# McKay Correspondence

A selected topic for a course on
representation theory of finite groups.
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## §1. Finite Subgroups of $SU(2)$

- **Conjugacy classes in $U(n)$**

Recall the following standard fact from linear algebra:

Any matrix in $U(n)$ is conjugate to a diagonal matrix, i.e. $\forall A \in U(n)$, $\exists B \in U(n)$ s.t. $BAB^{-1} = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$, where $\lambda_i \in S^1 \subseteq \mathbb{C}^*$. The $\lambda_i$'s are nothing but the eigenvalues of $A$.

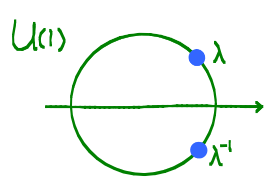
Note that the order of $\lambda_i$'s is unimportant since permuting $\lambda_i$'s can be realized by conjugating by a matrix in $U(n)$.

Consequently, the conjugacy class of a matrix $A \in U(n)$ is determined by the unordered set of its eigenvalues.

The same story applies to $SU(n)$ without much effort: the conjugacy class of a matrix $A \in SU(n)$ is determined by its unordered set of eigenvalues $\{\lambda_1, \cdots, \lambda_n \mid \lambda_1 \cdots \lambda_n = 1\}$. In fact, $U(n)$ is generated by $SU(n)$ and $U(1) = \{\lambda \cdot \mathrm{Id} \mid |\lambda| = 1\}$, but under conjugation, $U(1)$ acts trivially.

E.g. $SU(2)$

In this case, the conjugacy classes are parametrized by $\left\{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \,\middle|\, |\lambda| = 1\right\} \cong U(1)$, upto identification of $\lambda$ and $\bar{\lambda} = \lambda^{-1}$.




- **The conjugation representation of $U(n)$ on $Mat(n, \mathbb{C})$.**

Let $U(n)$ act on $Mat(n, \mathbb{C})$ by conjugation. We shall decompose it into (real) irreducible subrepresentations.

The first observation to make is that $U(n)$ preserves the subspaces of Hermitian and anti-Hermitian matrices :

$$\mathrm{Mat}(n,\mathbb{C}) \cong (\text{Hermitian}) \oplus (\text{Anti-Hermitian})$$
$$B \longmapsto \left( \frac{B+B^*}{2}, \frac{B-B^*}{2} \right)$$

and the $U(n)$ action preserves these subspaces:

$$B \text{ Hermitian}, A \in U(n) \Longrightarrow (ABA^*)^* = (A^*)^* B^* A^* = ABA^*$$
$$B \text{ anti-Hermitian}, A \in U(n) \Longrightarrow (ABA^*)^* = (A^*)^* B^* A^* = -ABA^*$$

Notation:

$W_+ \triangleq$ the space of Hermitian matrices

$W_- \triangleq$ the space of anti-Hermitian matrices

Note that these are only real representations of $U(n)$ and multiplication by $i$ is an isomorphism of real $U(n)$-modules:

$$W_+ \underset{\cdot i}{\overset{\cdot i}{\rightleftarrows}} W_-$$

We can do a little better, since conjugation preserves traces:

$$W_+ = \{\text{Trace } 0 \text{ Hermitian matrices}\} \oplus \{\lambda \mathrm{Id} \mid \lambda \in \mathbb{R}\}$$
$$B \longmapsto \left( B - \frac{\mathrm{tr} B}{n} \mathrm{Id}, \frac{\mathrm{tr} B}{n} \mathrm{Id} \right)$$

Denote $W_+^0 \triangleq \{\text{Trace } 0 \text{ Hermitian matrices}\}$, $W_-^0 \triangleq \{\text{Trace } 0 \text{ anti-Hermitian matrices}\}$.
We have shown that:

Lemma 1: As real representations of $U(n)$

$$\mathrm{Mat}(n,\mathbb{C}) \cong \mathbb{R}\cdot\mathrm{Id} \oplus W_+^0 \oplus i\mathbb{R}\mathrm{Id} \oplus W_-^0$$ □

- The double cover $SU(2) \longrightarrow SO(3)$

We shall try to relate finite subgroups of $SU(2)$ to finite subgroups of $SO(3)$, since $SO(3)$, being the rotational symmetry group of the unit sphere $S^2$ in $\mathbb{R}^3$, is more intuitive.

Consider a 2×2 invertible complex matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

whose inverse is

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Thus for such an $A$ to lie in $SU(2)$, we just need

$$\begin{cases} \det A = 1 \\ \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} \end{cases}$$

i.e. $\bar{a} = d$, $c = -\bar{b}$, and $\det A = ad - bc = |a|^2 + |b|^2 = 1$. Write $a = x_1 + ix_2$, $b = x_3 + ix_4$.
This identifies $SU(2)$ as

$$SU(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \middle| \; a, b \in \mathbb{C},\; |a|^2 + |b|^2 = 1 \right\}$$
$$\cong \{ x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \mid x_i \in \mathbb{R} \} = S^3 \subseteq \mathbb{R}^4,$$

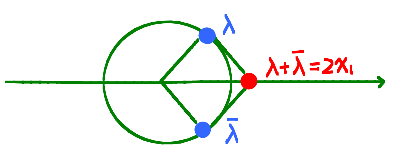
the unit 3-sphere in the 4-dim'l Euclidean space.

Continuting our earlier example of conjugacy classes of $SU(2)$, we see that, the conjugacy class of $A \in SU(2)$ is completely determined by

$$\mathrm{Tr}(A) = a + \bar{a} = 2x_1$$

Conversely, the set of eigenvalues $\{\lambda, \bar{\lambda}\}$ is determined by:

$$\{ x_1 \pm i\sqrt{1 - x_1^2} \}$$

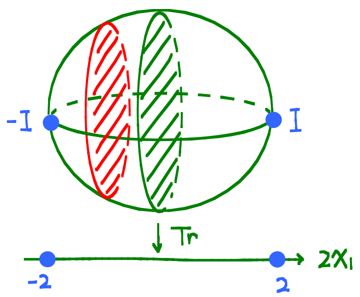



: $\lambda, \bar{\lambda}$ are roots of the equation
$x^2 - \mathrm{Tr}(A)x + \det A = x^2 - 2x_1 \cdot x + 1 = 0$

Thus, each conjugacy class of $SU(2)$ is just the set of matrices in $SU(2)$ with a fixed trace value $2x_1$ $(-1 \le x_1 \le 1)$, which is identified as

$$\{ x_2^2 + x_3^2 + x_4^2 = 1 - x_1^2 \} \cong S^2(\sqrt{1 - x_1^2})$$

a two sphere of radius $\sqrt{1-x_1^2}$. (When $x_1 = \pm 1$, this says that the matrices $\pm I$ form their own conjugacy class). Pictorially:




Imagine this to be the 3-sphere of $SU(2)$. The conjugacy classes are just preimages of $tr(A) = 2x_1$. Except for the "poles" $\pm I$, these conjugacy classes are geometrically 2-spheres with various radii.

Alternatively, $SU(2)$ can be described as the group of unit quaternions. In fact, quaternions $\mathbb{H}$ may be identified as

$$\mathbb{H} \cong \mathbb{R}\cdot Id \oplus W_-^0 \subseteq Mat(2,\mathbb{C})$$

$$1 \longmapsto Id$$

$$i \longmapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \triangleq \underline{i} \qquad j \longmapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \triangleq \underline{j} \qquad k \longmapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \triangleq \underline{k}.$$

Recall that we have shown: under the conjugation action of $SU(2)$, $W_-^0$ is invariant. Now

$$\begin{aligned} W_-^0 &= \{\text{Traceless anti-hermitian matrices}\} \\ &\cong \mathbb{R}\underline{i} \oplus \mathbb{R}\underline{j} \oplus \mathbb{R}\underline{k} \end{aligned}$$

Moreover

$$\begin{aligned} \{A \in SU(2) \mid TrA = 0\} &= \left\{ \begin{pmatrix} x_1+ix_2 & x_3+ix_4 \\ -x_3+ix_4 & x_1-ix_2 \end{pmatrix} \middle| \, 2x_1 = 0, \ x_1^2+x_2^2+x_3^2+x_4^2 = 1 \right\} \\ &= \left\{ x_2\underline{k} + x_3\underline{j} + x_4\underline{i} \,\middle|\, x_2^2+x_3^2+x_4^2 = 1 \right\} \end{aligned}$$

being a conjugacy class in $SU(2)$, is also the unit sphere in $W_-^0$.

$\Longrightarrow$ The conjugation action of $SU(2)$ on $W_-^0$ acts transitively on the "unit sphere" = {traceless elements in $SU(2)$}.

Now we are close to what we need: We have produced a representation $SU(2) \longrightarrow GL(W_-^0) \cong GL(3,\mathbb{R})$ so that $SU(2)$ preserves the "unit sphere" of $\mathbb{R}^3$. We shall impose a norm on $W_-^0$ so that $SU(2)$ preserves the norm, and the "unit sphere" becomes the genuine unit sphere under this norm.

By the same consideration as for $U(n) \cap Mat(n, \mathbb{C})$, a natural candidate for the norm is $(X, Y)' \triangleq Tr(XY)$ since $Tr$ on $W_-^0$ is preserved under conjugation action by $SU(2)$: $\forall A \in SU(2)$, $X, Y \in W_-^0$:

$$\begin{aligned}(A \cdot X, A \cdot Y)' &= Tr(AXA^{-1} \cdot AYA^{-1}) \\ &= Tr(AXYA^{-1}) \\ &= Tr(XY) \\ &= (X, Y)'\end{aligned}$$

and it's readily seen to be bilinear. Furthermore, we have:

$$\begin{cases} Tr(\underline{i} \cdot \underline{i}) = -2 & Tr(\underline{i} \cdot \underline{j}) = 0 \\ Tr(\underline{j} \cdot \underline{j}) = -2 & Tr(\underline{i} \cdot \underline{k}) = 0 \\ Tr(\underline{k} \cdot \underline{k}) = -2 & Tr(\underline{j} \cdot \underline{k}) = 0. \end{cases}$$

Hence if we rescale $(X, Y) \triangleq -\frac{1}{2}(X, Y)' = -\frac{1}{2} Tr(XY)$, we obtain a Euclidean inner product on $W_-^0$, w.r.t. which $\{\underline{i}, \underline{j}, \underline{k}\}$ forms an o.n.b.

Combining the above discussion, we have exhibited a map:

$$SU(2) \longrightarrow Aut(W_-^0, (\,,\,)) \cong Aut(\mathbb{R}^3, (\,,\,)) = O(3).$$

It remains to show that:

(i). The image of $SU(2)$ lies in $SO(3)$;

(ii). The map $SU(2) \xrightarrow{\gamma} SO(3)$ is surjective;

(iii). Analyze $Ker\,\gamma = ?$

(i) is easily guanranteed by the topology of $SU(2)$: The group homomorphism $SU(2) \longrightarrow O(3)$ is continuous (i.e. elements in $SU(2)$ close to Id moves vectors in $W_-^0$ only a little), and $SU(2)$ is connected, being a sphere. Thus its image in $O(3)$ must be connected. Since $SO(3)$ is the connected component of $O(3)$ containg $I$, $im(SU(2)) \subseteq SO(3)$. We shall denote the homomorphism:

$$\gamma : SU(2) \longrightarrow SO(3).$$

Exercise: Show that any continuous homomorphism $S^1 \longrightarrow G$, where $G$ is a discrete group, is the trivial one.

(ii). To show that $\nu$ is surjective, consider the action of

$$A(\varphi) = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} \qquad (0 \le \varphi < \pi)$$

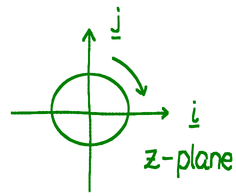
on $W^0$:

$$A(\varphi)\cdot \underline{k} = A(\varphi)\underline{k}A(-\varphi) = \underline{k}$$

and on the plane $\mathbb{R}\underline{i} \oplus \mathbb{R}\underline{j} = \left\{ \begin{pmatrix} 0 & i\bar{z} \\ iz & 0 \end{pmatrix} \middle| z \in \mathbb{C} \right\} \cong \mathbb{C}$, $A(\varphi)$ acts as

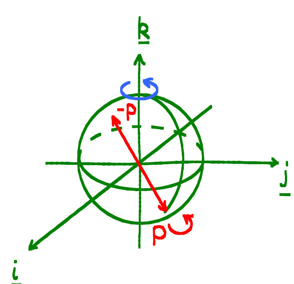
$$A(\varphi)\begin{pmatrix} 0 & i\bar{z} \\ iz & 0 \end{pmatrix} A(-\varphi) = \begin{pmatrix} 0 & i\overline{e^{-2i\varphi}z} \\ ie^{-2i\varphi}z & 0 \end{pmatrix}$$

i.e. it acts as clockwise rotation of the complex plane by the angle $2\varphi$.




In summary, $A(\varphi)$ acts on $W^0$ as the rotation about the $\underline{k}$-axis by an angle of $2\varphi$ $(0 \le 2\varphi < 2\pi$ : all the rotations$)$.

Next, we have shown that $SU(2)$ acts transitively on the unit sphere $S^2$ of $W^0$. Thus $\forall\, P \in S^2$, $\exists\, B \in SU(2)$ s.t. $B.\underline{k} = B\underline{k}B^{-1} = P$, and the subgroup $BA(\varphi)B^{-1} \subseteq SU(2)$ $(0 \le \varphi < \pi)$ consists of all rotations about the axis through $\{P, -P\}$:




Now we conclude from the well-known fact that $SO(3)$ consists of all rotations about various axis through the origin that $\nu$ maps $SU(2)$ surjectively onto $SO(3)$.

(iii). What's the kernel of $\nu: SU(2) \longrightarrow SO(3)$?

$A \in \operatorname{Ker}\nu \Leftrightarrow A$ acts trivially on $W_-^0$ : $\forall X \in W_-^0$, $AXA^{-1} = X$.

(But $A$ commutes with $i\mathbb{R}\mathrm{Id}$ trivially)

$\Leftrightarrow A$ acts trivially on $W_- = i\mathbb{R}\mathrm{Id} \oplus W_-^0$.

$( \cdot i : W_- \xrightarrow{\cong} W_+$ : an isomorphism of $SU(2)$ rep's)

$\Leftrightarrow A$ acts trivially on $\mathrm{Mat}(2, \mathbb{C}) \cong W_+ \oplus W_-$.

$\Leftrightarrow A \in Z(\mathrm{Mat}(2,\mathbb{C})) \cap SU(2) = \mathbb{C}\cdot\mathrm{Id} \cap SU(2) = \{\pm I\}$.

Ex. Show that $Z(SU(n)) = \{ \zeta^k \cdot \mathrm{Id} \mid \zeta = e^{\frac{2\pi i}{n}}, \ 0 \le k \le n-1 \}$.

In summary, we have shown:

Thm. 2. There exists a 2:1, surjective group homomorphism

$$\nu: SU(2) \longrightarrow SO(3)$$

of (Lie) groups, with $\operatorname{Ker}\nu = \{\pm \mathrm{Id}\}$. □

- Finite subgroups of $SO(3)$.

We shall use the following well-known:

Thm 3. Finite subgroups of $SO(3)$ are classified as follows:

There are two infinite families:

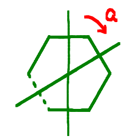
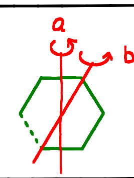
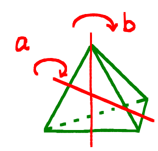
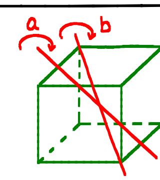
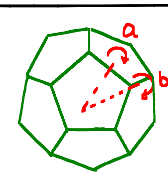
- $C_n$ : cyclic group of order $n$.
- $D_{2n}$ : dihedral group of order $2n$ ;

and 3 more exceptional cases:

- $A_4$ : the rotational symmetry group of a tetrahedron.
- $S_4$ : the rotational symmetry group of a cube / octahedron
- $A_5$ : the rotational symmetry group of an icosahedron / dodecahedron.

For a proof, see M. Artin : *Algebra*. □

More geometrically, we have the following presentation of these groups:

| $G \subseteq SO(3)$ | $\lvert G\rvert$ | Geometric description of generators |
|---|---|---|
| $C_n = \langle a \mid a^n = 1 \rangle$ | $n$ |   |
| $D_n = \langle a, b \mid a^2 = b^2 = (ab)^n = 1 \rangle$ | $2n$ |   |
| $A_4 = \langle a, b \mid a^2 = b^3 = (ab)^3 = 1 \rangle$ | 12 |   |
| $S_4 = \langle a, b \mid a^2 = b^3 = (ab)^4 = 1 \rangle$ | 24 |   |
| $A_5 = \langle a, b \mid a^2 = b^3 = (ab)^5 = 1 \rangle$ | 60 |   |

- Finite subgroups of $SU(2)$

Observe that in $SU(2)$, there is only one element of order 2, namely $-I$. This is because any matrix $A \in SU(2)$ can be conjugated to a diagonal matrix of the form $\begin{pmatrix}\lambda & \\ & \lambda^{-1}\end{pmatrix}$ and for it to be of order 2, $\lambda = -1$. In contrast, there are lots of elements of order 2 in $SO(3)$ (take any rotation by $\pi$ about any direction in $\mathbb{R}^3$!). Thus the preimages of these order 2 elements in $\mathbb{R}^3$ under $\gamma$ are all of order 4.

Now, let $G$ be a finite subgroup of $SU(2)$ and $H = \gamma(G)$ be its image in $SO(3)$. Since $\nu$ is $2:1$, there are two possibilites

(i). $|G| = |H|$, and $\nu|_G : G \xrightarrow{\cong} H$.

(ii). $|G| = 2|H|$, and $-I \in G$, $H \cong G/\{\pm I\}$

Also note that from our classification list for $SO(3)$, $|H|$ is even unless $H \cong C_{2k+1}$ is cyclic of odd order. Other than this $|H|$ is even $\Longrightarrow$ $|G|$ is even $\Longrightarrow$ $G$ has an order 2 element (elementary group theory!), and we are in case (ii).

We analyse case by case

(a). $H \cong C_n$. There are two possibilites:

(a.i) $n = 2k+1$. Then $G \cong H \cong C_{2k+1}$ or $G \cong C_2 \times H \cong C_{2(2k+1)}$

(a.ii) $n = 2k$. Then $G/\{\pm I\} \cong H \cong C_{2k}$. $\Longrightarrow$ $G \cong C_{4k}$ or $C_2 \times C_{2k}$. The latter is ruled out since there would be more than 1 order 2 elements in $G$.

Thus $G$ is always cyclic. Such $G$ is always conjugate to one of the form:

$$G = \left\{ \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \middle| \ \xi = e^{\frac{2\pi i k}{n}},\ 0 \le k < n \right\}$$

(b). $H \cong D_{2n} \Longrightarrow H \cong G/\{\pm I\}$. In this case $H \supseteq C_n$ as a (normal) subgroup $\Longrightarrow$ $\nu^{-1}(C_n) \cong C_{2n} \subseteq G$ (by case (a)), of index 2, and thus must be normal. Since $D_{2n} = C_n \amalg a \cdot C_n$ ($a$ of order 2) $\Longrightarrow$ $G = \nu^{-1}(H) = C_{2n} \amalg a' C_{2n}$, where $\nu(a') = a$ and $a'$ must have order 4. Then $G$ can be conjugated to the group generated by:

$$\left\{ \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = a' \ \middle| \ \xi = e^{\frac{\pi k i}{n}},\ 0 \le k < 2n \right\}$$

We denote this group by $D^*_{2n}$, called the binary dihedral group. Note that $D^*_{2n} \not\cong D_{4n}$ since the latter has many order 2 elements.

$$\begin{tikzcd} D^*_{2n} \arrow[r, "\nu"] & D_{2n} \\ C_{2n} \arrow[u, phantom, "\trianglelefteq" rotate=90] \arrow[r, "\nu"] & C_n \arrow[u, phantom, "\trianglelefteq" rotate=90] \end{tikzcd} \qquad \text{with } \nu\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 = 1$$

Rmk: It's not hard to figure out the structure of $D_{2n}^*$ directly from elementary group theory: Let $t$ be the generator of $C_{2n}$. $s = \nu(t) \in C_n$ ($\subseteq D_{2n}$) a generator. Then $\nu(a')\nu(t)\nu(a'^{-1}) = asa^{-1} = s^{-1} = \nu(t^{-1}) \Rightarrow a'ta'^{-1} = t^{-1}$ or $-t^{-1}$. But if $a'ta'^{-1} = -t^{-1}$ $\Rightarrow (a't)^2 = -t^{-1}a'a't = 1 \Rightarrow a't = \pm 1 = \mp (a')^2 \Rightarrow t = \mp a' \Rightarrow G$ is abelian. Contradiction. So $a'ta'^{-1} = t^{-1}$ and it's isomorphic to the group above.

(c). $H \cong A_4, S_4, A_5 \Rightarrow H \cong G/\{\pm I\}$. In these cases the corresponding $G$'s are denoted $A_4^*$, $S_4^*$, $A_5^*$, called the binary tetrahedron group, binary octahedron group, binary icosahedron group respectively.

Rmk: Note that $A_4^* \ncong S_4$, $A_5^* \ncong S_5$, since $S_4, S_5$ have more than 1 order 2 elements.

By now, we have classified all finite subgroups of $SU(2)$:

Thm 4. Finite subgroups of $SU(2)$ are classified as follows:

| $G \subseteq SU(2)$ | Presentation | $\lvert G \rvert$ |
|---|---|---|
| $C_n$ | $\langle a \mid a^n = 1 \rangle$ | $n$ |
| $D_{2n}^*$ | $\langle a, b \mid a^2 = b^2 = (ab)^n \rangle$ | $4n$ |
| $A_4^*$ | $\langle a, b \mid a^2 = b^3 = (ab)^3 \rangle$ | 24 |
| $S_4^*$ | $\langle a, b \mid a^2 = b^3 = (ab)^4 \rangle$ | 48 |
| $A_5^*$ | $\langle a, b \mid a^2 = b^3 = (ab)^5 \rangle$ | 120 |

$\square$

## §2. The McKay Graph

Let $V \cong \mathbb{C}^2$ be the 2-dimensional representation of $SU(2)$. By restricting it to any finite subgroup $G$ of $SU(2)$, we obtain a 2-dim'l representation of $G$, still denoted by $V$. Note that $V$ is irreducible unless $G \cong C_n$, the only finite abelian subgroups of $SU(2)$ (otherwise $V \cong U \oplus W$ is a sum of 2 1-dim'l representations $\Longrightarrow G \subseteq \mathbb{C}^* \times \mathbb{C}^*$ is abelian). This representation plays a pivotal role in what follows.

Lemma 5. $V$ is a self-dual representation.

Pf: $\forall g \in G$, $\exists B \in SU(2)$ s.t. $BgB^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, where $|\lambda| = 1$. Thus

$$\chi_V(g) = \mathrm{tr}_V(g) = \mathrm{tr}_V(BgB^{-1}) = \lambda + \lambda^{-1} = \lambda + \bar{\lambda} \in \mathbb{R}$$

$\Longrightarrow \chi_V$ is real $\Longrightarrow V$ is self-dual. $\square$

Rmk: Using character theory for connected compact Lie groups, we see that $V$ is a self-dual representation for $SU(2)$. Such an isomorphism $V \longrightarrow V^*$ is not hard to exhibit:

$\forall\, g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2) \Longrightarrow g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \Longrightarrow (g^{-1})^t = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$. Let $h = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then

$$h^{-1} g h = (g^{-1})^t = g^*.$$

Let $V_i, V_j$ be two irrep's of $G$. Consider the multiplicity of $V_i$ in $V_j \otimes V$.

$$m(V_i, V \otimes V_j) = \dim \mathrm{Hom}_G(V_i, V \otimes V_j) = \dim \mathrm{Hom}_G(V \otimes V_j, V_i).$$

Lemma 6. $m(V_i, V \otimes V_j) = m(V_j, V \otimes V_i)$.

Pf: Since $m(V_i, V \otimes V_j) \in \mathbb{Z}_{\geq 0}$, $m(V_i, V \otimes V_j) = \overline{m(V_i, V \otimes V_j)}$. $\Longrightarrow$

$$\begin{aligned} m(V_i, V \otimes V_j) &= \overline{m(V_i, V \otimes V_j)} \\ &= \overline{\frac{1}{|G|} \sum_{g \in G} \chi_i(g)\, \overline{\chi_V(g) \chi_j(g)}} \\ &= \frac{1}{|G|} \sum_{g \in G} \overline{\chi_i(g)}\, \chi_V(g) \chi_j(g) \\ &= \frac{1}{|G|} \sum_{g \in G} \overline{\chi_i(g) \chi_V(g)}\, \chi_j(g) \quad (V \text{ is self-dual}) \\ &= (\chi_j, \chi_i \chi_V) \end{aligned}$$

$$= m(V_j, V\otimes V_i). \qquad \square$$

Rmk: In general, it's true that $\forall X, Y, Z$ rep's of $G$,

$$\mathrm{Hom}_G(X, Z\otimes Y) \cong \mathrm{Hom}_G(X\otimes Z^*, Y)$$

If $X\cong V_i$, $Y\cong V_j$, $Z\cong V\cong V^*$, taking dimension of both sides, we obtain:

$$\begin{aligned} m(V_i, V\otimes V_j) &= \dim \mathrm{Hom}_G(V_i, V\otimes V_j) \\ &= \dim \mathrm{Hom}_G(V_i\otimes V, V_j) \\ &= m(V_j, V_i\otimes V). \end{aligned}$$

## • Construction of the graph

Notation: $a_{ij} \triangleq m(V_i, V\otimes V_j)$. Then $a_{ij} = a_{ji}$, by lemma 6.

Now to each finite subgroup $G$ of $SU(2)$, we associate with it a graph $\Gamma$ as follows:

Vertices: Irrep's $V_i$ of $G$.

Edges: The $i, j$ th vertices are connected by $a_{ij}$ edges.

Moreover, to each vertex, we assign to it an integer $d_i = \dim V_i$, called the weight.

E.g. $G\cong C_n = \langle a \mid a^n = 1\rangle$.

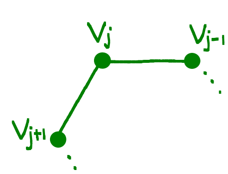
We know that in this case, Irrep's of $G$ are all 1 dimensional:

$$\mathrm{Irrep}(G) = \{V_0, V_1, \cdots, V_{n-1}\}$$

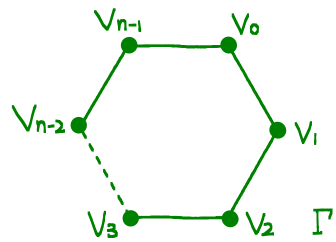
where $a$ acts on $V_k$ by multiplication by $\zeta^k = e^{\frac{2\pi k i}{n}}$, $0\le k<n$. Moreover, since $a = \begin{pmatrix}\zeta & \\ & \zeta^{-1}\end{pmatrix}$, we see that $V\cong V_1\oplus V_{-1}$ ( $V_i = V_{n+i}$ ). Thus $\forall V_j$

$$V_j\otimes V \cong V_j\otimes(V_1\oplus V_{-1}) \cong V_{j+1}\oplus V_{j-1}.$$

Hence in the graph:



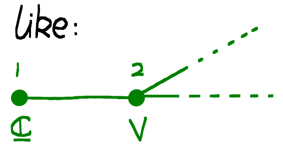
and the whole graph looks like:



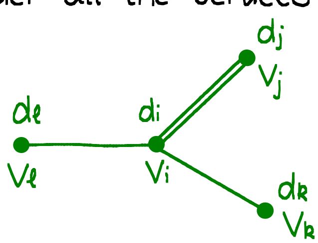

- Common features of McKay graphs.

Now we discuss general properties of the graph.

Note that for $G$ non-abelian, $V$ is irreducible. $\mathbb{C}\otimes V \cong V$. Thus $\Gamma$ always contains a portion like:




For any vertex $V_i$, consider all the vertices connected to it:




Then by definition, $V_i \otimes V \cong \oplus V_j^{a_{ij}}$. Taking dimensions of both sides, we get:

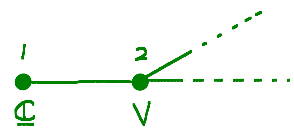
$$2d_i = \Sigma\, a_{ij} d_j.$$

Later we will show that, except for two degenerate cases, vertices in any McKay graph are connected by at most 1 edge.

Thm. 7. McKay graphs are connected.

Pf: By the example above, it suffices to prove for $G$ nonabelian. We shall prove by contradiction.

Assume for some $G$, $\Gamma$ is not connected. Then, by our discussion above, $\exists$ irrep $V_i$ of $G$ not contained in the connected component of the graph:




Note that the irreps of $G$ occuring in this component are precisely those irreps occuring inside $V^{\otimes n}$ for various $n \in \mathbb{Z}_{\geq 0}$ (by definition). Thus such $V_i$ must satisfy:

$$(\chi_i, \chi_{V^{\otimes n}}) = 0, \quad \forall n \geq 0$$
$$\Longleftrightarrow (\chi_i, \chi_V^n) = 0, \quad \forall n \geq 0$$
$$\Longleftrightarrow \frac{1}{|G|} \sum_g \chi_i(g) \overline{\chi_V(g)}^n = 0$$
$$\Longleftrightarrow \frac{1}{|G|} \sum_g \chi_i(g) \chi_V(g)^n = 0. \quad (V \text{ is self-dual})$$

By earlier discussion in §1, $\chi_V(g) \in [-2, 2]$ and $\chi_V(g) = -2$ iff $g = -I$, $\chi_V(g) = 2$ iff $g = I$. Since we have assumed that $G$ is non-abelian, $-I \in G$. Dividing both sides of the equation by $2^n$, and multiplying by $|G|$, we obtain:

$$\sum_g \chi_i(g) \left(\frac{\chi_V(g)}{2}\right)^n = 0, \quad \forall n \geq 0$$
$$\Longleftrightarrow \chi_i(I) + \chi_i(-I)(-1)^n + \sum_{|\chi_V(g)/2|<1} \chi_i(g) \left(\frac{\chi_V(g)}{2}\right)^n = 0, \quad \forall n \geq 0$$

Since $-I \in Z(G)$, by Shur's lemma, $-I$ acts on $V_i$ by a scalar matrix. Since $(-I)^2 = I$, it can only act as $\pm Id_{V_i}$. Hence $\chi_{V_i}(-I) = tr_{V_i}(\pm Id_{V_i}) = \pm d_i$. Now, divide both sides of the equation by $d_i$, we have:

$$1 + \varepsilon(-1)^n + \sum_{|\chi_V(g)/2|<1} \frac{\chi_i(g)}{d_i} \cdot \left(\frac{\chi_V(g)}{2}\right)^n = 0, \quad \forall n \geq 0,$$

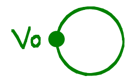
where $\varepsilon = \frac{\chi_i(-I)}{d_i} = \pm 1$ is fixed for $V_i$. Taking $n >> 0$, since $|\frac{\chi_V(g)}{2}| < 1$, the the summation term is very small and has to be an integer, and thus it must be 0. Hence we get an equation for all $n >> 0$:

$$1 + \varepsilon(-1)^n = 0$$

This is impossible and leads to the desired contradiction. $\square$

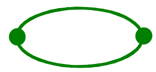
Cor. 8. $a_{ij} \leq 1$ unless $G \cong \{1\}$ or $C_2$.

Pf: $G \cong \{1\} \Longrightarrow \Gamma$ has only 1 vertex, namely $V_0 = \mathbb{C}$. $V \cong V_0 \oplus V_0 \Longrightarrow a_{00} = 2$. $\Gamma$ looks like

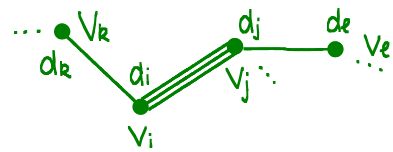



(the edge, considered leaving and entering, connects $V_0$ twice)

For $C_2$, we have shown that its graph is like:



Conversely, assume that $G \ncong \{1\}$, and there is a multiple edge between $V_i$ and $V_j$:




we have $a_{ij} = a_{ji} \geq 2$

$$\begin{cases} 2d_i = a_{ij}d_j + \sum a_{ik}d_k \\ 2d_j = a_{ji}d_i + \sum a_{j\ell}d_\ell \end{cases}$$

$\Longrightarrow 2d_i = 2d_j + (a_{ij}-2)d_j + \sum a_{ik}d_k = a_{ji}d_i + \sum a_{j\ell}d_\ell + (a_{ij}-2)d_j + \sum a_{ik}d_k$

$\Longrightarrow 2(a_{ij}-2)d_j + \sum a_{ik}d_k = 0$

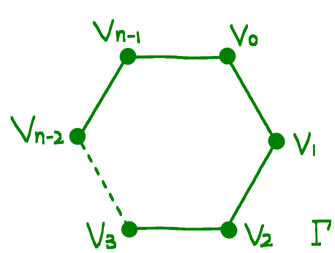
$\Longrightarrow d_k = 0$, $a_{ik} = 0$, $a_{ij} = 2$. i.e. no vertex other than $V_j$ connects to $V_i$.

By symmetry, this must also be true for $V_j$. Since we know that $\Gamma$ is connected, $\Gamma$ must then be:

and $G \cong \mathbb{Z}/2$ (the only group with only 2 conjugacy classes). $\square$

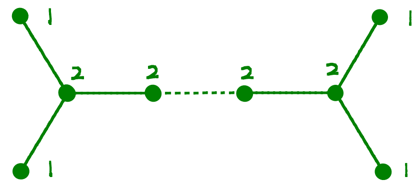
- List of McKay graphs

We have seen that the McKay graph for $C_n$ is




This graph is called $\widetilde{A}_{n-1}$

The graph for $D_{2n}^*$ is actually the following with $n+3$ vertices:

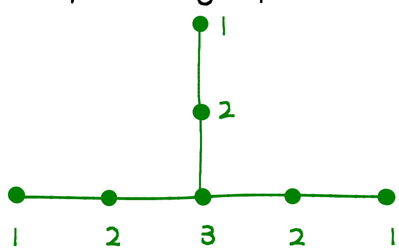



The graph is called $\widetilde{D}_{n+2}$. One can check the relation:

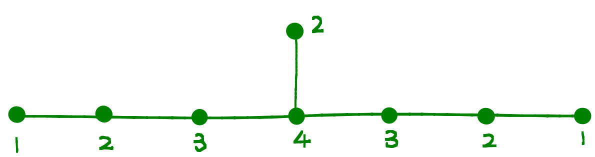
$$|G| = \Sigma d_i^2$$

from: $4n = 4\cdot 1^2 + (n-1)\cdot 2^2$.

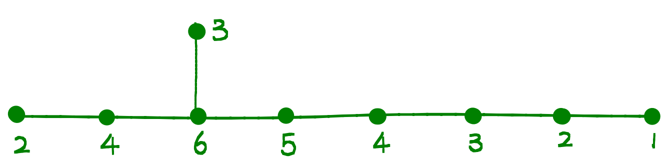
The exceptional groups:




$A_4^*$ : the graph $\widetilde{E}_6$




$S_4^*$ : the graph $\widetilde{E}_7$




$A_5^*$ : the graph $\widetilde{E}_8$

We shall prove, in the next section, that these are the only possibilities:

Thm. 9. Any connected graph $\Gamma$ with positive integral weights $d_i$ assigned to each vertex satisfying:

(i). $g.c.d(d_i) = 1$

(ii). $2d_i = \sum_{i-j} d_j$

is one of the graphs listed above.

We shall also show how to match the groups with their corresponding McKay graphs in the next section.

## §3. Classification

Our main goal in this section is to classify McKay graphs (i.e. to prove thm 9 of §2).

### • The associated inner product space of a graph.

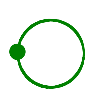
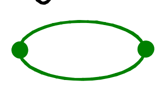
Let $\Gamma$ be a connected graph, whose vertices are $\{e_1, \cdots, e_n\}$, and between any two vertices there is at most one edge connecting them. To such a $\Gamma$ we associate a real vecter space $\mathbb{R}^\Gamma$ and an inner product on it, as follows:

$$\mathbb{R}^\Gamma \triangleq \bigoplus_{i=1}^n \mathbb{R} e_i$$

and the inner product on it, defined on the basis and extended bilinearly:

$$(e_i, e_j) \triangleq \begin{cases} 2 & \text{if } i=j \\ -1 & \text{if } i \neq j \text{ and } i, j \text{ are connected} \\ 0 & \text{otherwise} \end{cases}$$

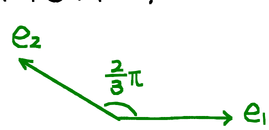
Recall our definition of the McKay graphs $\widetilde{A}_n, \widetilde{D}_n, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$ in thm. 9, together with the weights $\{d_i\}$ assigned to each vertex. Remark that in the above definition we exclude the degenerate cases of McKay graphs:

 $\widetilde{A}_0$  $\widetilde{A}_1$

E.g.

$\Gamma = \overset{e_1}{\bullet}$, then $\mathbb{R}^\Gamma = \mathbb{R}e_1$ and $(e_1, e_1) = 2$

$\Gamma = \overset{e_1}{\bullet}\!\!-\!\!-\!\!-\!\!\overset{e_2}{\bullet}$, then $\mathbb{R}^\Gamma = \mathbb{R}e_1 \oplus \mathbb{R}e_2$, with $e_1, e_2$ forming an angle of $\frac{2}{3}\pi$.

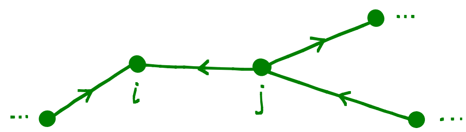



Lemma 10. If $\Gamma$ is among the McKay graphs $\widetilde{A}_n$ $(n \geq 2)$, $\widetilde{D}_{n+2}$ $(n \geq 2)$, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$, the associated inner product $(\ ,\ )$ on $\mathbb{R}^\Gamma$ is positive semi-definite, with a 1-dimensional null space spanned by the vector $\omega_0 = \sum_{i=1}^n d_i e_i$.

Pf: Indeed $\mathbb{R}\omega_0$ lies inside the null space of $(\ ,\ )$: $\forall i=1,\cdots,n$.

$$\begin{aligned}(\omega_0, e_i) &= (d_i e_i, e_i) + \textstyle\sum_{j\neq i} (d_j e_j, e_i)\\ &= 2d_i - \textstyle\sum_{i-j} d_j\\ &= 0\end{aligned}$$

To show that $(\ ,\ )$ is positive semi-definite, we assign an auxiliary orientation (arbitrarily) to all edges of $\Gamma$, so as to keep track of terms we are summing over:




Now, $\forall\ \omega = \sum_i x_i e_i$, $x_i \in \mathbb{R}$, we have:

$$\begin{aligned}0 &\le \textstyle\sum_{i\to j} d_i d_j \left(\frac{x_i}{d_i} - \frac{x_j}{d_j}\right)^2 \quad \text{(summing over all oriented edges)}\\ &= \textstyle\sum_{i\to j} d_i d_j \left(x_i^2/d_i^2 - 2x_i x_j/d_i d_j + x_j^2/d_j^2\right)\\ &= \textstyle\sum_{i\to j} \left(\frac{d_j}{d_i} x_i^2 - 2x_i x_j + \frac{d_i}{d_j} x_j^2\right)\\ &= \textstyle\sum_{i\to j} \frac{d_j}{d_i} x_i^2 - 2\sum_{i\to j} x_i x_j + \sum_{j\to i} \frac{d_j}{d_i} x_i^2\\ &= \textstyle\sum_i \left(\sum_{j:\, i\to j} \frac{d_j}{d_i} x_i^2 + \sum_{j:\, j\to i} \frac{d_j}{d_i} x_i^2\right) - 2\sum_{i\to j} x_i x_j\\ &= \textstyle\sum_i \left(\sum_{j:\, j-i} \frac{d_j}{d_i} x_i^2\right) - 2\sum_{i-j} x_i x_j\\ &= 2\textstyle\sum_i x_i^2 - 2\sum x_i x_j \quad (\text{since } \sum_{j:j-i} d_j = 2d_i)\\ &= (\omega, \omega),\end{aligned}$$

with "=" holding iff $x_i/d_i = x_j/d_j \equiv \lambda \in \mathbb{R}$ for all $i,j \in \{1,\cdots,n\}$, i.e. $\omega = \lambda\omega_0$. The result follows. $\square$

Next we introduce some standard definitions from combinatorics:

Def. Consider a connected graph $\Gamma$ as above (these graphs without multiple edges between any two vertices are sayed to be simply laced)

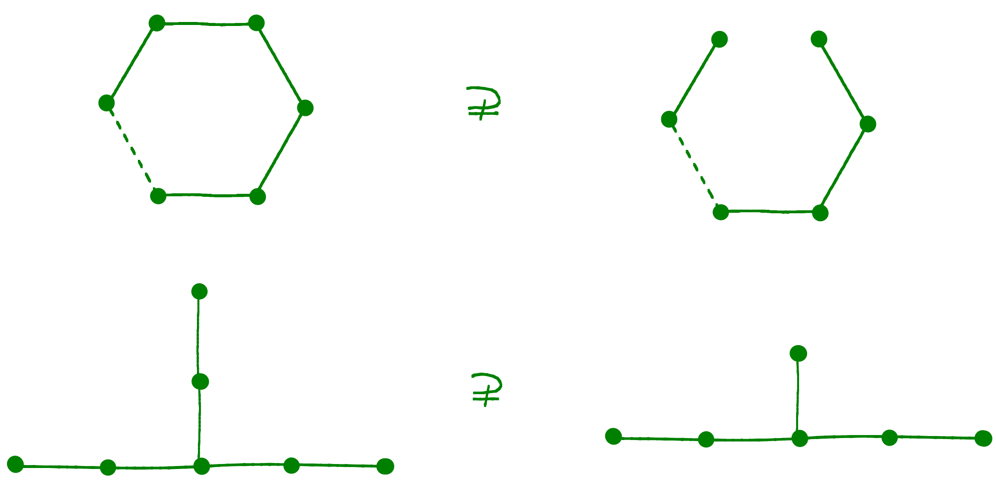
(i). $\Gamma$ is called affine if we can assign weights $d_i \in \mathbb{N}$ to its vertices s.t.

$$2d_i = \textstyle\sum_{j-i} d_j, \quad \forall i.$$

Rmk: Lemma 10 $\Longrightarrow$ all McKay graphs are affine. Furthermore, the result of the lemma actually holds for affine graphs. since in the proof we used nothing but the relation $2d_i = \sum_{i-j} d_j$.

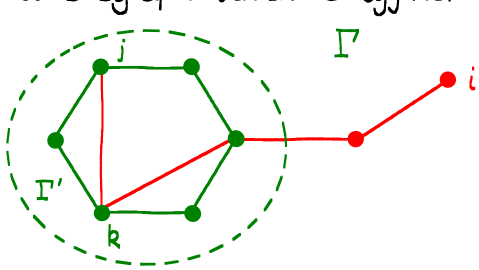
(ii). $\Gamma$ is called (finite) Dynkin if it's a proper subgraph of some affine graph.
Rmk: By slightly modifying the proof of lemma 10, it's readily seen that in this case the associated inner product is positive definite on $\mathbb{R}^\Gamma$ (C.f. the proof of lemma 11 below).

E.g.




(iii). $\Gamma$ is called indefinite if $\Gamma$ contains properly an affine graph.
Rmk: Lemma 11 below shows that in this case the associated inner product on $\mathbb{R}^\Gamma$ is indefinite.

Lemma 11. If $\Gamma$ is indefinite, then the associated inner product on $\mathbb{R}^\Gamma$ is indefinite.
Pf: Let $\Gamma' \subsetneq \Gamma$ be a subgraph which is affine. There are two possibilities:



(i). $\Gamma$ contains a vertex not in $\Gamma'$, say $e_i \in \Gamma$, $e_i \notin \Gamma'$

By def., $\exists \{d_j\}$ weights of $\Gamma'$ s.t. $2d_j = \sum_{(k-j)\in\Gamma'} d_k$ Let $\omega_0 = \sum_{j\in\Gamma'} d_j e_j$, and $\omega' = \omega_0 + \varepsilon e_j$. Then:

$$(\omega', \omega') = (\omega_0, \omega_0) + 2\varepsilon(\omega_0, e_i) + 2\varepsilon^2$$

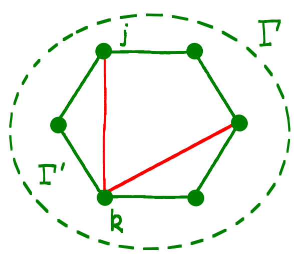
Note that $(\omega_0, \omega_0) \le 0$ : (to distinguish different inner products for $\Gamma$ and $\Gamma'$ we write $(\ ,\ )_\Gamma$, $(\ ,\ )_{\Gamma'}$)

$$\begin{aligned}(\omega_0, \omega_0)_\Gamma &= (\omega_0, \omega_0)_{\Gamma'} + \sum_{j,k\,(j-k)\notin\Gamma'} (d_j e_j, d_k e_k) \\ &= 0 - \sum_{j,k\,(j-k)\notin\Gamma'} d_j d_k \\ &\le 0\end{aligned}$$

However $(\omega_0, e_i) = \sum_{j\in\Gamma'} d_j (e_j, e_i) = -\sum_{j\in\Gamma'} d_j < 0$. Hence if we take $0<\varepsilon\ll 1$, $2\varepsilon^2 < -2\varepsilon(\omega_0, e_i)$.

$$\Longrightarrow (\omega', \omega') < 0$$

(ii). $\Gamma'$ is obtained from $\Gamma$ by removing more than one edges




Then

$$\begin{aligned}(\omega_0, \omega_0)_\Gamma &= (\omega_0, \omega_0)_{\Gamma'} + \sum_{(j-k)\notin\Gamma'} d_j d_k (e_j, e_k) \\ &= -\sum_{(j-k)\notin\Gamma'} d_j d_k \\ &< 0\end{aligned}$$ $\square$

Since the inner products associated with affine graphs are always positive semi-definite, we deduce that:

Cor 12. Affine graphs do not contain each other properly. $\square$

- Classification of affine graphs

We summarize the definitions we made into a table:

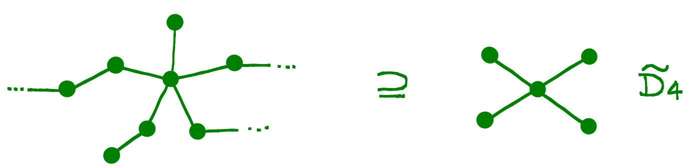
| Graph type | Definition | Associated inner product on $\mathbb{R}^{\Gamma}$ |
|---|---|---|
| Affine | $\Gamma'$ | positive semi-definite |
| Dynkin | $\Gamma \subsetneq \Gamma'$ | positive definite |
| Indefinite | $\Gamma \supsetneq \Gamma'$ | indefinite |

Claim (thm. 9): The McKay graphs $\widetilde{A}_n$ $(n \geq 2)$, $\widetilde{D}_n$, $(n \geq 4)$, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$ is a complete list of (simply laced) affine graphs.

Pf: We shall actually show that, if $\Gamma$ is neither Dynkin nor affine, it contains properly one of the McKay graphs, and thus is indefinite.

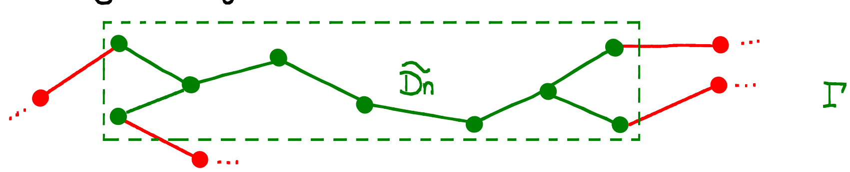
(i). If $\Gamma$ contains a cycle, then it contains $\widetilde{A}_n$ properly.

(ii). If $\Gamma$ contains a vertex of valency $\geq 4$, it's either $\widetilde{D}_4$, or it contains $\widetilde{D}_3$ properly.




By (i) and (ii), we may assume that $\Gamma$ has no loops (i.e. it's a tree), and all vertices have valency $\leq 3$.

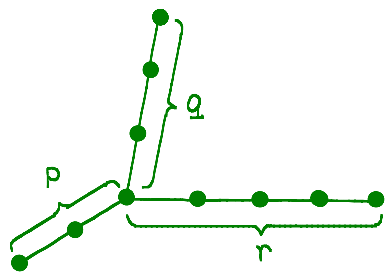
(iii). If $\Gamma$ contains more than 2 valency 3 vertices, choose a path between them. Then these two vertices, the vertices connected to them, together with the path connecting them form a $\widetilde{D}_n$:




Then $\Gamma$ is either $\widetilde{D}_n$, or it contains $\widetilde{D}_n$ properly.

(iv). If $\Gamma$ contains no valency 3 vertex, it is properly contained in $\widetilde{A}_n$ and is then Dynkin.

(v). It remains to discuss the case when $\Gamma$ contains exactly one vertex of valency 3. Let $p, q, r$ denote the number of vertices on each "antenna" of $\Gamma$. Without loss of generality, assume that $p \le q \le r$, and denote $\Gamma$ by $\Gamma_{p,q,r}$.




By lemma 10, $\Gamma_{3,3,3} = \widetilde{E}_6$, $\Gamma_{2,4,4} = \widetilde{E}_7$, $\Gamma_{2,3,6} = \widetilde{E}_8$ are affine.

By removing one of the weight one vertices, we obtain those $\Gamma_{p,q,r}$'s that are Dynkin:

$\widetilde{D}_{n+2} \longmapsto \Gamma_{2,2,n} \ (\triangleq D_{n+2})$ $\qquad$ $\widetilde{E}_6 \longmapsto \Gamma_{2,3,3} \ (\triangleq E_6)$

$\widetilde{E}_7 \longmapsto \Gamma_{2,3,4} \ (\triangleq E_7)$ $\qquad$ $\widetilde{E}_8 \longmapsto \Gamma_{2,3,5} \ (\triangleq E_8)$

Finally, any other values of $p, q, r$ other than those listed above will contain one of $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$ properly:

| $p$ | $q$ | $r$ | Result |
|---|---|---|---|
| 2 | 3 | $\ge 7$ | $\Gamma_{p,q,r} \supsetneq \Gamma_{2,3,6} = \widetilde{E}_8$ |
| 2 | 4 | $\ge 5$ | $\Gamma_{p,q,r} \supsetneq \Gamma_{2,4,4} = \widetilde{E}_7$ |
| 2 | $\ge 5$ | $\ge 5$ | $\Gamma_{p,q,r} \supsetneq \Gamma_{2,4,4} = \widetilde{E}_7$ |
| $\ge 3$ | $\ge 3$ | $\ge 3$ | $\Gamma_{p,q,r} \supsetneq \Gamma_{3,3,3} = \widetilde{E}_6$ |

□

Rmks.

(i). What we have shown is stronger than thm. 9, namely, we have partitioned all (simply-laced) graphs into 3 types:

| Dynkin | Affine | Indefinite |
|---|---|---|
| $A_n$<br>$D_n$<br>$E_6, E_7, E_8$ | $\tilde{A}_n$<br>$\tilde{D}_n$<br>$\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$ | All the other ones |

McKay graphs are exactly the same as (simply-laced) affine graphs.

(ii). Note that the types of $\Gamma_{p,q,r}$ is determined by the value of $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}$:

| $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}$ | $\Gamma_{p,q,r}$ |
|---|---|
| $>1$ | Dynkin |
| $=1$ | Affine |
| $<1$ | Indefinite |

We shall see some interesting application of this fact in the next section.

## • Matching groups with graphs

To fully establish McKay correspondence, we only need to match finite subgroups $G \subseteq SU(2)$ with its corresponding McKay (affine) graph $\Gamma$.

For $G \cong C_n$, we have shown in an example of §2 the associated McKay graph is $\tilde{A}_{n-1}$. Conversely, $\tilde{A}_{n-1}$ can only be associated with $C_n$ since all its weights being 1 implies that the associated group has all its irrep's 1-dim'l, and thus must be an abelian group.

To determine the McKay graph for $G = D^*_{2n}$, note that we have shown that $D^*_{2n} \supseteq C_{2n}$, a (normal) index 2 subgroup which is abelian. We need the following:

Lemma 13. If a finite group $G$ contains an index $r$ subgroup $H$ which is abelian, then any irrep $V$ of $G$ has $\dim V \le r$.
Pf: We know that the regular rep $\mathbb{C}[G]$ of $G$ contains all irreps of $G$, and

$$\begin{aligned}\mathbb{C}[G] &= \mathrm{Ind}_H^G(\mathbb{C}[H]) \\ &= \mathrm{Ind}_H^G\Big(\bigoplus_{\mu\in \mathrm{Irrep}(H)} V_\mu^{\oplus \dim V_\mu}\Big) \\ &= \bigoplus_{\mu\in\mathrm{Irrep}(H)} \mathrm{Ind}_H^G(V_\mu)^{\oplus \dim V_\mu}\end{aligned}$$

Now $H$ abelian $\Longrightarrow \dim V_\mu = 1 \Longrightarrow \mathrm{Ind}_H^G V_\mu$ has dimension $[G:H] = r \Longrightarrow$ Any irrep of $G$ is then contained in one of $\mathrm{Ind}_H^G V_\mu$. The result follows.

$\square$

Apply the lemma to $G = D_{2n}^* \supseteq H = C_{2n}$. We conclude that the irreps of $D_{2n}^*$ have $\dim \le 2$. Since $D_{2n}^*$ is nonabelian, it does have 2-dim'l irreps (for instance the fundamental $V$). Hence the McKay graph for $D_{2n}^*$ can't be $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$, since they contain vertices of weights $\ge 3$. Then, to identify which $\tilde{D}_k$ is associated with $D_{2n}^*$, we can use the relation

$$|G| = \sum d_i^2,$$

as we did before, to find the McKay graph $\tilde{D}_{n+2}$ for $D_{2n}^*$.

For $A_4^*$, $S_4^*$, $A_5^*$, recall that $\nu: SU(2) \longrightarrow SO(3)$ restricts to 2:1 surjective homomorphisms of them onto $A_4$, $S_4$, $A_5$. Recall from basic rep. theory that $A_4$, $S_4$, $A_5$ have irreps of $\dim \ge 3$. Thus their McKay graphs must be among $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$, the only graphs having some weights $d_i \ge 3$. Once again we apply the formula $|G| = \sum d_i^2$ to find the right graph for $G$.

In summary, we have shown:

Thm 14. Assigning each finite subgroup of $SU(2)$ its McKay graph gives a bijection of sets:

$$\{\text{Finite subgroups of } SU(2)\} \xleftrightarrow{1:1} \{\text{affine graphs}\} \qquad \square$$

• Application

Let $G$ be a nonabelian finite subgroup of $SU(2)$. By our classification of all such $G$'s, we know that $z = -I_{2\times 2} \in SU(2)$ lies in $G$. Let $V_i$ be an irrep of $G$, then $z \in Z(G) \Longrightarrow z$ acts as $\pm \mathrm{Id}_{V_i}$ on $V_i$, since $z^2 = I_{2\times 2}$. Since $z = -\mathrm{Id}_V$ on the fundamental $V$, if $V_j \subseteq V_i \otimes V$ (i.e. $j, i$ are connected in the McKay graph of $G$), we have:

$$\text{if } z \text{ acts as } \mathrm{Id}_{V_i} \text{ on } V_i,\ z = (-\mathrm{Id}_V) \otimes \mathrm{Id}_{V_i}|_{V_j} = -\mathrm{Id}_{V_j},$$

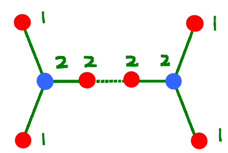
$$\text{if } z \text{ acts as } -\mathrm{Id}_{V_i} \text{ on } V_i,\ z = (-\mathrm{Id}_V) \otimes (-\mathrm{Id}_{V_i})|_{V_j} = \mathrm{Id}_{V_j}.$$

If we partition $\mathrm{Irrep}(G)$ into

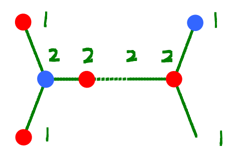
$$\mathrm{Irrep}(G) = \{V_i : z|_{V_i} = \mathrm{Id}_{V_i}\} \sqcup \{V_j : z|_{V_j} = -\mathrm{Id}_{V_j}\}$$

and mark them by different colors on the McKay graph, we have:

$$\boxed{\text{Any vertex on } \Gamma \text{ has its neighbors with a different color.}}$$



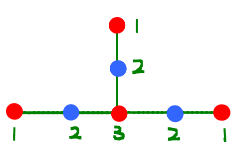

$D^*_{2n}$, $n$ even



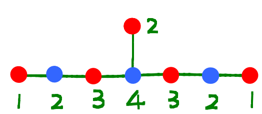

$D^*_{2n}$ : $n$ odd

● : $z$ acts as $-\mathrm{Id}$

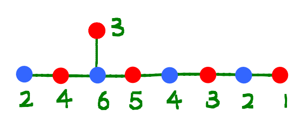
● : $z$ acts as $\mathrm{Id}$




$A_4^*$



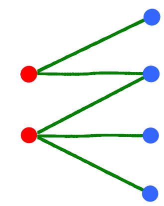

$S_4^*$




$A_5^*$

Note that, those $V_i$ with $z$ acting as $\mathrm{Id}$ are exactly those irrep's of $G$ that descend to $G/\{I, z\} \cong \mathrm{Im}\,\gamma(G)$. For instance, for $S_4^*$, we recover the result that there are 5 irreps of $S_4$, of dimensions 1, 1, 2, 3, 3 respectively.

The above remarks says that the McKay graphs are bipartite, i.e. it's a graph whose vertices can be partitioned into 2 classes such that the edges of the graph only connects vertices from different classes. A typical bipartite graph can be obtained as shown below:



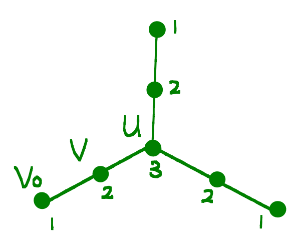
Finally, recall that tensoring with a 1-dim'l representation of $G$ induces an automorphism of $\mathrm{Irrep}(G)$. This induces an automorphism of the McKay graph.

E.g. Let's identify the vertices of $\widetilde{E}_6$ with specific representations of $A_4^*$. By the above discussion, we know that the 1-dim'l irrep's all come from that of $A_4$. Since 1-dim'l representations of any group $G$ forms a group $(G/[G,G])^\vee$, this says that the natural map of abelian groups

$$A_4^*/[A_4^*, A_4^*] \longrightarrow A_4/[A_4, A_4]$$

induces an isomorphism of the dual groups, and thus is an isomorphism itself (both $\cong C_3$)

Observe that the full symmetry group acts transitively on all weight 1 vertices (this is true for all McKay graphs!), so we can pick any of them to stand for $V_0 = \mathbb{C}$. Let $V_1$ and $V_2$ be the other two 1-dim'l irreps of $A_4^*$, then $V_1^{\otimes 2} \cong V_2$, $V_1^{\otimes 3} \cong V_0$. Let $V$ be the fundamental irrep.



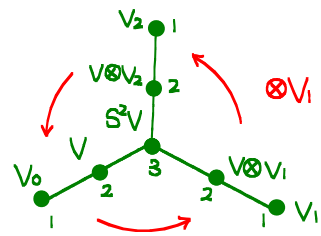

Then the central vertex $U$ satisfies $V^{\otimes 2} \cong V_0 \oplus U$. But we also have

$$V^{\otimes 2} \cong \Lambda^2 V \oplus S^2 V$$

and $SU(2)$ acts trivially on $\Lambda^2 V$. Hence $U \cong S^2 V$.

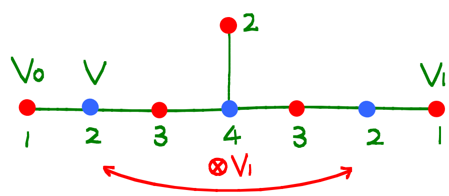
Now tensoring with $V_1$ induces an order 3 automorphism of $\widetilde{E}_6$

and it sends $V_0 \mapsto V_1$. Thus we obtain:




Note also that tensoring with 1-dim'l rep's do not give the full symmetries of the McKay graph here: $\mathrm{Sym}(\widetilde{E}_6) \cong D_6$ but here we only obtain $C_3$. Note that, however, these automorphisms permute transitively on weight 1 vertices, and the group has order exactly the number of weight 1 vertices. This is true for all finite subgroups of $SU(2)$.

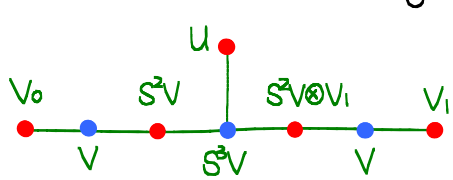
E.g. / Exercise. For $S_4^*$, the full symmetry group of $\widetilde{E}_7$ is the same as that induced by tensoring the non-trivial 1-dimensional irrep:




Note that from the diagram we have, similar as for $A_4^*$, that

$$\begin{aligned} (S_4/[S_4, S_4])^\vee &= \text{1-dim'l } \mathrm{Rep}(S_4) \\ &= \text{1-dim'l } \mathrm{Rep}(S_4^*) \\ &= (S_4^*/[S_4^*, S_4^*])^\vee \end{aligned}$$

$\Longrightarrow S_4^*/[S_4^*, S_4^*] \cong S_4/[S_4, S_4] \cong C_2$. $V_1$ comes from the sign rep of $S_4$.
We leave it as an exercise to check the diagram:




and $U$ is the 2-dim'l irrep of $S_4$: $S_4^* \xrightarrow{\gamma} S_4 \longrightarrow S_3 \curvearrowright \mathbb{C}_0^3$.

## §4. Fun with Graphs

Previously we have partioned all (simply-laced) graphs into 3 classes:

| Finite Dynkin | Affine | Indefinite |
|---|---|---|
| $\Gamma(\subsetneq\Gamma')$ | $\Gamma'$ | $\Gamma(\supsetneq\Gamma')$ |

As a corollary, we have:

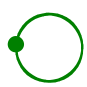
Cor. 15. There is a bijection:

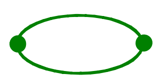
$$\left\{\begin{matrix}\text{Affine}\\ \text{graphs}\end{matrix}\right\} \xrightarrow{1:1} \left\{\begin{matrix}\text{Finite Dynkin}\\ \text{graphs}\end{matrix}\right\}$$

$$\Gamma' \longmapsto \Gamma \triangleq \Gamma' - \text{a weight } 1 \text{ vertex}$$

Note that removing any weight 1 vertex gives the same result since the group of automorphisms of $\Gamma'$ acts transitively on them. □

Rmk: We agree that for the special cases:

 $\tilde{A}_0 \longmapsto \phi$

 $\tilde{A}_1 \longmapsto$  $A_1$

Rmk: The Dynkin graphs $A_n, D_n, E_i$, $i=6,7,8$ first occured when people were trying to classify simple Lie groups / algebras. The history is much longer than McKay correspondence (~1980).

So far, what we have established is the following: Start with any finite subgroup $G \subseteq SU(2)$:

$$\underbrace{G \longrightarrow \text{McKay graph}}_{\text{McKay Correspondence}}\ \underbrace{\text{(affine)} \longrightarrow}_{\text{Removing any weight 1 vertices}} \text{Finite Dynkin}$$

We shall look at the nonabelian $G$'s. But let's summarize what we know:

| $G$ | $\lvert G \rvert$ | $H^*$ | Presentation of $G\,(H)$ | Dynkin graph |
|---|---|---|---|---|
| $D_{2n}^*$ | $4n$ | $D_{2n}$ | $a^2 = b^2 = (ab)^n \; (=1)$ | $\Gamma_{2,2,n}$ |
| $A_4^*$ | $24$ | $A_4$ | $a^2 = b^3 = (ab)^3 \; (=1)$ | $\Gamma_{2,3,3}$ |
| $S_4^*$ | $48$ | $S_4$ | $a^2 = b^3 = (ab)^4 \; (=1)$ | $\Gamma_{2,3,5}$ |
| $A_5^*$ | $120$ | $A_5$ | $a^2 = b^3 = (ab)^5 \; (=1)$ | $\Gamma_{2,3,5}$ |

$*$: $H = \nu(G)$ : $\nu : SU(2) \longrightarrow SO(3)$

Observation:

- The numbers $(p, q, r)$ occured twice : in the exponents of the group presentation and in the Dynkin graphs $\Gamma_{p,q,r}$.
- The relation $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 + \frac{4}{\lvert G \rvert}$ holds.

Question:

- Is there any other occurance of $(p, q, r)$?
- How do we explain the relation $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 + \frac{4}{\lvert G \rvert}$?

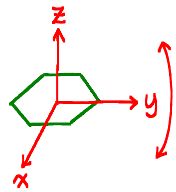
## • A Coxeter group $H'$

We shall introduce another group $H'$ associated with $H$. Recall that when classifying finite subgroups of $SO(3)$, we introduced them as rotational symmetries of regular $n$-gons and polyhedrons. Then, $G$ was introduced as the preimage $G = \nu^{-1}(H)$ under $\nu : SU(2) \longrightarrow SO(3)$. ("Spin" symmetry!)
However, the regular $n$-gons and polyhedrons also have "reflectional" symmetries, coming from $H \subseteq SO(3) \hookrightarrow O(3)$. Note that in dim 3,

$$O(3) \cong SO(3) \amalg (-I)SO(3) \cong SO(3) \times \mathbb{Z}/2 .$$

(This is not true for even dimensions : $\det(-I) = (-1)^{2k} = 1 \Rightarrow -I \in SO(2k)$).
Thus we would expect $H' \cong H \times \mathbb{Z}/2$. However, this is not true in general. Let's look at them case by case.

(1). Regular n-gon:



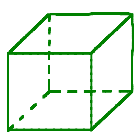

$A: z \longmapsto -z$, id on $x$-$y$ plane.

Note that in this case,

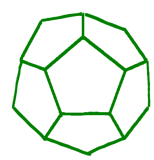
$$D_{2n} = \left\{ \begin{pmatrix} \cos\theta_k & -\sin\theta_k & 0 \\ \sin\theta_k & \cos\theta_k & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} R & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\theta_k & \sin\theta_k & 0 \\ -\sin\theta_k & \cos\theta_k & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; \begin{array}{l} \theta_k = \frac{k}{n}\cdot 2\pi,\ 0 \le k < n,\ R: \text{a} \\ \text{reflection in } x\text{-}y \text{ plane} \end{array} \right\}$$

Therefore $D_{2n}$ commutes with $A = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \in O(3) \setminus SO(3)$ ($A$ acts trivially on the n-gon!) and in this case it's $D_{2n} \times \mathbb{Z}/2$.

(2).



$\mathrm{Rot}(\text{cube}) \cong S_4$



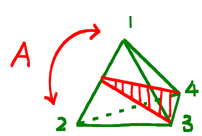
$\mathrm{Rot}(\text{Icosahedron}) \cong A_5$

Now it's easy to see that $-I \in O(3) \setminus SO(3)$ actually preserves these regular polyhedron. Since $-I \in Z(O(3))$, $H'$ in these cases are just

$S_4 \times \mathbb{Z}/2$ $\qquad\qquad$ $A_5 \times \mathbb{Z}/2$

respectively.

(3).



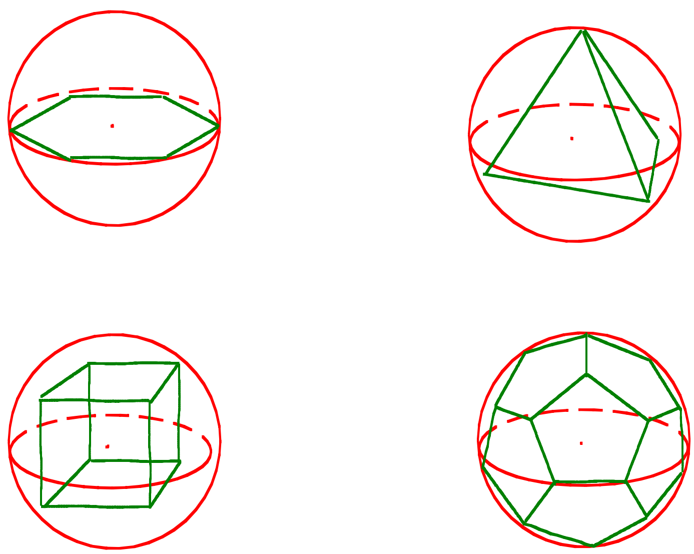

$\mathrm{Rot}(\text{tetrahedron}) \cong A_4$

Note that in this case $-I_{3\times 3}$ doesn't preserve the tetrahedron. Instead, the reflection $A$ acts as $(12)$. Similarly we have $(23)$, $(34)$ and thus in this case $H' \cong S_4$.

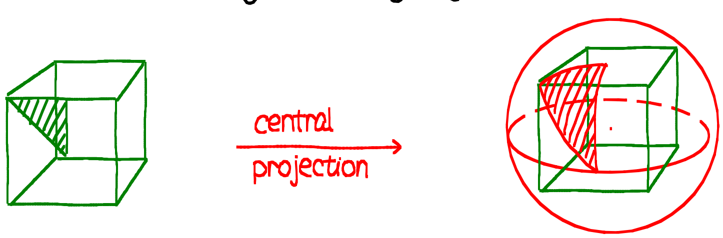
As a corollary, we see that $|H'| = 2|H| = |G|$. We add these data into the following table:

| $G$ | $\lvert G\rvert = \lvert H'\rvert$ | $H$ | $H'$ | Presentation of $H$ ($G$) | Dynkin graph |
|---|---|---|---|---|---|
| $D_{2n}^*$ | $4n$ | $D_{2n}$ | $D_{2n}\times\mathbb{Z}/2$ | $a^2=b^2=(ab)^n\ (=1)$ | $\Gamma_{2,2,n}$ |
| $A_4^*$ | 24 | $A_4$ | $S_4$ | $a^2=b^3=(ab)^3\ (=1)$ | $\Gamma_{2,3,3}$ |
| $S_4^*$ | 48 | $S_4$ | $S_4\times\mathbb{Z}/2$ | $a^2=b^3=(ab)^4\ (=1)$ | $\Gamma_{2,3,5}$ |
| $A_5^*$ | 120 | $A_5$ | $A_5\times\mathbb{Z}/2$ | $a^2=b^3=(ab)^5\ (=1)$ | $\Gamma_{2,3,5}$ |

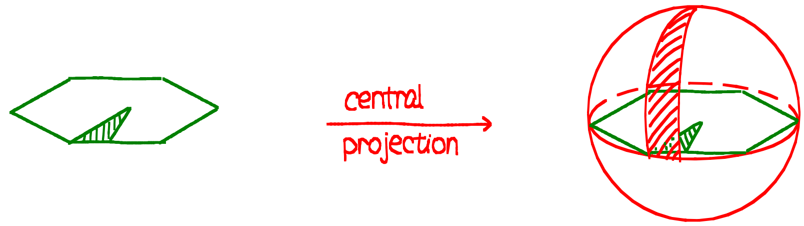
Next, we shall find nice presentations of these groups. To do this, we inscribe the regular n-gons and regular polyhedrons into the unit sphere $S^2$:



Since $H'$ preserves both the regular polyhedrons and the unit sphere, $H'$ will also preserve the central projection images of the regular polyhedrons onto the unit sphere $S^2$. Let $\Delta$ be a fundamental domain of the $H'$ action on the polyhedron, then its image on $S^2$ would be a spherical triangle, whose boundaries consist of arcs of great circles:




$\Delta$ and its spherical image

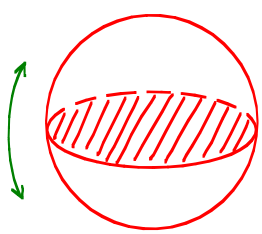



Note the slight difference here: the $\mathbb{Z}/2$ factor acts trivially on the regular n-gon but non-trivially on the sphere. Instead, we can think of the n-gon has some "thickness", so that its upper and lower faces are different.

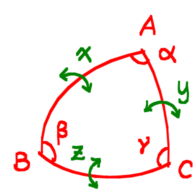
Since H' acts faithfully on the sphere, and transitively on all fundamental domains, it acts faithfully transitively on their spherical images. Thus

$$|H'| = \#\{\text{spherical fundamental domains}\}$$

Now using these spherical fundamental domains, it's easy to describe the generators of H' in terms of the spherical reflection about the sides of a spherical fundamental domain (triangle)

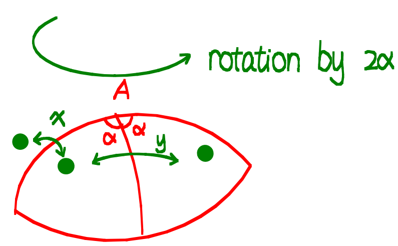


A spherical reflection: about the equator

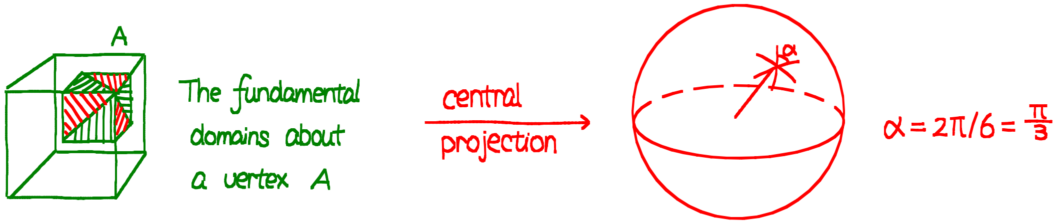



The spherical reflection generators $x, y, z$.

It's easy to see that, the composition of $xy$ is the rotation about the $\vec{AO}$ direction by $2\alpha$:

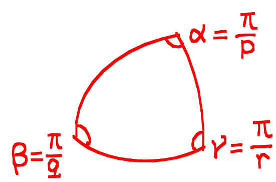



and thus $(xy)^n = 1$, where $n = 2\pi/\alpha$. For example, let's work out the cube case:




Thus $(xy)^3=1$. Similarly, it can be checked that the $\beta, \gamma$ angles are $2\pi/8=\frac{\pi}{4}$, $2\pi/4=\frac{\pi}{2}$ respectively, and thus $(yz)^4=1$, $(zx)^2=1$.

In general, we can check that, the angles of a spherical fundamental domain is $\pi/p$, $\pi/q$, $\pi/r$ respectively, where $p, q, r$ are the numbers in $\Gamma_{p,q,r}$ of the corresponding group $H$ (or $G$):




Moreover, it follows that $H'$ has the following presentation for $H'$ (possibly need to rename $x, y, z$).

$$H' = \langle x, y, z \mid x^2=y^2=z^2=1,\ (xy)^p=(yz)^q=(zx)^r=1 \rangle$$

It's also easy to see that the presentation of $H$ in terms of $a, b$ is also related to $x, y, z$ by:

$$a=xy,\quad b=yz,\quad (ab)^{-1}=zx$$

The above discussion then gives another occurence of $(p,q,r)$!

- Geometrization of $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1+\frac{4}{|G|}$

Actually, the title is a slight misnomer, and what we will "geometrize" is:

$$\boxed{\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1+\frac{4}{|H'|} \quad (*)}$$

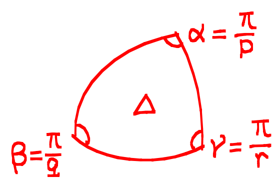
($|H'|=|G|$ anyway!)

To explain this we shall use the following:

Thm 16. (Area of a spherical triangle). A spherical triangle on the unit sphere with angles $\alpha, \beta, \gamma$ has area $\alpha+\beta+\gamma-\pi$.

The proof of the thm will be deferred. But using this thm and the previous disscussions about $H'$ and the spherical fundamental domains, we can give a satisfactory explanation of formula (*):

Since $H'$ acts simply transitively on the collection of all spherical fundamental domains (triangles), they all have the same area. Since




their angles are $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$ respectively, their areas are all equal to

$$Area(\Delta) = \frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} - \pi$$

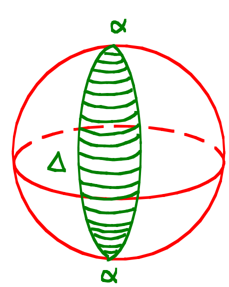
On the other hand, the sum of their total area is just the total area of the sphere, we then have:

$$Area(\Delta) = 4\pi / |H'|$$

Now (*) follows by equating these two.

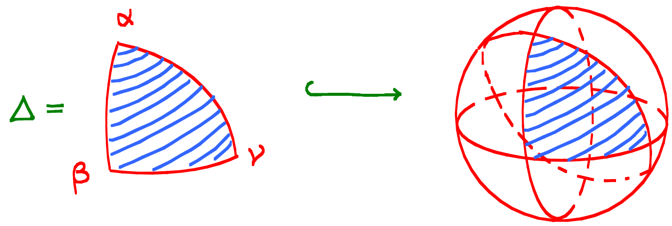
Proof of Thm 16.

We first prove this formula in the degenerate case where one of the angles is degenerate:

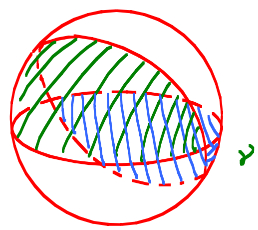



Observe that in this case $Area(\Delta)$ is propotional to $\alpha$, and when $\alpha = 2\pi$, $\Delta$ covers $S^2$ and $Area(\Delta) = 4\pi \implies Area = 4\pi \cdot \frac{\alpha}{2\pi} = 2\alpha$.

Next, let $\Delta$ be any spherical triangle and consider all the great circles forming its sides:




Note that any two great circles, say, those cutting out $\nu$, form a situation we considered above:




and thus the total shaded area is $2 \cdot 2\nu = 4\nu$. Similarly for $\alpha$ and $\beta$. Altogether, these shaded areas cover the whole unit sphere, but with $\Delta$ and its mirror image about the center counted 3 times. Thus:

$$4\alpha + 4\beta + 4\nu - 4 \cdot \mathrm{Area}(\Delta) = 4\pi$$

$$\Longrightarrow \mathrm{Area}(\Delta) = \alpha + \beta + \nu - \pi,$$

as claimed. $\square$

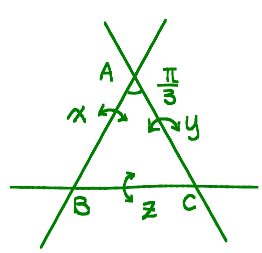
- Affine and indefinite cases

For the affine / indefinite graphs $\Gamma_{p,q,r}$ $\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 \,/\, < 1\right)$, we may consider the same presentation of Coxeter groups:

$$H' \triangleq \langle x, y, z \mid x^2 = y^2 = z^2 = 1,\ (xy)^p = (yz)^q = (zx)^r = 1 \rangle,$$

but things will be different: $H'$ won't be finite any more!

E.g. $\Gamma_{3,3,3}$.

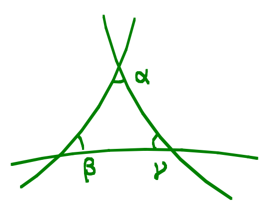



$x, y, z$: reflections on $\mathbb{R}^2$ about the sides of a regular triangle.

Then $xy$ is the rotation of $\mathbb{R}^2$ about $A$ by $\frac{2}{3}\pi$, and thus $(xy)^3 = 1$. Similarly $(yz)^3 = (zx)^3 = 1$. Clearly this group is infinite, but it contains a finite subgroup $H = \{(xy), (yz), (zx)\}$.

In general for affine $\Gamma_{p,q,r}$, the result is similar and $H'$ acts by affine transformations on $\mathbb{R}^2$, and that's why these graphs are called affine.

In the indefinite case, such a triangle no longer lives on $S^2$ or $\mathbb{R}^2$, but rather on $H^2$, the hyperbolic space, where the area of a triangle is given by $\pi - \alpha - \beta - \gamma$.




The Coxeter group defined this way will be very large (i.e. the number of group elements grows exponentially with respect to "length" of the group elements, it's like a free group). These groups are called hyperbolic.