Modern Algebra II, spring 2022

Homework 10, due Wednesday April 13.

1.(30 points) Recall that the commutator of elements a, b of a group G is defined by $[a, b] = aba^{-1}b^{-1}$. Define $[G, G]$ as the set of all possible products $c_1 \dots c_k$, where each $c_i = [a_i, b_i]$ is the commutator of some elements $a_i, b_i \in G$. Such a product can have any number k of terms.

(a) Prove that $[G, G]$ is a normal subgroup of G. First show that the inverse of a commutator is a commutator. Also check that conjugating the commutator by an element of G gives you a commutator (this was discussed in class). Then put these and other arguments together for a proof (you can first prove that $[G, G]$ is a subgroup and then prove that it's normal).

(b) Prove that $[G, G]$ is the trivial subgroup if and only if G is abelian.

(c) We proved in class that $[S_3, S_3] = A_3$. Mimic that proof to show that $[S_4, S_4] = A_4$. Here S_n, respectively A_n, is the symmetric group, respectively alternating group, of permutations of $\{1, \dots, n\}$. (In fact, $[S_n, S_n] = A_n$ for all n.)

2.(20 points) Review the definition of characters $\sigma : G \longrightarrow E^*$ of a group G in a field E (see class notes or Rotman p. 76 section "Independence of Characters").

(a) Take $G = C_3 = \{1, g, g^2 | g^3 = 1\}$ to be the cyclic group of order 3. Describe all characters of G in the field $\mathbb{C}$ of complex numbers. How many are there? (If you'd like a warm-up, first start with the cyclic group C_2 and its characters in the field $\mathbb{R}$ of real numbers. How many characters can you find?)

(b) Show directly that these characters of G are linearly independent over $\mathbb{C}$. (As an example, review our proof that characters of $\mathbb{Z}$ are linearly independent over $\mathbb{R}$. A very similar argument works here.) If you're not sure what to do, solve this problem first for the cyclic group C_2 and its characters in the field $\mathbb{R}$ of real numbers.

(c) (optional, extra credit, 10 points). Let $G = C_3$ and consider the field $\mathbb{F}_{16}$ of order 16. Pick a model for that field as $\mathbb{F}_2[\alpha]/(p(\alpha))$ for an irreducible polynomial $p(x) \in \mathbb{F}_2[x]$ of degree 4. Classify characters of G in $\mathbb{F}_{16}$. Rephrase Dedekind's lemma (Lemma 76 in Rotman) for these G and E in linear algebra terms in your model of $\mathbb{F}_{16}$.

3. (10 points). Given a ring R and an automorphism σ of R, the set

$$R^\sigma = \{a \in R \ : \ \sigma(a) = a\}$$

is a subring of R. Let $R = \mathbb{Z}[x_1, x_2]$ be the ring of polynomials in two variables with integer coefficients and σ the transposition of x_1 and x_2, so that $\sigma(x_1) = x_2, \sigma(x_2) = x_1$. For example, $\sigma(x_1^3 x_2 + x_1) = x_2^3 x_1 + x_2$.

Compute the action of σ on following polynomials:

$$10x_1 - x_2,\ x_1^2 x_2 - x_1 x_2^2,\ x_1^2 x_2^2 + x_1 x_2,\ x_1 + x_2,\ x_1^3 + x_2^3,\ (x_1 + x_2)x_1 x_2.$$

Which of these polynomials belong to R^σ? (Such polynomials are called *symmetric.*)

Remark: There's a theorem that any symmetric polynomial can be written as a polynomial in $x_1 + x_2$ and $x_1 x_2$. (You don't need this result to solve the problem above.)