Modern Algebra II, Spring 2022

Homework 4, due Wednesday February 16.

1. (10 points) Compute the following sums and intersections of ideals in $\mathbb{Q}[x]$ (first recall the relation between sums and intersections of ideas in $F[x]$ to gcd and lcm of polynomials). Note that all the ideals of $\mathbb{Q}[x]$ are principal. For each ideal, list the monic polynomial which generates the ideal.

$$(x)+(x+2), \quad (x^2)+(2x), \quad (3x^2+2x)+(4x^2+x),$$
$$(3x^2+x+5)+(0), \quad (x)\cap(2x+1), \quad (2x)\cap(3x^2).$$

2. (10 points) Compute the greatest common divisor of polynomials x^3-x^2-1 and x^2-x+1 in the field $\mathbb{F}_3=\mathbb{Z}/3$. (Hint: first divide one polynomial by the other with the remainder. Keep repeating this procedure until the remainder is 0.) What is the least common multiple of these polynomials in $\mathbb{F}_3$?

3. (15 points) (a) Find all monic irreducible polynomials of degree 2 in $\mathbb{F}_3$. (First think of a way to write down all monic degree 2 polynomials with coefficients in $\mathbb{F}_3$. How many are there? Then eliminate the ones with a root.)

(b) Find all irreducible polynomials of degrees $1,2,3$ in $\mathbb{F}_2$. (Note that in $\mathbb{F}_2$ any nonzero polynomial is monic.)

(c) Which of the following polynomials over $\mathbb{F}_5$ are irreducible?

$$3x^2+1, \quad x^4+x, \quad x^2+2x+2$$

4. (10 points) Consider the ring $R=\mathbb{F}_p[x]/(f(x))$, where $f(x)$ is a polynomial of degree n. This ring is the quotient of the ring $\mathbb{F}_p[x]$ of polynomials in x with coefficients in $\mathbb{F}_p$. (Recall that we use two notations, $\mathbb{Z}/p$ and $\mathbb{F}_p$, for the field of residues modulo a prime p.) Show that R is a finite ring with p^n elements. (Hint: in class we classified cosets of the ideal $(f(x))\subset F[x]$, for any field F. Count the cosets for the above ideal.)

5. (15 points) (a) Explain how to construct an infinite field of characteristic p. (Hint: start with the ring $\mathbb{F}_p[x]$ of polynomials in x with coefficients in $\mathbb{F}_p$. What properties does this ring have?)

(b) Explain why $(a+b)^p=a^p+b^p$ for any elements $a,b\in R$ where R is a (commutative, of course) ring in which $p=0$ (a ring which contains $\mathbb{F}_p$ as a subring). This was discussed in class; write your proof of this result based on our discussion.