necessary modifications. We first show that the image of $\ker(\mathcal{D}\text{ev} - \mathcal{D}\sigma)_\gamma$ under Q_γ is dense in the L^2 topology on the target. Suppose not, for a contradiction, and hence choose a non-zero element $V \in L^2$ which annihilates the image of $\ker(\mathcal{D}\text{ev} - \mathcal{D}\sigma)_\gamma$.

In particular, it annihilates the image under $\mathcal{D} := Q_\gamma(-,-,0,0)$ of the subspace consisting of all smooth configurations $(a, \phi, 0, 0)$ compactly supported away from E (see (iii) of Definition A.1). Now, $\mathcal{D}$ is an elliptic differential operator, so by elliptic regularity we obtain that V is in $L^2_{1,\text{loc}}$ on $([0,+\infty)\times Y)\setminus E$. In particular V is in L^2_1 on the collar neighbourhood $[0,1/2)\times Y$, where it satisfies the formal adjoint equation $\mathcal{D}^*V = 0$. By the unique continuation principle (see [49]: Lemma 7.1.3 for the cylindrical case, and the argument in Lemma 7.1.4 for arbitrary manifolds), because V is not identically zero over $K' = [0,+\infty) \times Y$ and E does not disconnect this set (see (iii) of Definition A.1), we know that V does not vanish identically on the collar $[0,1/2) \times Y$. The fact that $\mathcal{D}^*$ satisfies the unique continuation property follows from [[49], eq. (24.15)]. We thus obtain that the restriction of V to the boundary $0 \times Y$ is non-zero, again by the unique continuation principle [[49] , Lemma 7.1.3].

However, using the argument in the proof of [[49] ,Corollary 17.1.5] we can show that the restriction must be zero, by orthogonality of V to the image. This gives the desired contradiction.

Thus, the image under Q_γ of $\ker\mathcal{D}(\text{ev} - \sigma)_\gamma$ is dense in the L^2 topology. As we mentioned in the previous section, $\mathcal{D} : L^2_k \to L^2_{k-1}$ is surjective [[16], p.51], for which the argument is similar but simpler than this. Hence, the image of $\ker(\mathcal{D}\text{ev} - \mathcal{D}\sigma)_\gamma$ under Q_γ is of finite-codimension and dense in L^2, so (A.3) is surjective.

□

The previous lemma, together with the fact that Q_γ is also surjective on the bigger domain, and the surjectivity of $(\mathcal{D}\text{ev} - \mathcal{D}\sigma)_\gamma$, complete the proof of Proposition A.1 with the aid of the following observation from linear algebra:

Lemma A.5. *Suppose* $X \xrightarrow{q} Y$ *and* $X \xrightarrow{e} V$ *are linear maps of vector spaces. Assume that the following maps are surjective:* q *,* e *and the restriction* $\ker e \xrightarrow{q} Y$*. Then* $\ker q \xrightarrow{e} V$ *is also* *surjective.*

Proof. The cokernel of $\ker e \xrightarrow{q} Y$ is

$$\frac{V}{e(\ker q)} = \frac{e(X)}{e(\ker q)} \cong \frac{X}{e^{-1}\big(e(\ker q)\big)} = \frac{X}{\ker q + \ker e}$$
$$\cong \frac{X/\ker q}{\ker e/\ker q \cap \ker e} \cong q(X)/q(\ker e) = 0.$$

□

Remark A.2. The proof of Lemma A.4 shows that the surjectivity of (A.3) is already achieved by the tangent configurations $\{(a, \phi, 0, 0)\} \subset \ker(\mathcal{D}\mathrm{ev} - \mathcal{D}\sigma)_\gamma$. In particular, the map $\mathfrak{M}(K', \mathrm{ev}) \to \mathcal{P}$ is a submersion. If, in addition, $(\mathcal{D}\mathrm{ev} - \mathcal{D}\sigma)_\gamma$ achieves surjectivity without varying in T_pP (the "very good" condition), then the map $\mathfrak{M}(K', \mathrm{ev}) \to P \times \mathcal{P}$ will also be a submersion.

A.1.3 Proof of Propositions A.2 and A.3

For Proposition A.2 we need to establish the transversality of the fibre product. In other words, we need to check that the sum of the derivatives

$$(dR_+)_a + (d\mathfrak{R}_-)_b : T_aM^*([\mathfrak{a}], (-\infty, 0] \times Y) \oplus T_b\mathfrak{M}(K', \mathrm{ev}) \to T_{[\mathfrak{c}]}\mathcal{B}_{k-1/2}(Y)$$

is surjective for each (a, b) in the fibre product $\mathrm{Fib}(R_+, \mathfrak{R}_-)$, and $[\mathfrak{c}]$ the restriction to the boundary.

The sum $((dR_+)_a + (d\mathfrak{R}_-)_b)(-, -, 0, 0)$, i.e. acting on tangent directions which vanish on the P and $\mathcal{P}$ directions, is a Fredholm operator. This can be extracted from [16], Lemma 26 (see assertions (3),(4),(7),(8)). Thus, $(dR_+)_a + (d\mathfrak{R}_-)_b$ has finite dimensional cokernel. This, together with Lemma A.6, coming up next, shows that $(dR_+)_a + (d\mathfrak{R}_-)_b$ is surjective.

Lemma A.6. *Let* $\gamma = (A, \Phi, p, \mathfrak{p}) \in \mathfrak{M}([0, +\infty) \times Y, \mathrm{ev})_P$, *and let* $[\mathfrak{c}] = \mathfrak{R}_-(\gamma)$. *Then*

$$(d\mathfrak{R}_-)_\gamma : T_\gamma\mathfrak{M}(K', \mathrm{ev}) \to T_{[\mathfrak{c}]}\mathcal{B}_{k-1/2}(Y)$$

has dense image in the $L^2_{1/2}$ topology.

Proof. We follow the proof of [[49] ,Lemma 24.4.8]. The result will follow if we show that the following operator has dense image in the $L^2 \times L^2_{1/2}$ topology. It is the operator given by the restriction of $(\mathcal{D}\mathrm{sw}_\eta)_\gamma$ to $\ker(\mathcal{D}\mathrm{ev} - \mathcal{D}\sigma)_\gamma \oplus \mathcal{P}$, coupled with the derivative of the restriction $\tilde{\mathfrak{R}}_-$ to the configuration space of the boundary $C_{k-1/2}(Y)$, which has the form

$$\begin{aligned}\ker(\mathcal{D}\mathrm{ev} - \mathcal{D}\sigma)_\gamma \oplus \mathcal{P} \to & L^2_{k-1}(K', i\mathfrak{su}(S^+) \oplus S^-) \\ & \oplus L^2_{k-1/2}(Y; i\Lambda^1 \oplus S_Y).\end{aligned} \tag{A.4}$$

Here S_Y is the restriction of S^+ to $0 \times Y$, and note that

$$\ker(\mathcal{D}\mathrm{ev} - \mathcal{D}\sigma)_\gamma \subset L^2_k(K'; i\Lambda^1 \oplus S^+) \oplus T_p P.$$

We suppose for a contradiction that the image of this operator is not dense in $L^2 \times L^2_{1/2}$, and pick a non-zero $(V, v) \in L^2 \times L^2_{-1/2}$ which annihilates the image. By considering directions in $C_k(K')$ which are tangent to the gauge-orbit of γ (these are contained in $\ker(\mathcal{D}\mathrm{ev} - \mathcal{D}\sigma)_\gamma$ by (i) of Definition A.1) we see that v is orthogonal to the directions tangent to the gauge-orbit through $\gamma_{|Y}$.

Consider the restriction map r to the dt component of the connection form at the boundary. We couple the previous operator with r and the operator $\mathbf{d}^*_\gamma$ to obtain an operator on $\ker(\mathcal{D}\mathrm{ev} - \mathcal{D}\sigma)_\gamma$ by restriction of

$$\begin{aligned} L^2_k(K'; i\Lambda^1 \oplus S^+) \oplus T_p P \oplus \mathcal{P} \xrightarrow{\mathfrak{Q} \oplus r} & \\ & L^2_{k-1}(K'; i\mathfrak{su}(S^+) \oplus S^-) \oplus L^2_{k-1}(K'; i\mathbb{R}) \\ & \oplus L^2_{k-1/2}(Y; i\Lambda^1 \oplus S_Y) \oplus L^2_{k-1/2}(Y; i\mathbb{R}).\end{aligned}$$

The image of $\ker(\mathcal{D}\mathrm{ev} - \mathcal{D}\sigma)_\gamma$ under this operator is orthogonal to $(V, 0, v, 0)$. The operator

$Q := \mathfrak{Q}(-, 0, 0)$ is elliptic. As in the proof of Lemma A.4, $(V, 0, v, 0)$ is orthogonal to the image of the smooth configurations $(a, \phi, 0, 0)$ that are compactly supported away E, which are contained inside $\ker(\mathcal{D}\mathrm{ev} - \mathcal{D}\sigma)_\gamma$ by (iii) of Definition A.1. Elliptic regularity then implies that V is in $L^2_{1,\mathrm{loc}}$ away from E; so V is in L^2_1 on the collar $[0, 1/2) \times Y$ since $E \subset (1/2, 1)$. Thus, V satisfies the formal adjoint equation $Q^* V = 0$ over the collar $[0, 1/2) \times Y$, and so V does not vanish identically over this region by the unique continuation principle (similar argument as in the proof of Lemma A.4).

From this point on, the argument of [[49] ,Lemma 24.4.8] carries through without modification. Namely, by integrating by parts we see that $V_{|Y} = -v$ (under standard identifications of the corresponding bundles), and combining this with the fact that v was orthogonal to the gauge orbit, an argument as in [[49] ,Lemma 15.1.4] shows that V is orthogonal to the gauge-orbit on every slice $t \times Y$. Finally, the argument of [[49] , Proposition 15.1.3] produces, because V does not vanish identically on the collar, a perturbation $\mathfrak{t} \in T_{\mathfrak{p}}\mathcal{P} = \mathcal{P}$ for which the derivative of (A.4) in the direction of $(0, 0, 0, \mathfrak{t})$ is not orthogonal to (V, v), a contradiction. □

If ev is very good, then no variation in the P direction (horizontal) will be needed to achieve transversality in the previous Lemma. This, together with the Remark at the end of the previous subsection, gives us the stronger result of Proposition A.3.

A.2 Compactness

Here we briefly describe some of the compactness results that lead to the construction of the compactified moduli spaces by broken configurations. Large part of the material presented here is a straightforward adaptation of results found in [47], [81] and [49].

The main moduli spaces that will concern us in this section are $\mathfrak{M}(Z^+)$ and $\mathcal{M}(Z^+)$, the latter because it contains the parametrised evaluation moduli $\mathcal{M}(U, Z^+; \tau)$ and $\mathcal{M}(\gamma, Z^+; \kappa)$. For simplicity in notation we state all results for $\mathcal{M}(Z^+)$ or $\mathcal{M}(U, Z^+; \tau)$, but we note that the corresponding results for $\mathfrak{M}(Z^+)$ are analogous and simpler. As before, we use the notation $\mathcal{M}(\sigma)$ for the fibre product of $\mathcal{M}(Z^+) \to \mathcal{CM}(Y, \xi_0) \times \mathcal{P}$ and a C^2 singular simplex $\sigma : \Delta^n \to \mathcal{CM}(Y, \xi_0) \times \mathcal{P}$. $M(\sigma)$ is a C^2 manifold with corners provided transversality holds, and its points consist of gauge-equivalence classes $[(A, \Phi, t, s)]$ of Seiberg–Witten monopoles with $t \in \Delta^n$, $s \in \mathbb{R}$. The projection to the $s \in \mathbb{R}$ coordinate is denoted $\pi_{\mathbb{R}} : \mathcal{M}(\sigma) \to \mathbb{R}$. The simplex $\sigma : \Delta^n \to \mathcal{CM}(Y, \xi_0) \times \mathcal{P}$ will be kept fixed throughout this section.

It is also convenient to introduce the moduli space $\mathcal{M}_{\text{loc}}(Z^+)$ of gauge-equivalence classes of solutions to the same equations as $\mathcal{M}(Z^+)$, also approaching the canonical configurations in L^2_k on the conical end, but with *no* asymptotics to critical points on the cylindrical end. The relevant gauge group involved in the quotient is now the topological group $\mathcal{G}_{\text{loc}}$ of locally L^2_{k+1} gauge transformations which along the conical end approach the identity in L^2_k. The moduli space $\mathcal{M}_{\text{loc}}(Z^+)$ is not a Banach manifold, but carries a natural topology – that of convergence in L^2_k away from infinite cylindrical regions $(-\infty, l) \times Y$. We use the notation $\mathcal{M}_{\text{loc}}(\sigma)$ for the corresponding space obtained by a fibre product as above.

A.2.1 A local compactness result

The exponential decay estimates of Theorem 5.13 can be interpreted as telling us that certain energy along the conical end for configurations $(A, \Phi, t, s) \in \mathcal{M}(\sigma)$ is uniformly bounded, by a constant depending on σ, and hence that the conical end K behaves like a compact end for the purpose of the compactness analysis of $\mathcal{M}(\sigma)$. We now introduce the relevant notion of energy along the cylindrical energy, and describe the main local compactness result.

We fix $r \geq 1$. Later we will require that r is large enough, depending on σ only, so that for all configurations $(A, \Phi, t, s) \in \mathcal{M}(\sigma)$ we have $|\alpha| \geq 1/2$ (using the notation of (5.16-5.15)) over the portion $[r, +\infty) \times Y$ of the conical end K. That this can be done follows from the exponential decay estimate of Theorem 5.13. Let $Z^+_r = Z^+ \setminus (r, +\infty) \times Y$.

Throughout the article we have been fixing an admissible perturbation $\mathfrak{q}$ of the Chern-Simons-Dirac functional on $(Y, \mathfrak{s}_{\xi_0, \alpha_0, j_0})$. By the construction in [[49], §10.1], the admissible perturbation $\mathfrak{q}$ is the formal L^2-gradient of some gauge-invariant function f on the configuration space $\mathcal{C}(Y)$.

Definition A.3. The *cylindrical energy* of a configuration $\gamma = (A, \Phi, t, s) \in \mathcal{M}_z([\mathfrak{a}])$ is

$$\begin{aligned}\mathcal{E}_r(\gamma) =& \frac{1}{4}\int_{Z_r^+} F_{\hat{A}} \wedge F_{\hat{A}} - \int_{r\times Y} \langle \Phi|_{r\times Y}, D_B\Phi|_{r\times Y} \rangle \\ &+ \int_{r\times Y} (H/2)|\Phi|^2 + 2f([\mathfrak{a}]\,). \end{aligned} \tag{A.5}$$

Above, B denotes the restriction of A to the boundary $\partial Z_r^+ = r \times Y$. By H we denote the mean curvature vector field of the boundary $\partial Z_r^+ = r \times Y$.

That one should just integrate over $Z_r^+ \subset Z^+$ was proposed by B.Zhang (see p.54 [81]). The point of cutting off at r is that $\mathcal{E}_r(\gamma)$ approches $+\infty$ as r grows. This can be deduced from Lemma A.7 below. In [49] this type of energy is called *topological*: the analogous integral over a compact manifold with a cylindrical end attached only depends on the critical point $[\mathfrak{a}]$, the homotopy class z and the chosen perturbation $\mathfrak{q}$. This interpretation is lost in our case, due to the cutting off that is forced upon us, but we do have the identity

$$\mathcal{E}_r(\gamma) = 2\mathrm{CSD}_{\mathfrak{q}}(\mathfrak{a}) - 2\mathrm{CSD}(\gamma|_r) + \frac{1}{4}\int_{Z_r^+} F_{\hat{A}_0} \wedge F_{\hat{A}_0} \tag{A.6}$$

whose terms we describe now. First recall that for a closed oriented 3-manifold Y with a spin-c structure $\mathfrak{s}$, the Chern-Simons-Dirac functional (see [49], §4.1) is defined on the configuration space of pairs (B, Ψ) by

$$\mathrm{CSD}(B, \Psi) = -\frac{1}{8}\int_{r\times Y} (\hat{B} - \hat{B}_0) \wedge (F_{\hat{B}} + F_{\hat{B}_0}) + \frac{1}{2}\int_{r\ \times Y} \langle D_B\Psi, \Psi\rangle. \tag{A.7}$$

The above formula needs the choice of a base spin-c connection B_0. Then in formula (A.6), $\mathrm{CSD}_{\mathfrak{q}} = \mathrm{CSD} + f$ is the $\mathfrak{q}$-perturbed Chern-Simons-Dirac functional for $(Y, \mathfrak{s}_{\xi_0,\alpha_0,j_0})$ and some choice of base connection B_0. The term $\gamma|_r$ is the restriction of γ onto the slice $r \times Y$. We have chosen a spin-c connection A_0 over Z_r^+ which becomes translation invariant over the cylindrical end with the form $A_0 = d/dt + B_0$, and we use the restriction of A_0 onto the slice $r \times Y$ as base connection for the function CSD on configurations on the slice $r\times Y$. The identity (A.6) is obtained by integrating by parts as in [[49], §4.1].

That the cylindrical energy provides a good notion of energy along the cylindrical end is provided by the fact that it controls the L^2 norms of $F_{\hat{A}}$, Φ and $\nabla_A \Phi$ over compact sets:

Lemma A.7. *There exists a constant $C > 0$ depending on σ, such that for any configuration $\gamma = (A, \Phi, t, s) \in \mathcal{M}(\sigma)$ we have the following estimate: for any $l \leq 0$*

$$\mathcal{E}_r(\gamma) \geq \frac{1}{16} \int_{[l,r\ ] \times Y} (|F_{\hat{A}}|^2 + (|\Phi|^2 - C)^2 + |\nabla_A \Phi|^2) - C(r - l + 1).$$

The proof of the above is analogous to that of Lemma 24.5.1 in [49]. By an argument as in [[47], pp. 26-27], we can combine Theorem 5.13 and Lemma A.7 and obtain, following the standard compactness argument (based on the proof of Theorem 5.1.1 in [49]), the following local compactness result:

Proposition A.8. *For any sequence $\gamma_n \in \mathcal{M}(\sigma)$ with uniform bounds $\mathcal{E}_r(\gamma_n) \leq C$ and $-C \leq \pi_{\mathbb{R}}(\gamma_n) \leq C$, there exist a subsequence which converges in $\mathcal{M}_{\mathrm{loc}}(\sigma)$.*

At this point the compactness of the moduli spaces of broken configurations $\mathcal{M}_z^+([\mathfrak{a}], U, \sigma; \tau)$, $\mathcal{M}_z^+([\mathfrak{a}], \gamma, \sigma; \kappa)$ or $M_z^+([\mathfrak{a}], \sigma)$ follows. We state the result for the first. The broken configurations that can appear in the 1-dimensional case were listed in Proposition 5.20, and in the general one may see further breaking on the cylindrical end. The statement that we obtain is the following, and its proof follows the arguments of §16.1 and §24.6 of [49]:

Corollary A.9. *For a fixed $[\mathfrak{a}]$ and $C > 0$, the space of broken configurations $\gamma \in \bigcup_z \mathcal{M}_z^+([\mathfrak{a}], U, \sigma; \tau)$ with $\mathcal{E}_r(\gamma) \leq C$ is compact. In particular, each $\mathcal{M}_z^+([\mathfrak{a}], U, \sigma; \tau)$ is compact.*

Above, the cylindrical energy $\mathcal{E}_r$ has been extended to broken configurations γ as in [49]: by adding up the energies of each component of γ. We recall that the energy of a configuration γ in the cylinder moduli space $M_z([\mathfrak{a}], [\mathfrak{b}])$ is $2 \cdot (\mathrm{CSD}_{\mathfrak{q}}(\mathfrak{a}) - \mathrm{CSD}_{\mathfrak{q}}(\mathfrak{b}))$ provided γ approaches $\mathfrak{a}$ and $\mathfrak{b}$ on the corresponding ends. The second assertion in Corollary A.9 uses that $\mathcal{E}_r$ is bounded on $\mathcal{M}_z^+([\mathfrak{a}], U, \sigma; \tau)$, which can be seen from (A.6) combined with Lemma A.10 below.

Lemma A.10. *There is a constant $C > 0$ depending on σ such that for any $\gamma = (A, \Phi, t, s) \in \mathcal{M}(U, \sigma; \tau)$ one has $|\mathrm{CSD}(\gamma|_r) - \mathrm{CSD}((A_t, \Phi_t)|_r)| \leq C$.*

Proof. This follows from the exponential decay estimates in Theorem 5.13 and Corollary 5.14. □

A.2.2 Finiteness results

We now outline how to deduce the finiteness result below. This result is the input needed to conclude that the counts of zero dimensional moduli in this paper are indeed finite. We state our results for $\mathcal{M}_z([\mathfrak{a}], U, \sigma; \tau)$ but the same holds for $\mathcal{M}^+_z([\mathfrak{a}], \gamma, \sigma; \kappa)$ or $M^+_z([\mathfrak{a}], \sigma)$.

Proposition A.11. *Suppose that the moduli spaces $\mathcal{M}_z([\mathfrak{a}], U, \sigma; \tau)$ of expected dimension at most 1 are transversely cut out. Then there exist only finitely many pairs $([\mathfrak{a}], z)$ such that the compactified moduli spaces $\mathcal{M}^+_z([\mathfrak{a}], U, \sigma; \tau)$ are non-empty and of dimension ≤ 1.*

Remark A.3. The reason why the dimension is cut to at most 1 has to do with the fact that we have been working with C^l contact structures and C^2 simplices. This poses a problem if we want that all moduli spaces of all dimensions are transversely cut out after perturbing σ, due to the assumptions of the Thom-Smale transversality theorem. Raising the differentiability of our data would allow us to conclude the above result for moduli of higher dimensions.

The main estimate one needs to prove the above is

Lemma A.12 (Bounds on energy by dimension)**.** *There exists constants $r \geq 1$ $C > 0$ depending on σ such that the following holds. For any $\gamma \in \mathcal{M}_z([\mathfrak{a}], U, \sigma; \tau)$ we have*

$$e - C \leq \mathcal{E}_r(\gamma) + 4\pi^2(\mathrm{gr}_z([\mathfrak{a}], U, \sigma; \tau) - 2\iota([\mathfrak{a}])) \leq e + C$$

where $\mathrm{gr}_z([\mathfrak{a}], U, \sigma; \tau)$ denotes the expected dimension of $\mathcal{M}_z([\mathfrak{a}], U, \sigma; \tau)$, $\iota([\mathfrak{a}])$ is defined in [[49], p.286] and $e \in \mathbb{R}$ is a constant only depending on σ and the image of the critical point $[\mathfrak{a}]$ under the blow-down map $\mathcal{B}^\sigma_{k-1/2}(Y) \to \mathcal{B}_{k-1/2}(Y)$.

Proof. The corresponding result for the topological energy over a compact manifold with boundary would state that the quantity in the middle, denote it $Q(\gamma)$, only depends on the blow-down of $[\mathfrak{a}]$ (see Proposition 24.6.6 in [49] and its proof). This time, given two configurations $\gamma \in \mathcal{M}_z([\mathfrak{a}], U, \sigma; \tau)$, $\tilde{\gamma} \in \mathcal{M}_{\tilde{z}}([\tilde{\mathfrak{a}}], U, \sigma; \tau)$ with $[\mathfrak{a}]$ and $[\tilde{\mathfrak{a}}]$ having the same blow-down, their difference in Q can be computed using (A.6) and we see

$$Q(\gamma) - Q(\tilde{\gamma}) = -2\mathrm{CSD}(\gamma|_r) + 2\mathrm{CSD}(\tilde{\gamma}|_r) \tag{A.8}$$

We want to establish that $|Q(\gamma) - Q(\tilde{\gamma})| \leq C$ for a constant C only depending on σ, and this follows from Lemma A.10. □

Lemma A.13. *Suppose the moduli spaces $\mathcal{M}_z([\mathfrak{a}], U, \sigma; \tau)$ of dimension ≤ 1 are transversely cut out. Then for fixed $[\mathfrak{a}]$ there are only finitely many z for which the compactification $\mathcal{M}^+_z([\mathfrak{a}], U, \sigma; \tau)$ is non-empty and of dimension ≤ 1.*

Proof. We note that Lemma A.12 also holds for broken trajectories, with identical proof. For $[\mathfrak{a}]$ and z with $\mathcal{M}^+_z([\mathfrak{a}], U, \sigma; \tau)$ non-empty, and transversely cut out, we obtain from Lemma A.12 that any broken trajectory γ in the moduli space has

$$\mathcal{E}_r(\gamma) \leq C - \mathrm{gr}_z([\mathfrak{a}], U, \sigma; \tau) + 8\pi^2\iota(\mathfrak{a}) \leq C + 8\pi^2\iota(\mathfrak{a})$$

where the second inequality follows from $\mathrm{gr}_z([\mathfrak{a}], U, \sigma; \tau) \geq 0$ because the moduli is non-empty and transverse. Since $\mathfrak{q}$ is an admissible perturbation there are finitely many critical points in the blow-down, and the quantity $\iota([\mathfrak{a}])$ depends on the blow-down of $[\mathfrak{a}]$ only. So we obtain a uniform bound $\mathcal{E}_r(\gamma) \leq C$. Then Corollary A.9 yields finitely many such z. □

Proof of Proposition A.11. If the first Chern class of contact structure ξ_0, or equivalently that of the spin-c structure $\mathfrak{s}_{\xi_0}$, is non-torsion, then there are only finitely-many critical points $\mathfrak{a}$ and the result follows from Lemma A.13.

In the torsion case, we can still argue that there is a bound, independent of $[\mathfrak{a}]$ or z, on the cylindrical energy of all broken configurations. Indeed, consider just the case of an unbroken configuration $[\gamma] \in \mathcal{M}_z([\mathfrak{a}], U, \sigma; \tau)$ and the identity (A.6) for $\mathcal{E}_r(\gamma)$. Since the Chern-Simons-Dirac function is fully gauge-invariant in the torsion case, then there is a bound $|\mathrm{CSD}_{\mathfrak{q}}(\mathfrak{a})| \leq C$, since $\mathrm{CSD}_{\mathfrak{q}}$ only depends on the blow-down of the critical point $\mathfrak{a}$, for which there are only finitely-many possibilities. Also there is a bound $|\mathrm{CSD}(\gamma|_r)| \leq C$ from applying Lemma A.10. The remaining term in (A.6) can also be bounded, so this shows that $\mathcal{E}_r(\gamma)$ is bounded. The case of a broken configuration is no different.

Now, Lemma A.12 provides upper and lower bounds on

$$\mathcal{E}_r(\gamma) + 4\pi^2(\dim \mathcal{M}_z([\mathfrak{a}], U, \sigma; \tau) - 2\iota([\mathfrak{a}])).$$

Since we have upper and lower bounds on both the energy and dimension, we obtain $|\iota([\mathfrak{a}])| \leq C$. This gives finitely-many choices for $[\mathfrak{a}]$ again. □

A.3 Orientations

We described in §4.2.3.4 and §5.2.3.1 the rule for orienting all the moduli spaces in this article, which we called the *canonical orientations*. Whenever these moduli are 0-dimensional and we use them to make counts of points, each point is counted with a sign corresponding to its canonical orientation (relative to the natural orientation of a point). The compactifications $M_z^+([\mathfrak{a}], \sigma)$ and $\mathcal{M}_z^+([\mathfrak{a}], U, \sigma; \tau)$ of 1-dimensional moduli are 1-dimensional stratified spaces with a codimension-1 δ-structure near its boundary – a more general form than a manifold with boundary structure (see [49], Definition 19.5.3). In this situation each boundary point inherits a *boundary orientation* (see [49], Definition 20.5.1) generalising the usual outward-normal first convention for orienting the boundary of a manifold. The total enumeration of the boundary points of the compactified 1-dimensional moduli equals zero, provided the boundary points are counted with their boundary orientation.

The next two results compare the canonical and boundary orientations for the relevant moduli spaces. These provide the final touch to the proofs of Proposition 4.13 and Proposition 5.22.

Lemma A.14. *Let $M_z([\mathfrak{a}],\sigma)$ be a 1-dimensional moduli. For each of its codimension-1 stratum components listed in Proposition 4.14, the difference between the canonical and boundary orientation is given by the sign*

(a) $+1$

(b) $(-1)^{\dim M_{z_1}([\mathfrak{b}],[\mathfrak{c}])} = -1.$

(c) $(-1)^{n-1}(-1)^i$ *for the moduli over the face* $\Delta^{n-1}_i \subset \Delta^n$.

Proof. (a) and (b) are analogous to cases (i) and (iii) in Proposition 25.2.2 of [49].

For (c) we sketch the main idea. The key result is the following (see [49], p.379, formula (20.3)): if P_1 and P_2 are two Fredholm linear maps of Banach spaces, and the determinant lines $\det P_1$ and $\det P_2$ are oriented, then both $\det(P_1 \oplus P_2)$ and $\det(P_2 \oplus P_1)$ inherit orientations in a natural way, which under the obvious isomorphism $\det(P_1 \oplus P_2) = \det(P_2 \oplus P_1)$ differ by the sign

$$(-1)^{\mathrm{ind} P_1 \times \dim \mathrm{coker} P_2 + \dim \mathrm{coker} P_1 \times \mathrm{ind} P_2}.$$

Suppose now $P_2 = 0_N : N \to 0$ is the zero map out of a finite-dimensional oriented vector space N. We also assume $N = \mathbb{R} \times B$ is a product of oriented vector spaces, and that the orientation on N is the product orientation. Then we write $0_N = 0_{\mathbb{R}} \oplus 0_B$, and the previous result now gives us that the orientations of $\det(P_1 \oplus 0_{\mathbb{R}} \oplus 0_B)$ and $\det(\oplus_{\mathbb{R}} \oplus P_1 \oplus 0_B)$ differ by the sign $(-1)^{\dim \mathrm{coker} P_1}$.

Going back to our case of interest, what we want is to compute the boundary orientation (relative to the canonical orientation) of the boundary stratum component $M_z([\mathfrak{a}],\partial\sigma)$ of $M_z([\mathfrak{a}],\sigma)$, where $\sigma : \Delta^n \to \mathcal{CM}(Y,\xi_0) \times \mathcal{P}$ is a singular simplex of dimension n and $M_z([\mathfrak{a}],\sigma)$ is 1-dimensional. The deformation operator for a configuration γ in $M_z([\mathfrak{a}],\sigma)$ (say lying over the point $u \in \Delta^n$) can be transformed by a homotopy to an operator $P \oplus 0_N$ where P is the deformation operator for the configuration γ in the moduli over a point $M_z([\mathfrak{a}],\sigma(u))$ and $N := T_u\Delta^n$ is the tangent space to the simplex. When u lies on the interior of the face Δ_i^{n-1}, whose boundary orientation is given by the sign $(-1)^i$, we decompose $N = \mathbb{R} \oplus B$ where $B = T_u \Delta_i^{n-1}$ (with boundary orientation) and $\mathbb{R}$ is the outward-normal direction. The number $\dim \operatorname{coker} P$ is $n-1$, which by the above arguments gives the sign (c). □

Lemma A.15. *Let $\mathcal{M}_z([\mathfrak{a}], U, \sigma; \tau)$ be a* 1*-dimensional moduli. For each of its codimension-*1 *stratum components listed in Proposition 5.20, the difference between the canonical and boundary orientation is given by the sign*

(a) $+1$

(b) $(-1)^n(-1)^i$ *for the moduli over the face* $\Delta_i^{n-1} \subset \Delta^n$

(c) $+1$

(d) $(-1)^{\dim M_{z_1}([\mathfrak{b}],[\mathfrak{c}])} = -1$

(e) -1

(f) $(-1)^{\dim M_{z_1}([\mathfrak{b}],[\mathfrak{c}])+1} = +1$

(g) $(-1)^{\dim M_{z_2}([\mathfrak{a}],[\mathfrak{b}])+\dim M_{z_1}([\mathfrak{b}],U,[\mathfrak{c}])} = -1$

Proof. (a) is clear, and (b) is analogous to (c) of the previous Lemma. (c) and (d) are analogous to cases (i) and (iii) in Proposition 25.2.2 of [49], whereas (e), (g) and (f) are analogous to cases (i),(ii) and (iii) in Proposition 26.1.7 of [49]. □

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