Modern Algebra II, spring 2022

Homework 2, due Wednesday February 2. All rings are assumed commutative unless specified otherwise.

1. (10 points) For each of the following statements, either briefly explain why it is true or give a counterexample.

(a) Any subring of a field is an integral domain.

(b) Ring $\mathbb{Z}/49$ is an integral domain.

(c) Direct product $F_1 \times F_2$ of two fields is a field.

(d) Element ab of a ring R is invertible if and only if both a and b are invertible.

(e) Ring $\mathbb{Z} \times \mathbb{Z}$ has exactly four idempotents. (Hint: first find all idempotents in the ring $\mathbb{Z}$. Idempotent is an element e such that $e^2 = e$.)

2. (10 points) Find all zero divisors in the following rings:
(a) $\mathbb{Z}$, (b) $\mathbb{Z}/10$, (c) $\mathbb{Z}/17$, (d) $\mathbb{Z}/2 \times \mathbb{Z}/2$, (e) $\mathbb{R}$.

3. (20 points) (a) State the definition of a homomorphism $\alpha : R \longrightarrow S$ of rings.

(b) Prove that composition of homomorphisms is a homomorphism.

(c) Determine which of the following maps are homomorphisms:

1. $f : \mathbb{Z} \longrightarrow \mathbb{Z}$, $f(n) = -n$ for $n \in \mathbb{Z}$.
2. $f : \mathbb{Z} \longrightarrow \mathbb{Z}$, $f(n) = 2n$ for $n \in \mathbb{Z}$.
3. $f : \mathbb{R}[x] \longrightarrow \mathbb{R}$, $f(a_0 + a_1 x + \cdots + a_n x^n) = a_0$.
4. $f : \mathbb{C} \longrightarrow \mathbb{C}$, $f(z) = \overline{z}$ (complex conjugation).
5. $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(a) = a^2$ for $a \in \mathbb{R}$.
6. Projection from the direct product onto the first summand, $p : R_1 \times R_2 \longrightarrow R_1$, $p(a, b) = a$ for $a \in R_1, b \in R_2$. Here R_1, R_2 are rings.
7. Inclusion into the direct product, $\iota : R_1 \longrightarrow R_1 \times R_2$, $\iota(a) = (a, 0)$ for $a \in R_1$.

4. (10 points) Suppose that R is an integral domain. Show that any invertible element of $R[x]$ has degree 0 (thus, it is a constant polynomial). Conclude that $(R[x])^* = R^*$ and explain why $R[x]$ is not a field.

5. (20 points) Read through the incomplete proof of the Theorem in lecture 3 (pages 5-6) that $\mathrm{Frac}(R)$ is a field for an integral domain R. We defined $\mathrm{Frac}(R)$ via a suitable equivalence relation $\sim$ on the set $S = \{(a,b) | a, b \in R, b \neq 0\}$.

(a) Prove that multiplication is a well-defined operation in $\mathrm{Frac}(R)$. (In class we proved that for addition).

(b) Show that elements of the form $(0, b)$, $b \in R, b \neq 0$ constitute an equivalence class in S and give the zero element 0 of $\mathrm{Frac}(R)$. (In particular, show that no other element of S is in this equivalence class.)

(c) Show that elements of the form (a, a), $a \in R$, $a \neq 0$ constitute an equivalence class in S and give the identity (unit element) 1 of $\mathrm{Frac}(R)$.

(d) Prove that $\mathrm{Frac}(R)$ is associative under addition (this is part of the statement that $\mathrm{Frac}(R)$ is an abelian group under addition).

6. An element x of a ring R is called *nilpotent* if $x^n = 0$ for some $n > 0$. Note that $0 \in R$ is always nilpotent. (Remark: A nonzero nilpotent element is a zero divisor, while a zero divisor does not have to be nilpotent.)

(a) (5 points) Show that 0 is the only nilpotent element of an integral domain R.

(b) (5 points) Find all nilpotent elements in the following rings:

$$\mathbb{Z}, \quad \mathbb{Q}, \quad \mathbb{Z}/9, \quad \mathbb{Z}/12, \quad \mathbb{Q}[x].$$

(c) (optional, 10 points) Show that if x, y are nilpotent then $x + y$ is nilpotent (assume that $x^n = 0, y^m = 0$, use that the ring is commutative and apply the binomial theorem from lecture 1 to some large power of $x + y$).