$F_0 = \mathbb{Q}$ start here. Keep adding $\sqrt[n]{c}$, various c and n
(Rodman, p. 71)

Def B/F is a pure extension of type m if $B = F(\alpha)$, $\alpha^m \in F$ some m

$F = B_0 \subset B_1 \subset \dots \subset B_t$ radical tower if each B_{i+1}/B_i is a pure extension.
Call B_t/F radical extension.

Want to show $Gal(B_t/F)$ is not too complicated. (solvable group).

if $B = F(\alpha)$ pure of type m & F contains m-th roots of unity,
then B/F is a splitting field of $x^m - \alpha^m$ $x^m - c$, $c = \alpha^m$

Def $f \in F[x]$, <u>f is solvable by radicals over F</u> if
$\exists$ a radical extension which contains a splitting field E of f/F.

Thm Let $f(x) \in F[x]$ be solvable by radicals over F, char $F = 0$
and let E/F be its splitting field. Then $G = Gal(E/F)$ is a
solvable group.

Solvable groups (Rotman, appendix B)

$G \longrightarrow G^{(1)} = [G, G]$ commutator subgroup

$G/G^{(1)}$ - abelian

$G \xrightarrow{\varphi} H$ - abelian

any such homomorphism has the property $\varphi|_{[G,G]} = 1$ trivial hom

$\varphi([G,G]) = \{1\}$

$G^{(2)} = [G^{(1)}, G^{(1)}]$,

$G^{(n)} = [G^{(n-1)}, G^{(n-1)}]$

Example : Invertible upper-triangular matrices $n \times n$

S_5, A_5 - not solvable. $A_5 = [A_5, A_5]$, $S_5 \xrightarrow{[,]} A_5$ stuck

smallest non-abelian simple group. $|A_5| = 60$ rotations of dodecahedron

other examples of finite simple groups

$GL(n, \mathbb{F}_p)$ or q

PGL (mod out by center)

most n, p.

Prop If $H \triangleleft G$ normal then

G - solvable $\Longleftrightarrow$ H - solvable, G/H solvable.

add roots of 1

$F_0 \subset F_1 \subset F_2 \subset F_3 \ldots \subset F_n$

$G \supset G_1 \supset G_2 \quad \ldots\ldots \quad G_n = \{1\}$

$G = Gal(F_n/F_0)$
$\downarrow \varphi_0$
$Gal(F_1/F_0)$
abelian

$\ker \varphi_0 = G_1 = Gal(F_n/F_1)$
$\downarrow \varphi_1$
$Gal(F_2/F_1)$

$\ker \varphi_1 = G_2 = Gal(F_n/F_2)$
$\downarrow \varphi_2$
$Gal(F_3/F_2)$

$G_{i+1} \triangleleft G_i$ normal,

Thm There is a degree 5 polynomial $f \in \mathbb{Q}[x]$ not solvable by radicals.

Proof $f(x) = x^5 - 4x + 2$.

Eisenstein crit $\Rightarrow$ irreducible $/\mathbb{Q}$

Let $E/\mathbb{Q}$ be splitting field, $E \subset \mathbb{C}$ (use that $\mathbb{C}$ is alg. closed).

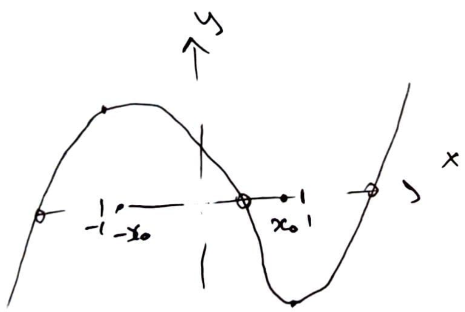
$G = \mathrm{Gal}(E/\mathbb{Q})$ $\qquad \alpha$ - root of $f(x)$

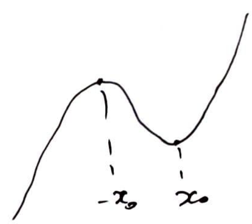
$E \supset \mathbb{Q}(\alpha) \overset{5}{\supset} \mathbb{Q}$ $\qquad [E:\mathbb{Q}] = [E:\mathbb{Q}(\alpha)][\mathbb{Q}(\alpha):\mathbb{Q}] = 5[E:\mathbb{Q}(\alpha)]$

$5 \mid |G| = [E:\mathbb{Q}]$.

$f'(x) = 5x^4 - 4$ crit pts $x^4 = \frac{4}{5}$, $x = \pm\sqrt[4]{\frac{4}{5}} \approx \pm 0.946$

$\pm x_0$



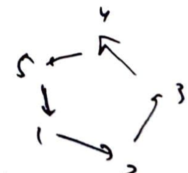


3 real roots, $\sim -1.52,\ 0.5,\ 1.24$ $\Rightarrow$ complex conj. in $\mathbb{C}$ restricts to a transposition β.

G contains el't of order 5 $\Rightarrow$ contains a 5-cycle.

Lemma (Rotman G. 39. p. 126) If $\alpha \in S_5$ a 5-cycle, β a transposition, then $\langle \alpha, \beta \rangle = S_5$.

Proof $H = \langle \alpha, \beta \rangle \subset S_5$ subgroup. Can assume $\alpha = (12345)$



$\tau = (1i)$ $\qquad \alpha^k(i) = 1$ some k

$\alpha^k (1i) \alpha^{-k} = (j1)$ some j, $j = \alpha^k(1)$

$\underline{i \neq j}$ $\quad \alpha, (1i)$ do not commute

$(1i)(1j) = (1ji)$ order 3.

$2, 3, 5 \mid |H| \Rightarrow 30 \mid |H|$
$|S_5| = 120$

$1 \xrightarrow{\alpha^k} i \xrightarrow{(1i)} 1 \xrightarrow{\alpha^k} j$

$30 \mid |H|$ $\quad |H| \mid |S_5| = 120$

$|H|: 30, 60, 120$

30, 60: no such subgroups in S_5

120: $H = S_5$

$(H \cap A_5) \subset H$ proper since H contains a transposition (odd permutation)

$K \subset S_5$ $\quad [S_5 : K] = 4 \to S_5 \to S_4$ homomorphism

$A_5 \subset S_5 \xrightarrow{\gamma} S_4$ order 24.

$\quad\quad\quad \cup$

$A_5 \cap \ker\gamma \quad \ker\gamma$

$\quad \|$

$\quad L \quad\quad |\ker\gamma| \geq 5$

$|L| \geq \frac{5}{2} = 3$. But A_5 is simple. (no normal subgroups)