Modern Algebra II, spring 2022

Homework 6, due Wednesday March 9.

1. (10 points) Which of the following extensions have finite degree? Which are algebraic?
(a) $\mathbb{R} \subset \mathbb{C}$.
(b) $\mathbb{F}_p \subset F$, where F is a finite field.
(c) $\mathbb{C} \subset \mathbb{C}(t)$, where $\mathbb{C}(t)$ is the field of rational functions in a variable t with complex coefficients.
(d) $\mathbb{Q} \subset \mathbb{R}$.
(e) $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}, \sqrt[3]{3}, \dots, \sqrt[p]{p}, ...)$, union over all primes p.
(f) $\mathbb{Q}(\sqrt[3]{3})/\mathbb{Q}$.

2. (10 points) Suppose we are given field extensions E/B and B/F and the degree $[E : F]$ is a prime number. Show that either $B = E$ or $B = F$.

3. (10 points) Determine the irreducible polynomial $\mathrm{irr}(\alpha, F)$ for $\alpha = \sqrt{2} + \sqrt{3}$ and the following fields F:
(a) $\mathbb{Q}$, (b) $\mathbb{Q}[\sqrt{2}]$, (c) $\mathbb{Q}[\sqrt{2}, \sqrt{3}]$, (d) $\mathbb{R}$.
Case (a) was discussed in class. For part (b), use that $\sqrt{3} \notin \mathbb{Q}[\sqrt{2}]$. To solve part (b), you can first figure out the irreducible polynomials of $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{3} + 1$ are over the field $\mathbb{Q}[\sqrt{2}]$, and then do $\sqrt{2} + \sqrt{3}$.

4. (10 points) Prove that an n-th root $\sqrt[n]{\alpha}$ of an algebraic number $\alpha \in \mathbb{C}$ is algebraic. By an n-th root of a complex number z we mean any complex number w such that $w^n = z$. If $z \neq 0$, there are n such numbers. A complex number is *algebraic* if it's a root of a polynomial $x^m + a_{m-1}x^{m-1} + \cdots + a_0$ with rational coefficients. (Hint: consider the subfield of $\mathbb{C}$ generated by $\mathbb{Q}$ and α, and then the subfield generated by $\sqrt[n]{\alpha}$.)

5. (10 points) Consider the field $\mathbb{F}_8 = \mathbb{F}_2[\alpha]/(\alpha^3 + \alpha + 1)$. Check that the polynomial $x^3 + x + 1$ splits in this field as

$$x^3 + x + 1 = (x + \alpha)(x + \alpha^2)(x + \alpha^2 + \alpha).$$

6. (10 points) (a) Prove that the sum of an algebraic and a transcendental complex number is always transcendental.
(b) Give an example of two transcendental complex numbers α, β such that $\alpha\beta$ is algebraic. You can assume that π is transcendental.