Research Statement: Symplectic Properties of Milnor Fibres
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Overview. My research pertains to an area of pure mathematics called symplectic geometry. It is a rapidly developing field, with modern tools drawing from many different areas of mathematics. Modern geometry is centred around the study of manifolds, smooth objects that at small enough scale look like the standard space of a fixed dimension; for instance, the surface of a ball is a 2D-manifold, standard space-time is a 4D-manifold, and the parameter space for a biological experiment might be an 18D-manifold. Symplectic manifolds are equipped with extra structure that generalizes conservation laws from classical mechanics. Also, some models in string theory, a branch of physics, allow any symplectic manifold in lieu of space-time. This has been a tremendous stimulus; duality ideas in physics have led to mirror-symmetry, a booming field that relates symplectic geometry with a very different looking part of mathematics.

Two major open questions are: ‘What are the transformations (that is, global symmetries) of a symplectic manifold?’ and ‘What subspaces can one fit into it?’ The start of my thesis gives answers to the first question for infinitely many new examples; these have more internal structure than the ones previously studied. As for the second question, we care about smooth subspaces that have a physical interpretation: Lagrangian submanifolds – e.g. all possible positions of a particle at rest, or all possible velocities of a particular point in space. The most fickle and interesting ones are exact, an extra rigidity condition. In the second part of my thesis, I construct exact Lagrangian submanifolds in an infinite collection of examples; many were ‘invisible’ to the structures we previously knew of – an exciting development. Also, I solve mirror symmetry conjectures for a family of these examples.

Dehn twists. Some of the few known transformations are called Dehn twists. Let’s first describe these for 2D surfaces, all of which are symplectic. Start with a closed curve without self-intersections (e.g. curve $a$ or $b$ on figure 1). Cut the surface open along it. The resulting object has two boundary components, each a circle. Twist each of the boundaries to the right by 180°, and glue the edges back (see figure 1). Jumping from the first to the third image, we get a transformation, the Dehn twist about $a$.

Curves on surfaces are Lagrangian circles (1D-spheres). Symplectic manifolds of any dimension can have Lagrangian spheres, and one can define a Dehn twist about any one of these analogously to the 2D case. When two curves, or spheres, do not intersect, the corresponding two twists commute: if you perform both twists, you will get the same result irrespective of which one you do first. When two curves intersect exactly once, the twists satisfy a more complicated relation, which has also been generalized to higher dimensional twists.

When the curves intersect twice or more, you can take any sequence of positive or negative twists, and unless you use exactly the inverse sequence (for instance, $a$, then $b$, then inverse-$b$, then inverse-$a$), you can never get back to where you started. Mathematically, we say the Dehn twists about those curves have no relations: things are as complicated as possible.

In an article to be published in the Journal of Topology, submitted here, I generalize this result to higher-dimensional Dehn twists. To prove this, I extend some of the ideas used for surfaces; however, the heart of the proof is different: it uses not only the major tool in the field, called Floer cohomology, but also recent and delicate refinements of that invariant, whose syntax belong to a branch of mathematics called homological algebra. Additionally, I show the

![Figure 1. Dehn twist about the curve $a$.](image-url)
theorem applies to a vast collection of examples (in particular, symplectic manifolds associated to most polynomials).

**Lagrangian submanifolds.** The symplectic manifolds we consider are also at the heart of singularity theory, a field tied to the parts of mathematics that explain discontinuities and abrupt changes.

Let’s start with an example. Consider the function $f(x, y) = x^2 - y^2$. The pre-image of 0 – that is, the collection of points $(x, y)$ such that $f(x, y) = 0$ – consists of two lines crossing at the origin, a singular space (see figure 2). However, the pre-image of 0.5 is a smooth space: two hyperbolas. The same is true for all other values, e.g. 0.8, and the shape is the same. We say that $f$ represents an isolated singularity at the origin.

If we instead use complex numbers, the smooth pre-image of $x^2 - y^2$ becomes two-dimensional, and looks like a stretched cylinder; think of taking the hyperbolas of figure 2, and spinning them around the $y$–axis. This is called the Milnor fibre of the singularity. One can do this for any isolated singularity. Milnor fibres encode information about the singular spaces, and are studied of their own right as symplectic objects.

The first interesting case is when singularities involve three variables (e.g. $x, y$ and $z$). Here Milnor fibres are four-dimensional, and their Lagrangians are surfaces. We’ll care about tori, closed surfaces with one hole – e.g. crusts of doughnuts – that are arguably the simplest surfaces after spheres.

Previous work had concentrated on the geometry of a special family known as ‘simple’ or ‘modality-zero’ singularities. The only known exact Lagrangians in any Milnor fibre were spheres. Moreover, we knew that there were no other possibilities for simple singularities, and, for a larger collection, that all exact Lagrangians could be “built” from spheres – where “built” has technical meaning.

I construct exact Lagrangian tori in the Milnor fibres of all non-simple singularities. Further, for the most elementary of these (‘modality-one’), I show the tori cannot be “built” from spheres. To the study of the geometry of Milnor fibres, this is akin to discovering that a house has a floor that you were previously unaware of – and opens many new questions.

I also resolve existing mirror-symmetry conjectures about for modality-one singularities. As well as fine-tuning the large duality picture, this should allow us to turn complicated questions on one side into easier questions on the other.