Modern algebra II, Review problems for the final exam, part 1.

First few problems are related to the material covered in the last lecture. I recommend using a combination of Rotman, Howie and Fridman's notes to prepare for the exam, as well as class notes.

1. (a) Take the formula $\Delta = -4p^3 - 27q^2$ for the discriminant of $f(x) = x^3 + px + q$. Set $q = 0$ and check that the coeficient of p^3 is indeed -4 by explicitly writing down all roots of $f(x)$ in $\mathbb{C}$ and computing Δ via the product of differences of roots.

(b) Suppose $f(x) \in \mathbb{Q}[x]$ is an irreducible polynomial of degree 3 which has only one real root. Show that the Galois group $Gal(E/\mathbb{Q})$ of its splitting field over $\mathbb{Q}$ is S_3.

(c) Work through the proof of Theorem 104 (ii) in Rotman (page 101) for the structure of the Galois group when f has three real rooots. See Example 38 (Rotman, p.102) for Galois groups of several degree three polynomials.

2. Exercises 100, 101, 102 in Rotman (page 100). Note how complicated the discriminant becomes (exercise 102 ii) if the coefficient a at x^2 is nonzero. Check that all terms in the expression for D are homogeneous (in terms of degrees, as we've discussed). Note that Rotman's D is Friedman's Δ.

3. Determine Galois groups of splitting fields over $\mathbb{Q}$ of the following degree 3 polynomials

$$x^3 - 3x^2 - x - 1, \quad x^3 - x^2 + x - 1, \quad x^3 + 27x - 4, \quad x^3 - 21x + 7$$

4. Determine the Galois group over $\mathbb{Q}$ of the polynomial $x^5 - 12x^2 + 2$.

5. Let $f(x), g(x) \in \mathbb{Q}[x]$ be two irreducible polynomials, each of degree two. Form the splitting field E of the product polynomial $f(x)g(x)$. Describe possible Galois groups $Gal(E/Q)$. Try to generalize your result to the case of a product of n quadratic polynomials and the corresponding splitting field.

6. Show that the Galois group $Gal(\mathbb{R}/\mathbb{Q})$ is trivial (or go through the proof, for instance, in `https://math.stackexchange.com/questions/555661/how-to-get-the-galois-gr`

7. (a) Explain why the angle $25°$ cannot be constructed using a ruler and compass.

(b) Given a regular n-gon and a regular m-gon such that n and m are coprime, show that you can create a regular nm-gon using only a ruler and compass.

8. (a) Let G be the Galois group of a degree n polynomial in $F[x]$. Explain why $|G|$ divides $n!$.

(b) What are possible Galois groups of a degree three polynomial $f(x) \in F[x]$? Include the case when $f(x)$ is reducible in your answer.

9. (a) Give an example of a normal inseparable field extension.

(b) Give an example of a finite degree extension E/F which contains infinitely many intermediate subfields (necessarily an inseparable extension).

10. (a) Write down a presentation for the field $\mathbb{F}_{27}$ (similar to what we did for $\mathbb{F}_8$ and $\mathbb{F}_{16}$.) Write down a basis of $\mathbb{F}_{27}$ as an $\mathbb{F}_3$-vector space. What are the subfields of $\mathbb{F}_{27}$?

(b) Pick your generator of $\mathbb{F}_{27}$ and write down its orbit under the action of the Galois group $\mathrm{Gal}(\mathbb{F}_{27}/\mathbb{F}_3)$. How many orbits of the Galois group action $Gal(\mathbb{F}_{27}/\mathbb{F}_3)$ are there in $\mathbb{F}_{27}$?

(c) How many degree 3 monic irreducible polynomials over $\mathbb{F}_3$ are there? Do this counting without explicitly writing them all down.