Modern Algebra II, spring 2022

Homework 12, due Sunday, May 1.

For solvable groups, one possible reference is Rotman, Appendix B, pages 118-128.

1.(20 points) In Monday's lecture (see Lecture notes or your own notes or that of a friend) we started with a characteristic 0 field F and first passed to the splitting field K of $x^n - 1$, adding all n-th roots of unity. Then we formed the splitting field E of the polynomial $x^n - c$, for some $c \in F^*$. There is a chain of inclusions $F \subset K \subset E$. Review our construction and write in your own words and explanations why the Galois group $G = Gal(E/F)$ is a subgroup of the group of affine symmetries $A = Aff(\mathbb{Z}/n)$. Recall that the latter group A consists of "affine" transformations of $\mathbb{Z}/n$. These are transformations that take $u \in \mathbb{Z}/n$ to $au + b$ for fixed a, b, with invertible $a \in (\mathbb{Z}/n)^*$ and $b \in \mathbb{Z}/n$. Such a transformation is associated to a pair (a, b) as above. Explain why G is a subgroup of A (examine the action of G on the set of roots of $x^n - u$ and match this action to the action of A on $\mathbb{Z}/n$).

2. (10 points) Let H be a normal subgroup of some group G. Prove that G is solvable iff both H and G/H are solvable. (Alternatively, you can review the proof Nguyen gave last week and write it down in your own words.)

3. (10 points) Show that the group A in Problem 1 is solvable (this was discussed in class). This implies the Galois group G in that problem is solvable.

4. (20 points) (a) Consider Galois extension $\mathbb{F}_2 \subset \mathbb{F}_{16}$. Write down its Galois group, intermediate fields B and the correspondence between subgroups of the Galois group and subfields of $\mathbb{F}_{16}$.

(b) Do the same for the extension $\mathbb{Q}[\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3}]/\mathbb{Q}$, where p_1, p_2, p_3 are distinct prime numbers. How many intermediate subfields of degree 2 and of degree 4 over $\mathbb{Q}$ did you find? Pick one subfield of degree 4 over $\mathbb{Q}$ and list its Galois symmetries.

5 (*optional, extra 10 points*) Last week Nguyen showed you the Galois correspondence for the extension $\mathbb{Q}[\sqrt[4]{2}, i]/\mathbb{Q}$ with the dihedral Galois group D_4, which is also the splitting field extension for $x^4 - 2$ (this example is also worked out in Friedman's notes starting at the bottom of page 3). Explain why this implies that the dihedral group D_4 and the affine group $Aff(\mathbb{Z}/4)$ in problem 1 above are isomorphic. Can you construct an explicit isomorphism between these two groups? (Hint compare the action of D_4 on the vertices of the square and the action of $Aff(\mathbb{Z}/4)$ on residues mod 4.)