Modern Algebra II, spring 2022

Homework 11, due Wednesday April 20.

We finished on Wednesday by stating the main result of Galois theory – a bijective order-reversing correspondence between intermediate fields B, $F \subset B \subset E$ of a Galois extension E/F and subgroups $H \subset G$ of the Galois group $G = \mathrm{Gal}(E/F)$. Rotman (page 83) states an even fancier version of this correspondence. For the version discussed in class see Friedman's notes Galois theory II (section 4 on page 23; Friedman uses K for an intermediate field instead of B we used in class).

1.(30 points)

(a) Consider Galois extension $\mathbb{Q}[\sqrt{10}, \sqrt{3}]/\mathbb{Q}$. Determine its Galois group. List all intermediate subfields and write down explicitly the bijection between the subfields and subgroup of G. Don't forget fields F and E themselves (for extension E/F) – which subgroups do they correspond to? Draw the diagrams of subfields and their inclusions and the corresponding diagram of subgroups of G.

(b) The same as (a) for the Galois extensions $\mathbb{F}_{16}/\mathbb{F}_2$ and $\mathbb{F}_{64}/\mathbb{F}_2$.

(c) Same as (a) for the splitting field extension $x^3 - 7$ over $\mathbb{Q}$ (it's very similar to the example done in class).

2.(20 points) In class, we wrote down matrices of multiplication and matrices of Galois symmetries acting on $\mathbb{Q}[\sqrt{2}]$ over $\mathbb{Q}$ in the basis $\{1, \sqrt{2}\}$.

(a) Do the same for the field extension $\mathbb{F}_8/\mathbb{F}_2$ using the model

$$\mathbb{F}_8 = \mathbb{F}_2[\alpha]/(\alpha^3 + \alpha + 1).$$

Use the basis $\{1, \alpha, \alpha^2\}$ of $\mathbb{F}_8$ over $\mathbb{F}_2$. Consider the Galois symmetry σ, where $\sigma(x) = x^2$ (here $x \in \mathbb{F}_8$) is the Frobenius automorphism. Write down the matrix of σ in the above basis.

(b) Write down the matrices of multiplication by α and α^2 in the above basis. What is the matrix of the linear operator that takes $x \in \mathbb{F}_8$ to $\alpha\, \sigma^2(x)$?

(c) (*optional, extra 10 points*) In class we restated the theorem $[E : E^G] = |G|$ as saying that any linear operator on E as a vector space over $F = E^G$ has a unique presentation as a sum $\sum_i a_i \sigma_i$, where $a_i \in E$ and $\sigma_i \in G$. Write down the corresponding statement in the special case of the extension $\mathbb{F}_8/\mathbb{F}_2$ using (a),(b) above and explicit matrices for the actions of G and E on E.