Modern Algebra II, spring 2022

Homework 7, due Wednesday March 23.
This week we've covered the following topics: irreducible polynomials over $\mathbb{Q}$ (references: Rotman, section Irreducible Polynomials (pages 38-43) and Howie, section 2.4 or lecture notes), uniqueness of finite fields (Howie, chapter 6 or Friedman Finite fields or lecture notes), formal derivative and multiple roots (Friedman Multiple roots).

1-3. (20 points each) Exercises 64, 65, 66 in Rotman, page 43. Use Exercise 63, which we also proved in class, to solve exercise 64. That result limits possible rational roots of a polynomial in $\mathbb{Q}[x]$.

4. (20 points) For which of the following polynomials can we use the Eisenstein criterion to conclude that they are irreducible over $\mathbb{Q}$?

$$x^7 - 180, \quad x^7 - 4x^4 + 6, \quad x^5 + 3x^2 + 3x + 9,$$
$$x^4 - 15x^2 + \frac{25}{2}x - 20, \quad x^5 - \frac{3}{5}x^3 + 27x - 6.$$

5. (20 points) Check that polynomial $x^8 - x$ factors over $\mathbb{F}_2$ into

$$x(x+1)(x^3+x+1)(x^3+x^2+1).$$

Note that these factors include both irreducible monic degree three polynomials over $\mathbb{F}_2$. Now form the field $\mathbb{F}_8 = \mathbb{F}_2[\alpha]/(\alpha^3+\alpha+1)$ and check that x^8-x factors into linear terms over $\mathbb{F}_8$. What are these linear terms? Which of these terms combine to give a factorization of x^3+x+1 and which factorize x^3+x^2+1?

6. (20 points) Recall that a field F of characteristic p is called *perfect* if any element $a \in F$ has a p-th root in F, that is there exists $b \in F$ such that $b^p = a$.
(a) Following the proof given in class, explain why any finite field F is perfect.
(b) Let field F has characteristic p and consider the field $F(t)$ of rational functions in a formal variable t with coefficients in F. Any element of $F(t)$ has the form $f(t)/g(t)$ subject to the usual manipulation and cancellation rules, where polynomials f and g have coefficients in F.
Show that t has no p-th root in $F(t)$. Hint: assume otherwise, $b^p = t$ for some $b = f(t)/g(t)$. Find a polynomial relation on f and g. Use unique factorization in the polynomial ring $F[t]$ to get a contradiction.