Meaning of associativity (informal guide).

Comes from sets and maps of sets

Sets X, Y, Z, W, maps $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$

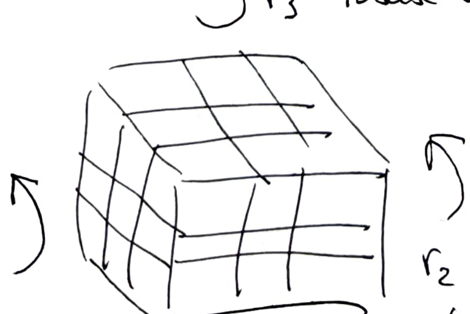
$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend left, "gf"] \arrow[rrr, bend left=60, "h(gf)"] \arrow[rrr, bend right=60, "(hg)f"'] & Y \arrow[r, "g"] \arrow[rr, bend right, "hg"'] & Z \arrow[r, "h"] & W
\end{tikzcd}$$

$(hg)f = h(gf)$ same map $x \in X \longmapsto h(g(f(x)))$

Meaning of commutativity

1) Operations on "far-away" or disjoint parts of a structure commute.

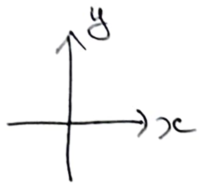
a) Symmetries of Rubik's Cube



permuting 2 disjoint subsets of 9 little cubes $\Rightarrow$

$r_1 r_2 = r_2 r_1$ commute

$r_1 r_3 \neq r_3 r_1$ do not commute.

b) 

r_1: reflect about y-axis

$(x, y) \to (-x, y)$

r_2: reflect about x-axis

$(x, y) \longrightarrow (x, -y)$

$r_1 r_2 = r_2 r_1$ act on $\mathbb{R}^2$ on different coordinates

$$\mathbb{R}^2 = \underset{\underset{r_1}{\circlearrowright}}{\mathbb{R}} \oplus \underset{\underset{r_2}{\circlearrowright}}{\mathbb{R}}$$

similar to (a)

2) commutativity of $+$ in $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$

operator $n \longmapsto n+1$ (add 1) $A(n) = n+1$

A - operator $\Rightarrow$ powers of A, $A^m = A \dots A$

powers of A commute

A^{-1} inverse (act on $\mathbb{Z}$) $n \mapsto n-1$

$\sqrt[m]{A} = A^{1/m}$ powers of $A^{1/m}$ commute $A^{1/m}(x) = x + \frac{1}{m}$

acts on rationals $\mathbb{Q}$

$A^{n/m}$, $\rightsquigarrow$ complete to $\mathbb{R}$

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ "small" structures, built out of powers of a single operator $\Rightarrow$ commutativity.