Modern Algebra II, Spring 2022, Instructor M. Khovanov

Homework 1, due Wednesday Jan 26.

Use notes for lecture 1 or read Rings section in Rotman. (In Rotman all rings are commutative starting from page 8). For more examples, you can read Howie Section 1.1 or Judson Section 16.1 (see a link to Judson's book at the bottom of our webpage.).
In this homework set, rings are not necessarily commutative and they contain 1. Note that $1 = 0$ in R if and only if R is the zero (trivial) ring.

1. (15 points) (a) For a ring R we defined

$$R^* = \{a \in R | \exists b, ab = ba = 1\}$$

as the set of invertible elements of R. Prove that R^* is naturally a group under multiplication.

(b) Determine the groups of invertible elements R^* in the rings $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$.

(c) Give an example of a ring R such that the group R^* is not commutative.

2. (10 points) (a) Recall from the lecture that we defined $\mathbb{Z}/n$ as the ring of residues modulo n. Review the definition of $\mathbb{Z}/n$ via cosets $a + n\mathbb{Z}$ from the first semester of the course. Explain why the multiplication in $\mathbb{Z}/n$ is associative by manipulating cosets.

(b) Write down the group of invertible elements in the ring $\mathbb{Z}/10$. Is this group cyclic? Can you find n such that the group of invertible elements $(\mathbb{Z}/n)^*$ is not cyclic? (Hint: try various powers of 2 for n.)

3. (10 points) We say that $S \subset R$ is a subring of R if S is an abelian group (under addition in R), contains $1 \in R$ and closed under multiplication.

(a) Prove that the intersection $R_1 \cap R_2$ of two subrings of R is a subring of R.

(b) Prove that $\mathbb{Z}\left[\frac{1}{n}\right] = \{\frac{m}{n^k} | m \in \mathbb{Z}, k \in \mathbb{N}\}$ is a subring of $\mathbb{Q}$. Here we fix $n > 1$. (We briefly discussed this example in class.)

4. (15 points) We worked out in class that $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} | a, b \in \mathbb{Q}\}$ is a ring, under the usual addition and multiplication of real numbers (and it's a subring of $\mathbb{R}$). Likewise, determine which of the

following subsets of $\mathbb{R}$ are rings, with respect to addition and multiplication of real numbers. Give brief justifications or explanations.

(a) $\mathbb{Z}$,

(b) $5\mathbb{Z} = \{5n | n \in \mathbb{Z}\}$,

(c) $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \ : \ a, b \in \mathbb{Z}\}$,

(d) $\mathbb{Q}[\sqrt{5}] = \{a + b\sqrt{5} \ : \ a, b \in \mathbb{Q}\}$,

(e) $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \ : \ x \geq 0\}$,

(f) $R = \{a + b\sqrt[3]{3} \ : \ a, b \in \mathbb{Q}\}$,

(g) $\mathbb{Q}[\sqrt[3]{3}] = \{a + b\sqrt[3]{3} + c\sqrt[3]{9} : a, b, c \in \mathbb{Q}\}$.

5. (a) (10 points) Give an example of an object X whose symmetry group $\mathrm{Sym}(X)$ is the cyclic group C_n of order n. (Hint: the symmetry group of a regular n-gon is the dihedral group D_n. Can you break the symmetry down to C_n?) This exercise is purposely vague, not specifying what we mean by an object. You are free to use graphs, polygons, higher-dimensional shapes, sets with additional structure, etc. as objects.

(b) (optional) Can you give an example of an object with the symmetry group $\mathbb{Z}$? With the symmetry group $C_n \times C_m$?