HOMOTOPIC TOPOLOGY

by

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PREFACE TO THE ENGLISH EDITION

This book was written on the basis of lectures held at the Moscow State University during the mid-sixties. They were preceded by a number of striking discoveries of general importance to mathematics, first of all, by the Atiyah–Singer theorem on the index of elliptic operators. At this time topology attracted a good many mathematicians from fields such as analysis and differential equations. They suddenly felt a new-born interest in the subject which they had considered a somewhat obsolete and rather useless field before. Lecture halls, as if at a stroke, became overcrowded when the topic dealt with there was topology. However, we should mention here that the lecturers themselves had been brought up on the homotopic topology of the 'fifties—and they were strongly influenced by its algebraic approach. Albeit due to a different reason, but the index formula of elliptic operators was a strange and distant idea to them just as to the majority of their audience. For them the calculation of the homotopy groups of spheres was the main subject of topology (or mathematics as a whole?) Why? It would be hard to answer this question now in retrospect. Nevertheless, the lectures referred to above were overburdened with calculation. A lecturer's main aim was to dig a tunnel for the ignorant from the basic terms to "the height of heights"—the Adams spectral sequence, and it was only a lucky chance that this tunnel led through a few reefs of gold.

To the reader, the book offers a wide range of topics: singular homology, obstruction theory, spectral sequences of fibre bundles, Steenrod squares. We hope that he or she will not be confused by the naive accentuation of some of them, and the bulky calculations of homotopy groups at the end of the book will prove a useful source for practice. As to other chapters of topology having more in common with geometry, the reader may consult other books on the subject. (Milnor's works in the literature are recommended.)

The book is fully illustrated by *A. Fomenko*'s pictures. One could hardly imagine the Russian original without them—they are an organic part of it. A well-known mathematician (and a renowned artist) today, Fomenko was a young student at the time the book was written, and his drawings give the feeling of a beginner's creative reaction to a fresh and promising subject. I have no doubt that they offer a useful guide to many readers who otherwise would have been, perhaps, lost in the "labyrinth of zeros and arrows" which algebraic topology was thought of some time ago.

The authors are greatful to *Károly Málуusz* for translating the book into English and to *Aliz Fialowsky* whose contribution to this edition was a great help.

D. Fuchs

CONTENTS

CHAPTER I

HOMOTOPY

§1. HOMOTOPY AND HOMOTOPY EQUIVALENCE

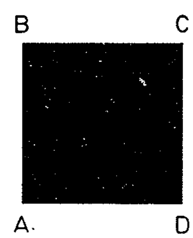
Some basic constructions of topological spaces

1. *Product space.* Let X and Y be topological spaces, $M = X \times Y$. A subset of M is open if it is the product of a pair of open subsets of X and Y, respectively, or if it is the union of an arbitrary number of such subsets.

★ *Exercise.* Prove that the axioms of topology are satisfied.

2. *Quotient space.* Let R be an equivalence relation on a space X. We consider the set of equivalence classes, denoted by X/R, and choose the weakest among the topologies for which the natural mapping f: $X \to X/R$ is continuous (i. e. a subset of X/R is open if and only if its pre-image in X is open). It is called the quotient topology in X/R. Whenever X/R will be mentioned, we shall always mean this particular topology.

Examples. Let X be a square and let us introduce the following equivalence relations R_i, $i = 1, \ldots, 5$. We consider as equivalent with respect to



R_1: points of the segments AB and DC if they lie on the same horizontal line (i. e. parallel with AD);

R_2: points of AB and CD if they lie on the same line passing through the centre of the square;

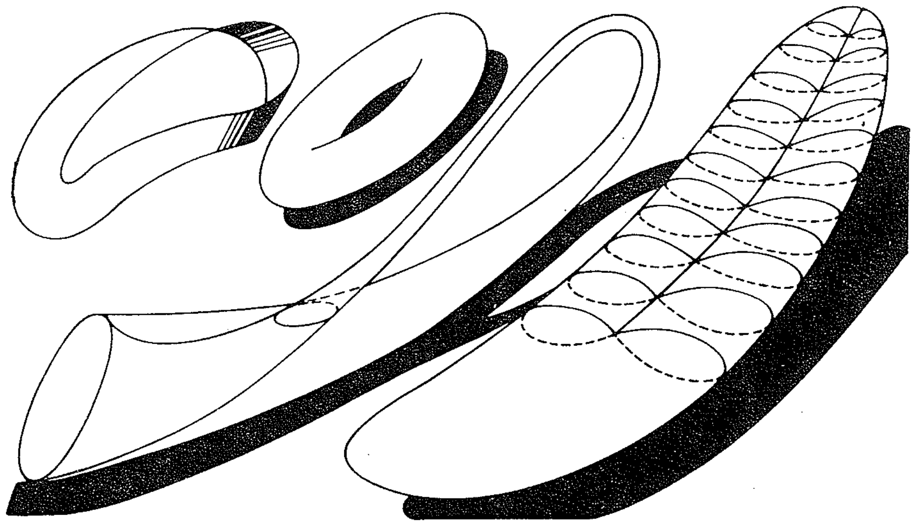
R_3: points of AB and CD according to R_1, and points of BC and AD analogously;

R_4: points of AB and CD according to R_2, and points of BC and AD according to R_1 and

R_5: points of AB and CD as well as points of BC and AD according to R_2.

Clearly X/R_1 is the annulus, X/R_2 is the Möbius band, X/R_3 is the two-dimensional torus, X/R_4 is the Klein bottle and X/R_5 is the projective plane.

3. *Attaching.* Let $A \subset X$ and $B \subset Y$ be topological spaces and f a mapping of A onto B. We define R on $X \cup Y$ in the following way: each $b \in B$ is equivalent with any $a \in A$ such that $f(a) = b$; points which are not involved in the mapping (i. e. points of

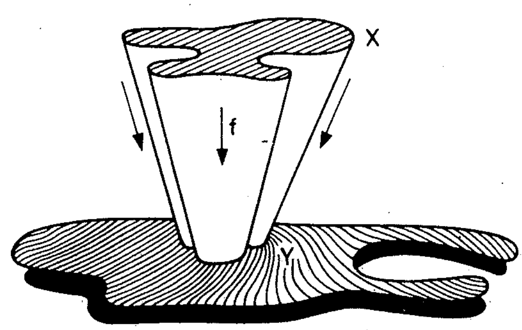


$(X\setminus A)\cup(Y\setminus B)$) are inequivalent. We denote the quotient space $X\cup Y/R$ by $X\cup_f Y$ and say that we obtained it by attaching X to Y along f.

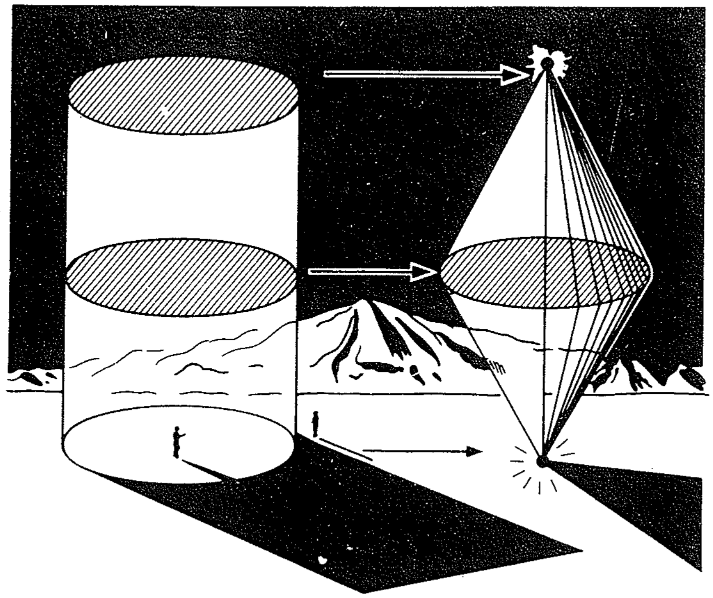
4. *Wedge*. Let x_0 and y_0 be points of X and Y, respectively, and let f: $x_0\to y_0$ be the mapping of the point x_0 into y_0. We shall call $X\cup_f Y$ the *wedge* or *union* of the pointed spaces X and Y, and denote it by $X\vee Y$.

For example, $S^1\vee S^1$, where S^1 is the circle, the figure looks like the number eight:

5. *Mapping cylindre*. Let X and Y be spaces and f: $X\to Y$ a continuous mapping. Assuming that f is also mapping $X\times(1)\to Y$, we obtain $(X\times I)\cup_f Y$. It will be called the mapping cylindre of f and denoted by C_f.



6. *Suspension*. Let X be a space and $X\times I$ its product with the interval $I=[0, 1]$. We collapse the upper face $X\times 1$ into a point and the lower face $X\times 0$ into another point. The result is called the suspension over X and is denoted by ΣX.



Example. $S^n = \Sigma S^{n-1}$ (S^n is the n-dimensional sphere).

7. *Mapping space.* Let $H(X, Y)$ be the set of all continuous mappings of the space X into the space Y. We shall always assume $H(X, Y)$ to be equipped with the following topology.

Let $\mathscr{C}$ be the family of all compact subsets of X and $\mathscr{U}$ be the family of all open subsets of Y. Let $[c, u]$, $c \in \mathscr{C}$, $u \in \mathscr{U}$ be the subset of $H(X, Y)$ consisting of all mappings f with $f(c) \subset u$. We take the subsets $[c, u]$ as a basis of a topology on $H(X, Y)$ which we call the *compact-open* topology.

Exercise. Show that if Y is a metric space, the compact-open topology is the same as the topology of uniform convergence on compact sets.

Exercise. Let X, Y and Z be three topological spaces. Show that if X, Y are Hausdorff spaces, and X is locally compact, then $H(X \times Z, Y)$ and $H(Z, H(X, Y))$ are homeomorphic. As the space $H(X, Y)$ is sometimes denoted by Y^X, this statement can be written as $Y^{X \times Z} = (Y^X)^Z$. Hence it is called the *exponential law.*

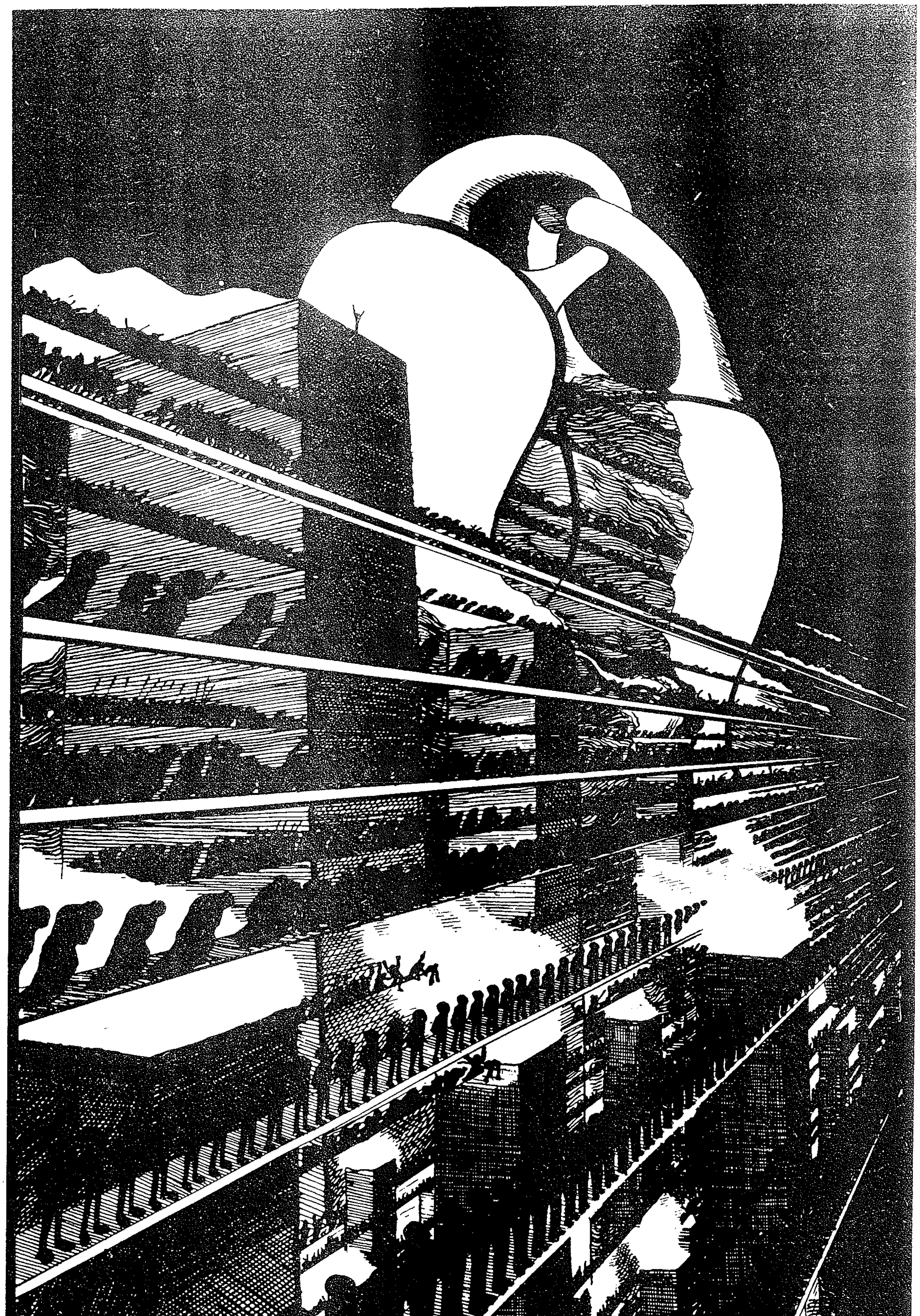
Examples. If $Y = *$ is a single point, then $H(X, Y)$ consists of one element.

If $X = *$ is a single point, then $H(X, Y) = Y$.

If $X = I = [0, 1]$, then $H(I, Y)$ is called the *path-space* of Y.

Let $y_0 \in Y$ be an arbitrary fixed point; the subspace $\Omega \subset H(I, Y)$ consisting of all mappings $f\colon I \to Y$ such that $f(0) = f(1) = y_0$, is the *loop space* of Y.

If any two points of a space can be connected by a path, we have a *pathwise-connected* space.



Homotopy

Let X and Y be topological spaces, f: $X \to Y$ and g: $X \to Y$ two continuous mappings. We say that f and g are homotopic and write $f \sim g$ whenever there exists a family φ_t: $X \to Y, t \in I$ of mappings such that

1) $\varphi_0 \equiv f$, $\varphi_1 \equiv g$;
2) the mapping Φ: $X \times I \to Y$, $\Phi(x, t) = \varphi_t(x)$ is continuous.

The continuous mapping Φ is called a *homotopy* between f and g.

The relation of homotopy is a relation of equivalence. (Prove it!)

Example. All mappings f: $X \to I$ of an arbitrary space X to the interval $I = [0, 1]$ are homotopic.

Indeed, for any f: $X \to I$ the mappings $\varphi_t = (1-t)f$ form a homotopy between f and the zero mapping $\varphi_1(x) \equiv 0$.

There is another way of defining homotopy as a path in the space which connects the point $f \in H(X, Y)$ with $g \in H(X, Y)$.

The relation of homotopy divides $H(X, Y)$ into a set of equivalence classes. It is denoted by $\pi(X, Y)$.

Examples. 1) $\pi(X, I) = *$ (it consists of a single element).

2) $\pi(*, Y)$ is the set of the pathwise-connected components of Y.

Let X, X' and Y be spaces and h: $X \to X'$ a mapping; we define h^*: $\pi(X', Y) \to \pi(X, Y)$ in the following way: for every class $\bar{\alpha} \in \pi(X', Y)$ we choose a representative $\alpha \in H(X', Y)$ and assign to $\bar{\alpha}$ the class $h^*(\bar{\alpha})$ of the mapping $\alpha \circ h \in H(X, Y)$ (i. e. the composition $X \xrightarrow{h} X' \xrightarrow{\alpha} Y$).

Let X, Y and Y' be spaces and h: $Y \to Y'$ a mapping; we define h_*: $\pi(X, Y) \to \pi(X, Y')$ in the following way. We take $\bar{\alpha} \in \pi(X, Y)$ and choose an arbitrary representative $\alpha \in H(X, Y)$ in it. Then assign to $\bar{\alpha}$ the class $h_*(\bar{\alpha})$ generated by $h \circ \alpha \in H(X, Y')$.

★ *Exercise.* Prove that the two definitions are correct.

Homotopy equivalence

We first give three equivalent definitions of the homotopy equivalence.

Definition 1. The spaces X_1 and X_2 are homotopy equivalent: $X_1 \sim X_2$ if there exist mappings f: $X_1 \to X_2$ and g: $X_2 \to X_1$ such that the composite mappings $g \circ f$: $X_1 \to X_1$, $f \circ g$: $X_2 \to X_2$ are homotopic to the identity mappings.

Remark. If $g \circ f$ and $f \circ g$ are not only homotopic to but also equal to the respective identity mapping, then f and g are homeomorphisms, moreover, they are inverses to each other. The notion of homotopy equivalence therefore generalizes the notion of homeomorphism.

Definition 2. $X_1 \sim X_2$ if for any space Y there exists a one-to-one correspondence $\varphi_Y: \pi(X_1, Y) \to \pi(X_2, Y)$ such that for every continuous mapping h: $Y \to Y'$ the diagram

$$\begin{array}{ccc} \varphi_Y: \pi(X_1, Y) & \to & \pi(X_2, Y) \\ \downarrow h_* & & \downarrow h_* \\ \varphi_{Y'}: \pi(X_1, Y') & \to & \pi(X_2, Y') \end{array}$$

is commutative, i. e.,

$$\varphi_{Y'} \circ h_* = h_* \circ \varphi_Y.$$

Definition 3. $X_1 \sim X_2$ if for every space Y there exists a one-to-one correspondence $\varphi^Y: \pi(Y, X_1) \to \pi(Y, X_2)$ such that for every continuous mapping h: $Y \to Y'$ the diagram

$$\begin{array}{ccc} \varphi^Y: \pi(Y, X_1) & \to & \pi(Y, X_2) \\ \uparrow h^* & & \uparrow h^* \\ \varphi^{Y'}: \pi(Y', X_1) & \to & \pi(Y', X_2) \end{array}$$

is commutative, i. e.,

$$\varphi^Y \circ h^* = h^* \circ \varphi^{Y'}.$$

Theorem. The definitions 1, 2 and 3 are equivalent.

Proof. We prove the equivalence of definitions 1 and 2.

Suppose that $X_1 \sim X_2$ in the sense of definition 2, then there exists a one-to-one correspondence $\varphi_{X_2}: \pi(X_1, X_2) \leftrightarrow \pi(X_2, X_2)$. We write $\bar{f} = \varphi_{X_2}^{-1}$ (id X_2) and choose $f \in \bar{f}$ (we shall keep the notation that $\bar{h}$ denotes the homotopy class of the mapping h). There exists, moreover, a one-to-one correspondence $\varphi_{X_1}: \pi(X_1, X_1) \to \pi(X_2, X_1)$. Put $\bar{g} = \varphi_{X_1}$ and choose $g \in \bar{g}$. We show that f and g satisfy the conditions of the definition, i. e., $f \circ g \sim$ id X_2 and $g \circ f \sim$ id X_1. The diagram

$$\begin{array}{ccc} \varphi_{X_2}: \pi(X_1, X_2) & \to & \pi(X_2, X_2) \\ \uparrow f_* & & \uparrow f_* \\ \varphi_{X_1}: \pi(X_1, X_1) & \to & \pi(X_2, X_1) \end{array}$$

is commutative by definition 2. Hence $\varphi_{X_2} \circ f_* = f_* \circ \varphi_{X_1}$. We consider the images of the element id X_1 under the mappings in the diagram. By the definition of f_* we have $f_*(\overline{\text{id } X}) = \overline{f \circ \text{id} X_1} = \bar{f}$; $\varphi_{X_2}(\bar{f}) = \overline{\text{id} X_2}$ by the choice of f. Therefore $\varphi_{X_2} \circ f_*(\overline{\text{id} X_1}) = \overline{\text{id} X_2}$. On the other hand $f_* \circ \varphi_{X_1} \overline{\text{id} X_1} = \overline{f \circ g}$. Since the diagram is commutative,

$$\overline{\text{id } X_2} = \overline{f \circ g}, \quad \text{i. e.,} \quad f \circ g \sim \text{id } X_2.$$

It can be proved similarly that $g \circ f \sim \text{id } X_1$. We have shown that 2 implies 1.

Let us now assume that there exist mappings f: $X_1 \to X_2$ and g: $X_2 \to X_1$ with $fg \sim \text{id} X_2$ with $gf \sim$ id X_1, and take an arbitrary space Y. We put $\varphi_Y = g^*$ and consider the mappings

$$\varphi_Y = g^*: \pi(X_1, Y) \to \pi(X_2, Y),$$

$$f^*: \pi(X_2, Y) \to \pi(X_1, Y).$$

We show that g^* and f^* are inverse to each other. By the definition of g^* and f^*, $g^*(\bar{\alpha})=\overline{\alpha\circ g}$ and $f^*(\overline{\alpha\circ g})=\overline{(\alpha\circ g)\circ f}=\overline{\alpha\circ(g\circ f)}=\alpha$, since $\overline{g\circ f}=\overline{\mathrm{id}\,X_1}$. It can be checked in the same way that $g^*\circ f^*=\mathrm{id}$.

We have verified that φ_Y has an inverse mapping, that is, φ_Y is one-to-one.

Let us now verify the second property of φ_Y.

Let Y' be a space and h: $Y\to Y'$ a continuous mapping. We consider the diagram

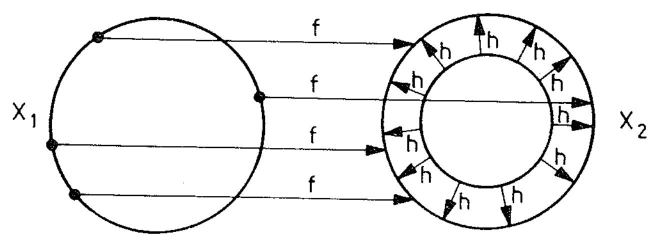
$$\begin{array}{ccc} \varphi_Y\equiv g^*\colon \pi(X_1, Y) & \leftrightarrow & \pi(X_2, Y) \\ \downarrow h_* & & \downarrow h_* \\ \varphi_{Y'}\equiv g^*\colon \pi(X_1, Y') & \leftrightarrow & \pi(X_2, Y') \end{array}$$

which is commutative. Indeed, if $\alpha\in\pi(X_1, Y)$, then $h_*(\bar{\alpha})=\overline{h\circ\alpha}$, $\varphi_{Y'}(\overline{h\circ\alpha})=g^*(h\circ\alpha)=$ $=\overline{(h\circ\alpha)\circ g}$; on the other hand $\varphi_Y(\bar{\alpha})=g^*(\bar{\alpha})=\overline{\alpha\circ g}$ and $h_*(\overline{\alpha\circ g})=\overline{h\circ(\alpha\circ g)}$. The statement is proved.

The equivalence of 1 and 3 can be proved in the same way. It is easy to show that homotopy equivalence is indeed an equivalence relation in the usual sense.

A class of homotopy equivalent spaces is a *homotopy type.*

Example for spaces of the same homotopy type: X_1 is a circle and X_2 a ring:



Here f: $X_1\to X_2$ is an imbedding and $g\equiv f^{-1}\circ h$: $X_2\to X_1$ (h is a contraction along the radii).

Obviously homeomorphic spaces are homotopy equivalent. As it is seen in the example, the converse statement is not true.

★ *Exercise.* Prove that a contractible space is homotopy equivalent to the single-point space. (A space X is called *contractible* if the identity mapping is homotopic to a mapping $X\to X$ which takes X into one point.)

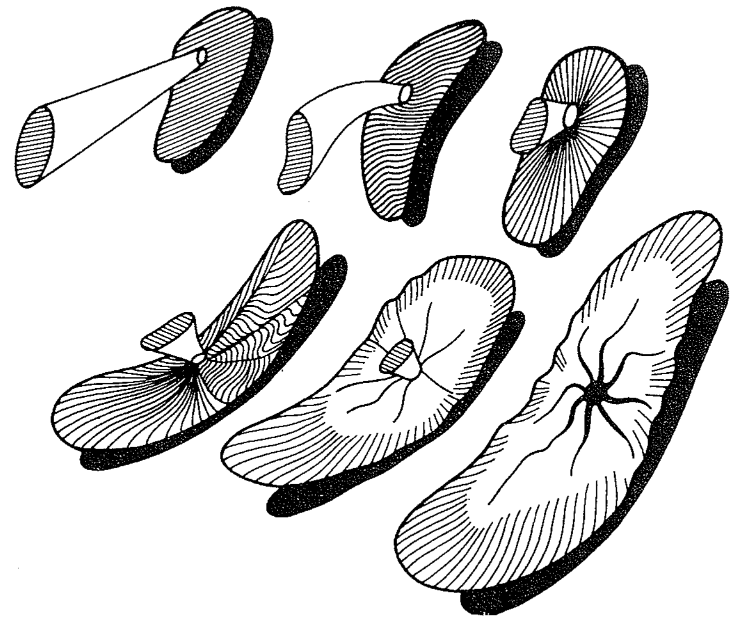
★ *Exercise.* Prove that the cylinder of a mapping f: $X\to Y$ is homotopy equivalent to Y.

★ *Exercise.* Construct two spaces X_1 and X_2 such that even though there exist one-to-one continuous mappings f: $X_1\to X_2$ and g: $X_2\to X_1$, the spaces are not homotopy equivalent.

A subspace $X\subset Y$ is called *contractible* in Y to the point y_0 if the inclusion mapping $X\subset Y$ and the mapping $X\to Y$, taking the whole X into y_0, are homotopic to each other.

★ *Exercise.* Let X be any space; $\mathrm{cat}_1 X$ (the Lyusternik–Schnirelmann category of X) is defined as the minimal cardinality of sets I with $X=\bigcup_{i\in I} X_i$ such that:

1. the sets X_i are closed,



2. the mapping of inclusion φ_{0i}: $X_i \subset X$ and the mapping φ_{1i}: $X_i \to X$ which maps X_i into one point are homotopic for every $i \in I$.

For spaces good enough, condition 2 is equivalent to

2′. for every $i \in I$ there exists a homotopy $\varphi_t^{(i)}$: $X \to X$ such that $\varphi_0^{(i)} = \mathrm{id}\, X$ and $\varphi_1^{(i)}$ sends X_i to one point.

aps X_i

d $\varphi_1^{(i)}$

We get the definition of the "strong category" $\text{cat}_2 X$ by substituting 2 by the following condition:

2″. every X_i is contractible to a point.

Exercise. Are $\text{cat}_1 X$ and $\text{cat}_2 X$ invariant to homotopy equivalence? (Are they "homotopy invariants" of X?)

Exercise. Let K be the two-dimensional sphere with three of its points identified. Is then $\text{cat}_1 K = \text{cat}_2 K$?

The relative case

A *topological pair* or simply a *pair* is a space with a specified subspace. A mapping of the pair (X, A) into (X', A') is a mapping $f\colon X \to X'$ such that $f(A) \subset A'$.

Homotopy, homotopy equivalence, and all related notions are naturally transferred to the class of pairs and their mappings. (Verify it!)

In the special case $A = \{x_0\}$ the pair $(X, A) = (X, x_0)$ is called a pointed space.

The space $H((X, x_0), (X', x_0'))$ of all mappings $(X, x_0) \to (X', x_0')$ will be denoted by $H_b(X, X')$. (In this notation X and X' are symbols for pointed spaces.) In the similar sense we use the notation $\pi_b(X, X')$. The index b (from the word *base*) shows that each space has a base point and only the mappings that carry base points into base points are considered.

We also may consider *triples* (X, A, B). It is then understood that $B \subset A \subset X$; a mapping $f\colon (X, A, B) \to (X', A', B')$ of triples is a mapping $f\colon X \to X'$ such that $f(A) \subset A'$ and $f(B) \subset B'$.

§2. NATURAL GROUP STRUCTURE ON THE SETS $\pi(X, Y)$

In homotopy theory we study invariants assigned to topological spaces and continuous mappings, whose values are taken from discrete sets. As a rule, these invariants coincide if the spaces are homotopy equivalent and the mappings are homotopic. We have a general procedure for constructing such kind of invariants. Namely, we fix a space Y and assign to any space X the set $\pi(X, Y)$ (or $\pi(Y, X)$). In many cases it is easier to study these sets than the spaces X. Information about $\pi(X, Y)$ can be turned into information about X.

We have already noticed an important property of the sets $\pi(X, Y)$. If $X' \to X''$ and $Y' \to Y''$ are mappings between spaces, there are mappings $\pi(X'', Y) \to \pi(X', Y)$ and $\pi(X, Y') \to \pi(X, Y'')$ that corresponds to them. In other words, $\pi(\cdot, \cdot)$ is a functor from the category of topological spaces into the category of sets; it is contravariant in the first argument and covariant in the second one.

Studying $\pi(X, Y)$ becomes considerably easier when it is equipped with a natural group structure. Before explaining this notion in detail let us agree on the form we choose for presenting the material. We shall study invariants of two kinds. We will fix a space Y and then we shall assign to X either $\pi(X, Y)$ or $\pi(Y, X)$. We shall prove theorems for either case. The theories remain parallel — more exactly, dual — for a good time. This is called the Eckman–Hilton duality. We are not going to expound it in the present book, nevertheless in this § we shall give emphasis to this notion by giving visibly parallel exposition of the dual definitions, statements and proofs.

Throughout the present §, spaces will be assumed to be equipped with base points, i. e. we shall consider pointed spaces.

Let us fix a space Y with base point y_0.

Suppose that for every X, the set $\pi_b(X, Y)$ is equipped with a group structure. Such structures are called natural if for any continuous mapping $\varphi\colon X' \to X''$, $\varphi^*\colon \pi_b(X', Y) \to \pi_b(X'', Y)$ is a homomorphism.

Definition. Y is a H-space if there are given mappings

$$\mu\colon Y \times Y \to Y$$

and

$$\nu\colon Y \to Y$$

such that

(i) the mappings

$$Y \xrightarrow{j_1} Y \times Y \xrightarrow{\mu} Y$$

and

$$Y \xrightarrow{j_2} Y \times Y \xrightarrow{\mu} Y,$$

where $j_1(y) = (y, y_0)$, $j_2(y) = (y_0, y)$ are homotopic to the identity mapping id $Y\colon Y \to Y$;

(ii) (homotopy associativity) the mappings

$$Y \times Y \times Y \xrightarrow{\text{id } Y \times \mu} Y \times Y \xrightarrow{\mu} Y$$

and

$$Y \times Y \times Y \xrightarrow{\mu \times \text{id } Y} Y \times Y \xrightarrow{\mu} Y$$

are homotopic;

Suppose that for every X, the set $\pi_b(X, Y)$ is equipped with a group structure. Such structures are called natural if for any continuous mapping $\varphi\colon X' \to X''$, $\varphi'_*\colon \pi(Y, X') \to \pi(Y, X'')$ is a homomorphism.

Definition. Y is a H'-space if there are given mappings

$$\mu\colon Y \to Y \vee Y$$

and

$$\nu\colon Y \to Y$$

such that

(i) the mappings

$$Y \xrightarrow{\mu} Y \vee Y \xrightarrow{\pi_1} Y$$

and

$$Y \xrightarrow{\mu} Y \vee Y \xrightarrow{\pi_2} Y$$

where $\pi_1(\pi_2)$ is the identity mapping on the first Y (on the second Y) and trivial on the second Y (on the first Y) are homotopic to the identity mapping id $Y\colon Y \to Y$.

(ii) (homotopy coassociativity) the mappings

$$Y \xrightarrow{\mu} Y \vee Y \xrightarrow{\text{id } Y \vee \mu} Y \vee Y \vee Y$$

and

$$Y \xrightarrow{\mu} Y \vee Y \xrightarrow{\mu \vee \text{id } Y} Y \vee Y \vee Y$$

are homotopic;

(iii) the mapping

$$Y \xrightarrow{\text{id } Y \times \nu} Y \times Y \xrightarrow{\mu} Y$$

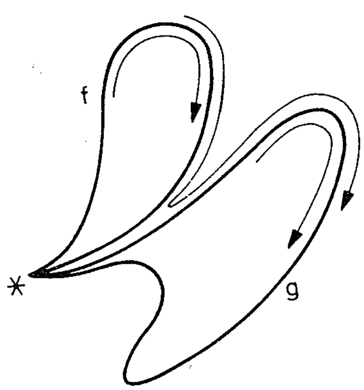
is homotopic to the constant mapping (into the single point).

An important *example* for a H-space. The space $Y_0 = \Omega Z$ of loops in Z, where Z is an arbitrary space.

The mapping μ: $\Omega Z \times \Omega Z \to \Omega Z$ is given by the formula

$$\mu(f, g)(t) = \begin{cases} f(2t), & 0 \leq t \leq \frac{1}{2} \\ g(2t-1), & \frac{1}{2} \leq t \leq 1, \end{cases}$$

i. e. we assign to f and g the loop we obtain by walking first along f and then along g.



The mapping ν: $\Omega Z \to \Omega Z$ is given by the formula

$$\nu(t) = f(1-t).$$

Another important *example* of a H-space is any topological group.

Theorem. The set $\pi_b(X, Y)$ may be equipped with a group structure natural in X if and only if Y is a H-space.

Proof. Necessity. Assume that for every X, there is a multiplication in $\pi_b(X, Y)$, which is natural in X.

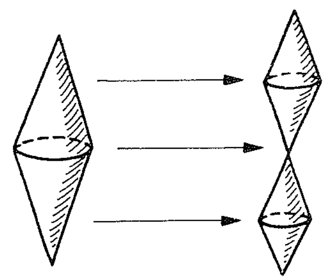
(iii) the mapping

$$Y \xrightarrow{\mu} Y \vee Y \xrightarrow{\text{id } Y \vee \nu} Y$$

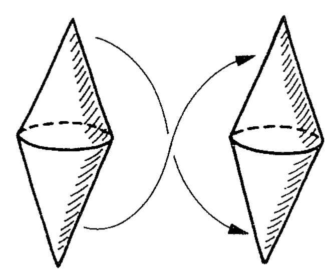
is homotopic to the mapping into the single point.

An important *example* for a H'-space. The suspension $Y_0 = \Sigma Z$ over an arbitrary space Z.

The mapping μ: $\Sigma X \to \Sigma X \vee \Sigma X$ is given in the following way:



The mapping ν: $\Sigma X \to \Sigma X$ is given as follows:



It is useful to assume the line segment over the base point of ΣX to be contracted into a single point, namely the base point of ΣX. Such modification does not alter the homotopy type of ΣX (for sufficiently good X).

Theorem. The set ~~$\pi_b(X, Y)$~~ $\pi_b(Y, X)$ may be equipped with a group structure natural in X if and only if Y is a H'-space.

Proof. Necessity. Assume that for every X, there is a multiplication in $\pi_b(Y, X)$, which is natural in X.

Let us choose $X = Y \times Y$ and consider the homotopy classes $\bar{\alpha}_1, \bar{\alpha}_2 \in \pi_b(Y \times Y, Y)$ of the projections of $Y \times Y$ onto the factors.

Set $\bar{\mu} = \bar{\alpha}_1 \cdot \bar{\alpha}_2$. (Here we use the multiplication in $\pi_b(Y \times Y, Y)$. Let μ: $Y \times Y \to Y$ be an arbitrary mapping for which $\mu \in \bar{\mu}$. We define v: $Y \to Y$ as a representative of the coset $\bar{v} \in \pi_b(Y, Y)$ given as the inverse in the group $\pi_b(Y, Y)$ to the coset of the identity mapping id Y: $Y \to Y$.

Properties (i–iii) of μ and v are automatically satisfied. Let us verify, for example, that $\mu \circ j_1 \sim \mathrm{id}\ Y$. The mapping j_1: $Y \to Y \times Y$ induces j_1^*: $\pi_b(Y \times Y, Y) \to \to \pi_b(Y, Y)$ which maps α_1 onto $\alpha_1 \circ j_1$ and α_2 onto $\alpha_2 \circ j_1$. Now $\alpha_1 \circ j_1(y) = *$ and $\alpha_2 \circ j(y) = y$, thus $\alpha_1 \circ j_1 = *$ (where $*$ is the constant mapping) and $\alpha_2 \circ j_1 = = \mathrm{id}\ Y$. Being natural in X, the product is carried into product, hence

$$j_1^*(\alpha_1 \circ \alpha_2) = \bar{*} \circ \overline{\mathrm{id}\ Y} = \overline{\mathrm{id}\ Y},$$

i. e. $\mu \circ j_1 \sim \mathrm{id}\ Y$.

(We have made use of the fact that the coset of the constant mapping, i. e. the mapping that sends the whole space Y into the base point y_0, is the identity of the group $\pi_b(Y, Y)$. This can immediately be proved by considering the single-point space for X and the mapping $Y \to X$; we obtain a homomorphism $\pi_b(X, Y) \to \pi_b(Y, Y)$ where the identity, i.e. the single element of the group is carried into the identity of $\pi_b(Y, Y)$.)

Sufficiency. Suppose that Y is a H-space. Let X be an arbitrary space. Then μ: $Y \times Y \to Y$ induces μ_*: $\pi_b(X, Y \times Y) \to \to \pi_b(X, Y)$. We compose it with the natural imbedding φ: $\pi_b(X, Y) \times \pi_b(X, Y)$

Let us choose $X = Y \vee Y$ and consider the homotopy classes $\bar{\alpha}_1, \bar{\alpha}_2 \in \pi_b(Y, Y \vee Y)$ of the imbeddings of Y into $Y \vee Y$.

Set $\bar{\mu} = \bar{\alpha}_1 \cdot \bar{\alpha}_2$. (Here we use the multiplication in $\pi_b(Y, Y \vee Y)$.) Let μ: $Y \to Y \vee Y$ be an arbitrary mapping for which $\mu \in \bar{\mu}$. We define v: $Y \to Y$ as a representative of the coset $\bar{v} \in \pi_b(Y, Y)$ given in the group as the inverse of the coset of the identity mapping id Y: $Y \to Y$.

Properties (i–iii) of μ and v are automatically satisfied. Let us verify, for example, that $j_1 \circ \mu \sim \mathrm{id}\ Y$. The mapping j_1: $Y \vee Y \to Y$ induces j_{1*}: $\pi_b(Y, Y \vee Y) \to \to \pi_b(Y, Y)$ which maps α_2 onto $j_1 \circ \alpha_2$ and α_1 onto $j_1 \circ \alpha_1$. Now $j_1 \circ \alpha_1(y) = *$ and $j_1 \circ \alpha_2(y) = y$, thus $j_1 \circ \alpha_1 = *$ (where $*$ is the constant mapping) and $j_2 \circ \alpha_2 = \mathrm{id}\ Y$. Being natural in X, the product is carried into product, hence

$$j_{1*}(\alpha_2 \circ \alpha_1) = \bar{*} \circ \overline{\mathrm{id}\ Y} = \overline{\mathrm{id}\ Y},$$

i. e. $j_1 \circ \mu \sim \mathrm{id}\ Y$.

(We made use of the fact that the coset of the constant mapping, i. e. the mapping that sends the whole space Y into the base point y_0, is the identity of the group $\pi_b(Y, Y)$. This can immediately be proved by considering the single-point space for X and the mapping $X \to Y$; we get a homomorphism $\pi_b(Y, X) \to \pi_b(Y, Y)$ where the identity, i. e. the single element of the group $\pi_b(Y, X)$ is carried into the identity of $\pi_b(Y, Y)$.)

Sufficiency. Suppose that Y is a H'-space. Let X be an arbitrary space. Then μ: $Y \to Y \vee Y$ induces μ^*: $\pi_b(Y \vee Y, X) \to \to \pi_b(Y, X)$. We compose it with natural imbedding φ: $\pi_b(Y, X) \times \pi_b(Y, X)$

$\xrightarrow{(=)} \pi_b(X, Y \times Y)$. We obtain a mapping which we denote by μ_*.

Similarly, ν: $Y \to Y$ induces ν_*: $\pi_b(X, Y) \to \to \pi_b(X, Y)$.

Multiplication μ_* and inversion ν_* define on $\pi_b(X, Y)$ a group structure natural in X, as the reader will easily verify.

★ *Exercise.* $\pi_b(X, \Omega\Omega Z)$ is an Abelian group.

Let $n \geq 1$. Since the n-dimensional sphere S^n is the suspension over S^{n-1}, $\pi_b(S^n, X)$ is a group. It will be called the n-th homotopy group of X and denoted by $\pi_n(X)$. It follows from the exercise that $\pi_n(X)$ is Abelian if $n \geq 2$.

$\xrightarrow{(=)} \pi_b(Y \vee Y, X)$. We obtain a mapping which we denote by μ^*.

Similarly, ν: $Y \to Y$ induces ν^*: $\pi_b(Y, X) \to \to \pi_b(Y, X)$.

Multiplication μ^* and inversion ν^* define on $\pi_b(Y, X)$ a group structure natural in X, as the reader will easily verify.

★ *Exercise.* $\pi_b(\Sigma\Sigma Z, X)$ is an Abelian group.

Let $n \geq 1$. There exists a space K_n (cf. §8) such that

(1) $\pi_i(K_n) = \begin{cases} 0 & \text{for } i \neq n, \\ \mathbf{Z} & \text{for } i = n; \end{cases}$

(2) $K_{n-1} \sim \Omega K_n$.

Then $\pi_b(X, K_n)$ is a group. It will be called the n-th integral cohomology group of X and denoted by $H^n(X)$.

Moreover,

$$H^n(S^m) = \begin{cases} 0 & \text{for } n \neq m, \\ \mathbf{Z} & \text{for } n = m \end{cases} \quad \text{(cf. §12)}$$

§3. CW COMPLEXES

A CW complex is a topological space which is represented as a disjoint union $K = \bigcup_{q=0}^{\infty} \bigcup_{i\in I_q} e_i^q$ of sets (cells) e_i^q, if there exists a family of continuous mappings $f_i^q\colon B^q \to X$ (where B^q is the q-dimensional ball), called the characteristic mapping for e_i^q, such that the restriction of f_i^q to Int B^q is a homeomorphism Int $B^q \approx e_i^q$ and $f_i^q(S^{q-1})$ is contained in the union of the cells of smaller dimensions: $f_i^q(S^{q-1}) \subset \bigcup_{p=0}^{q-1} \bigcup_{i\in I_p} e_i^p$. Further, the following axioms have to be satisfied:

(C) The closure of each cell meets only a finite number of cells;

(W) a subset $F \subset K$ is closed if and only if for each e_i^q the pre-image $(f_i^q)^{-1}(F) \subset B^q$ is closed in B^q.

A CW complex is *finite* if it consists of finitely many cells. A subcomplex of a complex K is a CW complex contained in K as a closed subset, whose cells are cells of K as well. For example, a subcomplex of K is its n-th *skeleton*, that is, the union of all of its cells of dimension $\leq n$.

A complex is *locally finite* if each point in it has a neighbourhood that belongs to a union of finitely many cells.

Exercise. Prove that any cell is contained in a finite subcomplex.

★ *Exercise.* Prove that the direct product of a locally finite CW complex with an arbitrary one is a CW complex. Its cells are the products of the cells of the two factors.

Exercise. Prove that the topology given by axiom (W) is the weakest one among the topologies for which the characteristic mappings are continuous.

★ *Exercise.* A function given on a CW complex is continuous if and only if it is continuous on every finite subcomplex.

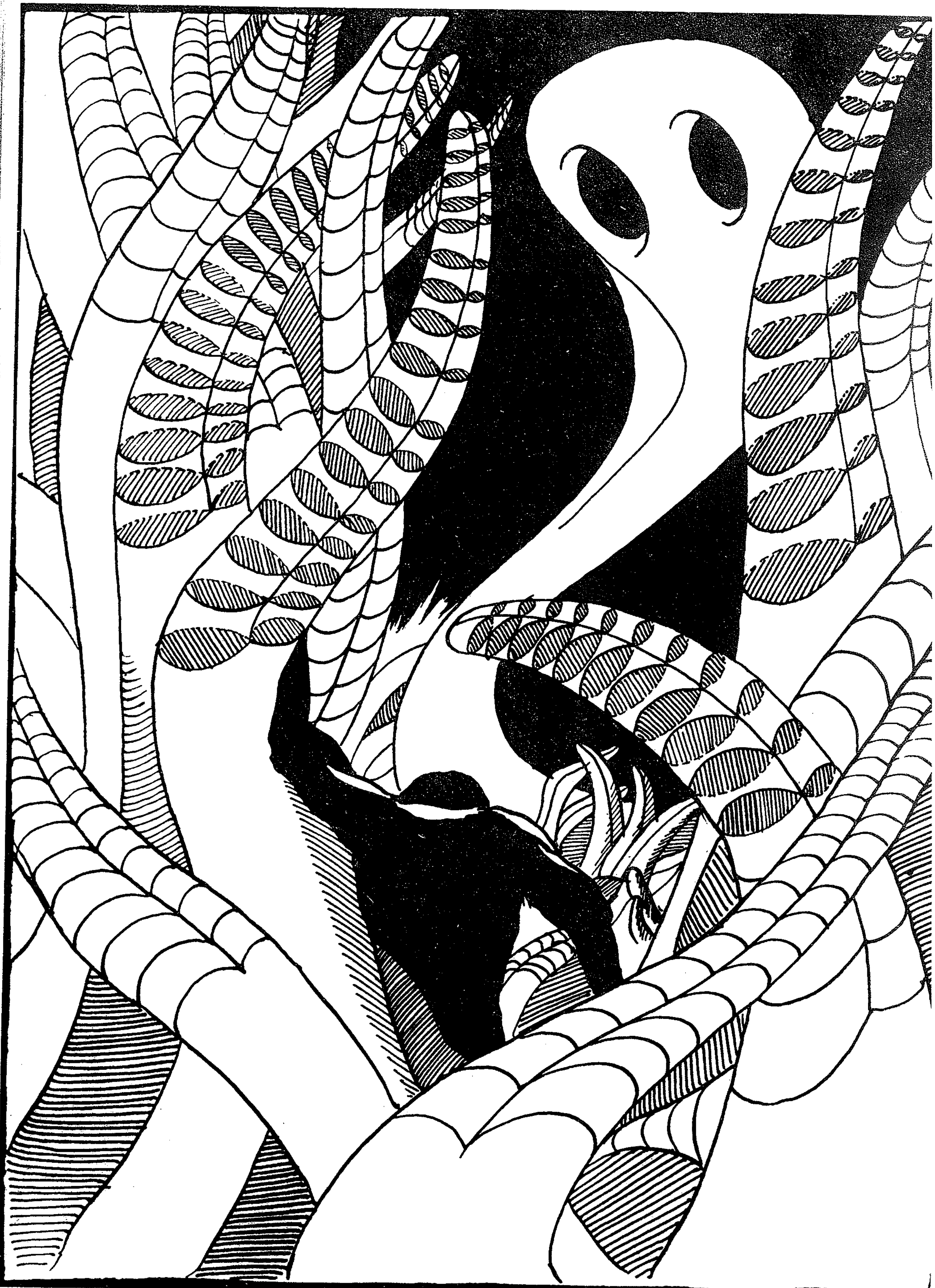
Axiom (C) does not imply (W). Indeed, let S^∞ be the set of sequences $(x_1, x_2, \ldots)$ of real numbers, satisfying the conditions (a) for sufficiently large i, $x_i = 0$, and (b) $\sum_{i=1}^{\infty} x_i^2 = 1$. The topology in S^∞ is defined by means of the usual metric $\rho(\{x_i\},\{y_i\}) = (\Sigma(x_i - y_i)^2)^{1/2}$. The topological space S^∞ can be represented as a union $\bigcup_{q=0}^{\infty} \bigcup_{i=1}^{2} e_i^q$ where

$$e_1^q = \{x = (x_1, x_2, \ldots) \mid x_i = 0 \quad \text{for} \quad i > q;\ x_q > 0\},$$

$$e_2^q = \{x = (x_1, x_2, \ldots) \mid x_i = 0 \quad \text{for} \quad i > q;\ x_q < 0\}.$$

This is a cell structure that satisfies (C) but does not satisfy (W). (Prove it!)

We note that if a space admits a division into cells which satisfies all the conditions except (W), one can always weaken the topology by applying the condition (W) so that the space becomes a CW complex.



Examples for CW complexes.

1. The n-dimensional sphere S^n.

It may be represented as a union $e^0 \cup e^n$ of a point e^0 and its complement $e^n = S^n \setminus e^0$. The characteristic mapping $f^n \colon B^n \to S^n$ of the cell e^n transfers the boundary of the ball B^n into the point, and homeomorphically maps the interior of B^n onto e^n.

Another cell structure can be defined on S^n similarly to the previous example.

2. The real projective space $\mathbf{RP}^n$ of dimension n. We choose in $\mathbf{RP}^n$ a sequence of projective subspaces

$$* = \mathbf{RP}^0 \subset \mathbf{RP}^1 \subset \ldots \subset \mathbf{RP}^n$$

and set $e^0 = \mathbf{RP}^0$, $e^1 = \mathbf{RP}^1 \setminus \mathbf{RP}^0, \ldots, e^n = \mathbf{RP}^n \setminus \mathbf{RP}^{n-1}$. The representation $\mathbf{RP}^n = \bigcup_{q=0}^{n} e^q$ clearly defines on $\mathbf{RP}^n$ a structure of a CW complex.

3. Similarly the n-dimensional complex projective space can be represented as a CW complex with one cell in each dimension $0, 2, 4, \ldots, 2n$. The n-dimensional projective space over the field of quaternions has an analogous cell structure with one cell in each dimension $0, 4, 8, \ldots, 4n$.

★ *Exercise*. Represent as CW complexes:

a) the torus,

b) the Klein bottle,

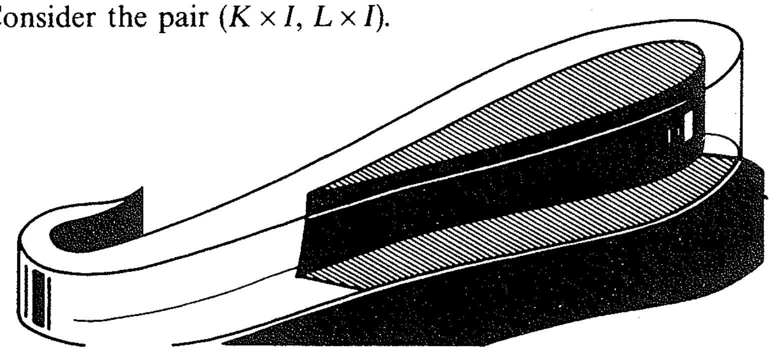
c) the suspension over a given CW complex.

★ *Exercise*. Prove that any finite CW complex can be imbedded into an Euclidean space of sufficiently large dimension.

Definition. A topological pair (A, B) is called a *Borsuk pair* (or *cofibration*) if for any X and $F: A \to X$, an arbitrary homotopy $f_t: B \to X$ with $f_0 = F|_B$ can be extended to a homotopy $F_t: A \to X$ such that $F_0 = F$ and $F_t|_B = f_t$.

Theorem (Borsuk). Any CW pair (K, L) (i. e. K is a CW complex and L is its subcomplex) is a Borsuk pair.

Proof. Consider the pair $(K \times I, L \times I)$.



Assume that there are given $\Phi: L \times I \to X$ (the homotopy f_t) and $F: K \times 0 \to X$, and $F|_{L \times 0} = \Phi|_{L \times 0}$. Extending homotopy f_t to F_t is the same as extending the mapping $F: K \times 0 \to X$ to a mapping $F': K \times I \to X$ for which $F'|_{L \times I} = \Phi$.

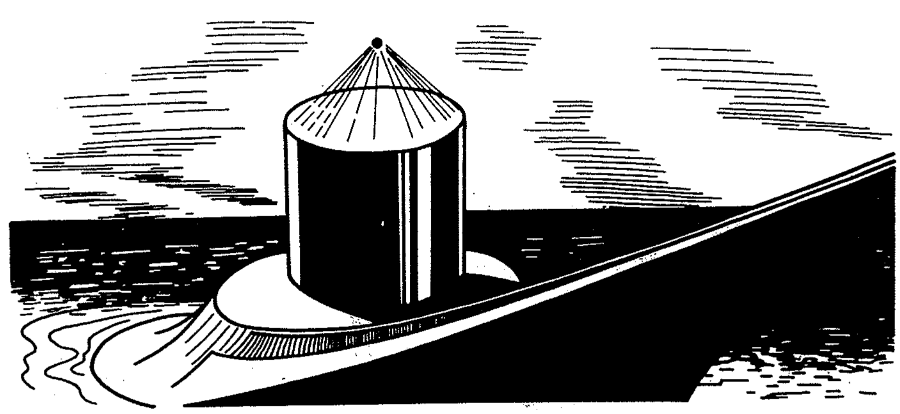
The construction will be carried out by induction on the dimensions of the cells. Let $n=0$. For null-dimensional cells e_i^0 we set

$$F'(a, t) = \begin{cases} F(a, 0) & \text{if} \quad a = e_i^0 \notin L, \\ \Phi(a, t) & \text{if} \quad a \in L. \end{cases}$$

Assume now that F' has been extended from $L \times I$ to $K^n \times I$, where K^n is the n-dimensional skeleton of K.

Let us take an arbitrary $(n+1)$-dimensional cell $e^{n+1} \notin L$. By induction, Φ is given on $(\overline{e^{n+1}} \setminus e^{n+1}) \times I$, because the boundary $\dot{e}^{n+1} = \overline{e^{n+1}} \setminus e^{n+1}$ of the cell e^{n+1} is the same as $f^{n+1}(\partial B^{n+1})$, thus it belongs to K^n by the definition of CW complexes. (Here f^{n+1} is the characteristic mapping of e^{n+1}.)

The next thing to do is to extend F' to the interior of the cylinder $f^{n+1}(B^{n+1}) \times I$ from the "wall" $f^{n+1}(\partial B^{n+1}) \times I$ and the bottom $f^{n+1}(B^{n+1})$.



Again by the definition of CW complexes it is clear that this is equivalent to extending $\psi\colon (\partial B^{n+1}\times I)\cup(B^{n+1}\times\{0\})\to K$ to a mapping $\psi'\colon B^{n+1}\times I\to K$.

Let us take a point outside the cylinder and near the ball $B^{n+1}\times\{1\}$. The mapping $\eta\colon B^{n+1}\times I\to(\partial B^{n+1}\times I)\cup(B^{n+1}\times\{0\})$ of projecting the cylinder from the point onto the boundary is the identity mapping on the boundary. So we define the mapping ψ' by $\psi'(a,t)=\psi\circ\eta(a,t)$.

The cells e_i^{n+1} do not intersect one another, thus the mapping may be defined this way on the whole $(n+1)$-skeleton K^{n+1}. Q. e. d.

Corollary 1. Let K be a CW complex and L its subcomplex. If L is contractible, then $K/L\sim K$.

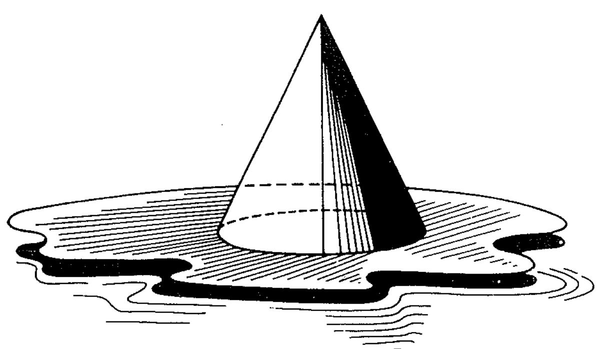
Proof. We construct $p\colon K\to K/L$ and $q\colon K/L\to K$, and show that 1) $q\circ p\cong \mathrm{id}\,K$ and 2) $p\circ q\sim \mathrm{id}\,(K/L)$.

1) Denote by p the projection $K\to K/L$. Since L is contractible, there exists a homotopy f_t such that $f_0\colon L\to L\subset K$ is the identity mapping, i. e. $\mathrm{id}\,(K/L)=f_0$ and $f_1|_L=*$.

By the Borsuk theorem there exists a homotopy $F_t\colon K\to K$ such that $F_0=\mathrm{id}\,K$ and $F_t|_L=f_t$. Then $F_1(L)=*$. Thus F_1 may be considered as a mapping given on K/L. More precisely, $F_1=q\circ p$ where $q\colon K/L\to K$ is some mapping. We obtain $F_1\equiv F_0$, i. e. $q\circ p\sim\mathrm{id}\,K$.

2) We show that $p\circ q\sim\mathrm{id}\,(K/L)$. As above, $p\circ F_t(L)=*$ implies $p\circ F_t=q_t\circ p$, where $q_t\colon K/L\to K/L$ is a homotopy, and $q_0=\mathrm{id}\,(K/L)$, $q_1=p\circ q$. Hence $p\circ q\sim\mathrm{id}\,(K/L)$. Q. e. d.

Corollary 2. If (K,L) is a Borsuk pair, then $K/L\sim K\cup CL$ where CL is the cone over L.



The proof is left to the reader.

Definition. A mapping $f\colon K\to L$ is *cellular* if $f(K^n)\subset L^n$ $(n=0,1,\ldots)$, where L^n and K^n are the n-skeletons of K and L.

The cellular approximation theorem

Let f: $K \to L$ be a continuous mapping between CW complexes K and L. Assume that f is cellular on a subcomplex $K_1 \subset K$. Then there exists a mapping g: $K \to L$ such that 1) $f \sim g$; 2) $f|_{K_1} = g|_{K_1}$; 3) g is cellular on K; 4) the homotopy connecting f and g may be chosen fixed on K_1.

In the proof of the theorem we shall need some notions from the theory of simplicial complexes. We notice that characteristic mappings of cells of CW complexes may be considered as mappings of simplexes Δ^q as well as closed balls. A CW complex is simplicial if the following two conditions are satisfied:

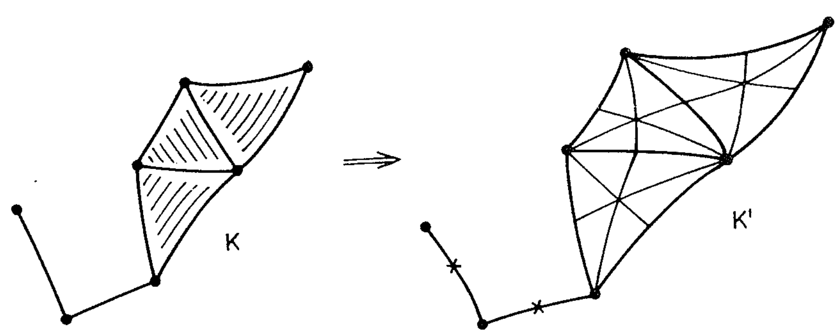
(i) Each characteristic mapping f_i^q: $\Delta^q \to K$ is a homeomorphism on the whole $\bar{e}_i^q$.

(ii) For any face $\Delta^r \subset \Delta^q$, $f_i^q(\Delta^r)$ coincides with the closure of one of the cells of K, and $f_i^q|_{\Delta^r}$: $\Delta^r \to K$ coincides with the characteristic mapping of this cell (up to affine transformation of Δ^r). We do not distinguish between the r-dimensional standard simplex and the r-dimensional face of the q-dimensional standard simplex. The necessary corrections will be left to the reader.

The closures of the cells of a simplicial complex are its *simplexes*. The null-dimensional cells are called *vertices*. The star of a vertex is the union of all simplexes containing the vertex. The star of a vertex a will be denoted by St (a).

A simplicial mapping between two simplicial complexes K and L is a continuous mapping which linearly maps simplexes of K onto simplexes of L (of the same, or smaller dimensions). In particular, a simplicial mapping sends vertices into vertices, thus two simplicial mappings coincide whenever they coincide on the vertices of K.

A simplicial complex K' is a *subdivision* of K if K and K' coincide as topological spaces and each simplex of K is a union of some complete simplexes of K' (in other words, K' is obtained by dividing the simplexes of K into smaller ones). An important case is the *barycentric subdivision.*



It is obtained as follows. After the $(q-1)$-skeleton of K has been divided, we find the centre of each q-dimensional simplex and divide it into pyramides with their tops at the centre and bottoms coinciding with one of the various simplexes of the barycentric subdivision of the boundary.

Exercise. Any finite CW complex is homotopy equivalent to a simplicial one.

★ *Exercise.* Any finite simplicial complex is the subcomplex of a simplex of sufficiently large dimension. In particular, it can be imbedded in the Euclidean space in such a way that the imbedding is linear on each simplex.

★ *Exercise.* The dimension of the Euclidean space in the previous exercise can be cut down to $2n+1$, where n is the dimension of the complex.

The simplicial approximation theorem

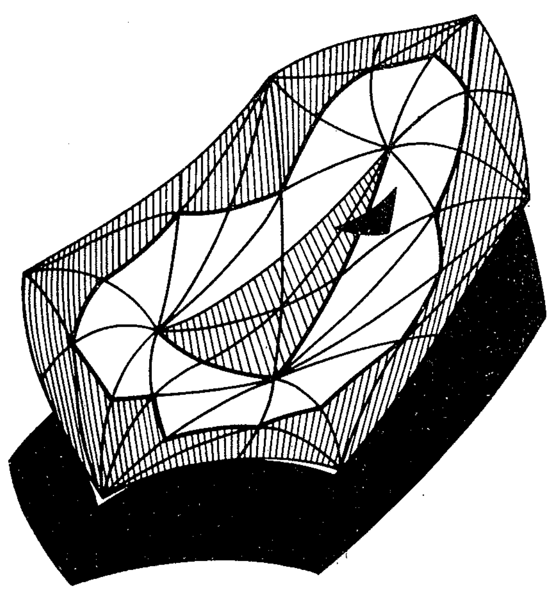
Theorem. Let $f\colon K \to L$ be a continuous mapping between finite simplicial complexes. Then there exists a refinement K' of K and a mapping $f'\colon K' \to L$ which is simplicial and homotopic to f.

Remark. We are going to construct a mapping which is not merely homotopic but also, in a certain sense, near to f. As simplicial approximations only play an auxiliary role in our investigations, we give no formal treatment to this property. It can be found in almost every textbook on topology (cf. for example, *Rochlin–Fuchs,* The beginner's course in topology, Springer, 1984).

Proof. Complexes L and K will be assumed as being imbedded in an Euclidean space.

Let σ be an arbitrary simplex of L, and let L' be its first barycentric subdivision. Let v be a vertex. If $v \notin \sigma$ then $\rho(\sigma, \mathrm{St}'(v)) > 0$, where ρ is the distance and the comma $'$ means that the item belongs to a barycentric subdivision. In the present case, the comma means belonging into the first subdivision L'. As L is a finite complex, we have $\min \varrho(\sigma, \mathrm{St}'(v)) = a > 0$ for $v \notin \sigma$.

Now f is a uniformly continuous mapping, thus there exists a subdivision K' of K with the property that $\mathrm{diam}\,(f(\sigma')) < a$ for any $\sigma' \subset K'$. Here diam denotes the diameter



of the set under consideration. (For K' we may choose a multiple barycentric subdivision.) For a vertex $w' \in K'$, we define $f'(w')$ to be equal to any of the vertices for which $f(w') \in \mathrm{St}'(v)$. As it can easily be seen, if vertices $w_0', w_1', \ldots$ belong to the same simplex of K', then $f'(w_0'), f'(w_1'), \ldots$ belong to the same simplex of L. Hence extending f' "by linearity" as a simplicial mapping $K' \to L$ is possible.

Next we show that $f' \sim f$. If $f(x) \in \sigma$, where σ is a simplex of L, then $f'(x) \in \sigma$ as well. Indeed, in the opposite case there exists at least one vertex of K', belonging to a simplex containing x, whose image by f belongs to the barycentric star of a vertex not contained in σ. This would, however, contradict the construction.

Finally we define the homotopy connecting f' with f by the formula $\varphi(x, t) = f(x) - [f(x) - f'(x)]t$. Q. e. d.

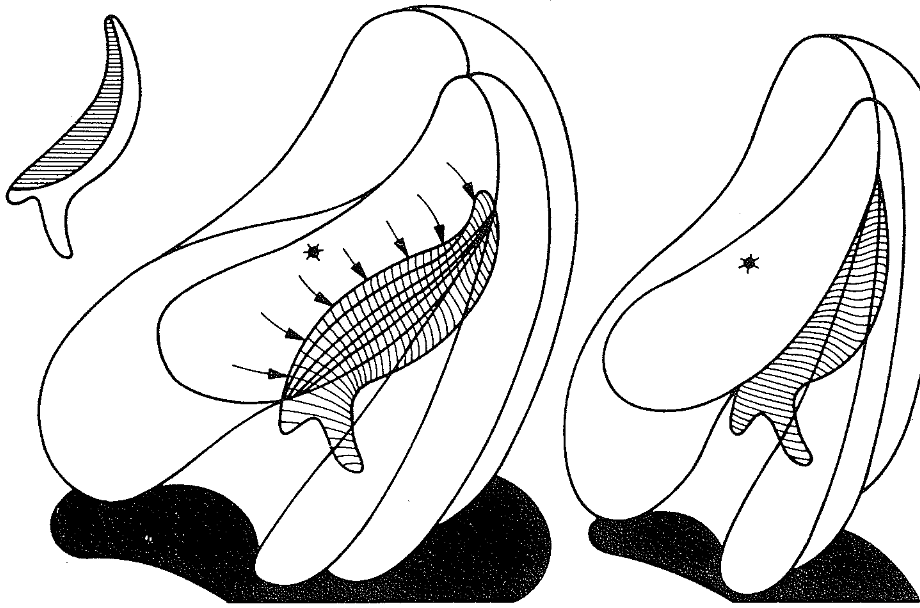
Proof of the cellular approximation theorem

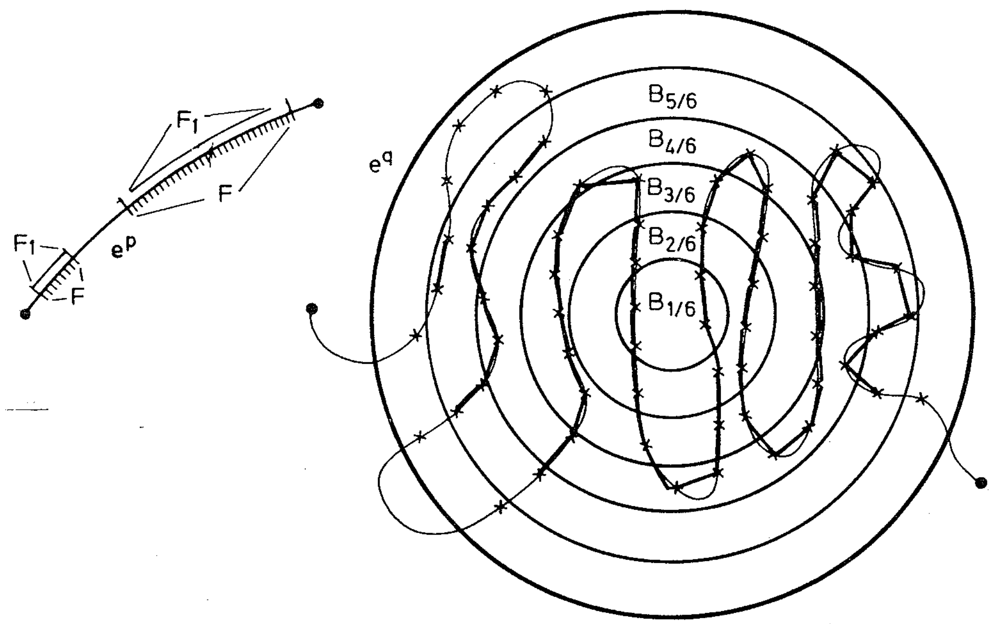
Suppose that the mapping is already cellular on the cells of the subcomplex K' as well as on all cells of K of dimensions smaller than p. Let $e^p = e_i^p$ be a p-dimensional cell of K. By axiom (C) its image $f(e^p)$ meets only finitely many cells of K. (Indeed, $f(\bar{e}^p)$ is compact, being the image of a compact set by a continuous mapping.) Let us consider one of these cells; assume that it has the highest dimension possible. We shall denote it by ε^q. There are two possibilities.

1) The image of e^p does not fill the whole ε^q. Then we can take a free point and pull the set $f(e^p)$, along the radii starting at this centre, to the boundary. (We have a homeomorphism between the cell and the open ball.) This deformation can be extended to a homotopy which is given on $K' \cup K^p$ and is constant outside $e^p \cap f^{-1}(\varepsilon^q)$. By the Borsuk theorem it can be extended on the whole K.

2) It may happen that there are no "free" points, i. e. $\varepsilon^q \subset f(e^p)$. In that case one can apply the following procedure: substitute f on a part of e^p, by a simplicial mapping whose image does not fill ε^q. (If the reader has already understood "everything", he may skip the end of this section.)

First of all we identify the interiors of e^p and ε^q with the open unit balls of the corresponding dimensions. Let $B_r \subset \varepsilon^q$ denote the closed concentric ball with radius r. Since $\bar{f}$ is cellular on the $(p-1)$-skeleton of K, we can take in e^p a finite simplicial simplex F containing $e^p \cap f^{-1}(B_{5/6})$. We take a subdivision of F into finer simplexes such that (i) whenever α is a simplex of F (in the new subdivision) and $f(\alpha) \cap B_{5/6} \neq \Phi$, we have $f(\alpha) \subset \varepsilon^q$; (ii) for any $f(\alpha) \subset \varepsilon^q$, $\operatorname{diam} f(\alpha) < 1/6$. (We recall that ε^q is the unit ball.) Now let us consider in F the minimal subcomplex F_1 that contains all simplexes whose images meet $B_{4/6}$. Then $B_{4/6} \cap f(e^p) \subset f(F_1) \subset B_{5/6}$. Together with f we consider another mapping $f_1\colon F_1 \to B_{5/6}$ which coincides with f on the vertices of F_1 and is linear on each simplex. (Again ε^q is the unit ball!) Now $f = f_0$ and f_1 are connected with the homotopy $f_t\colon F_1 \to B_{5/6}$ moving each point $f_t(x)$ with constant speed from $f(x)$ to $f_1(x)$ along a corresponding line segment.





The necessity of "being sewn" is not clear from this figure. It appears for $p > 1$.

Mappings f and f_1 can be "sewn" together in the following way. Let $\bar{f}$: $K' \cup K^p \to L$ be defined by

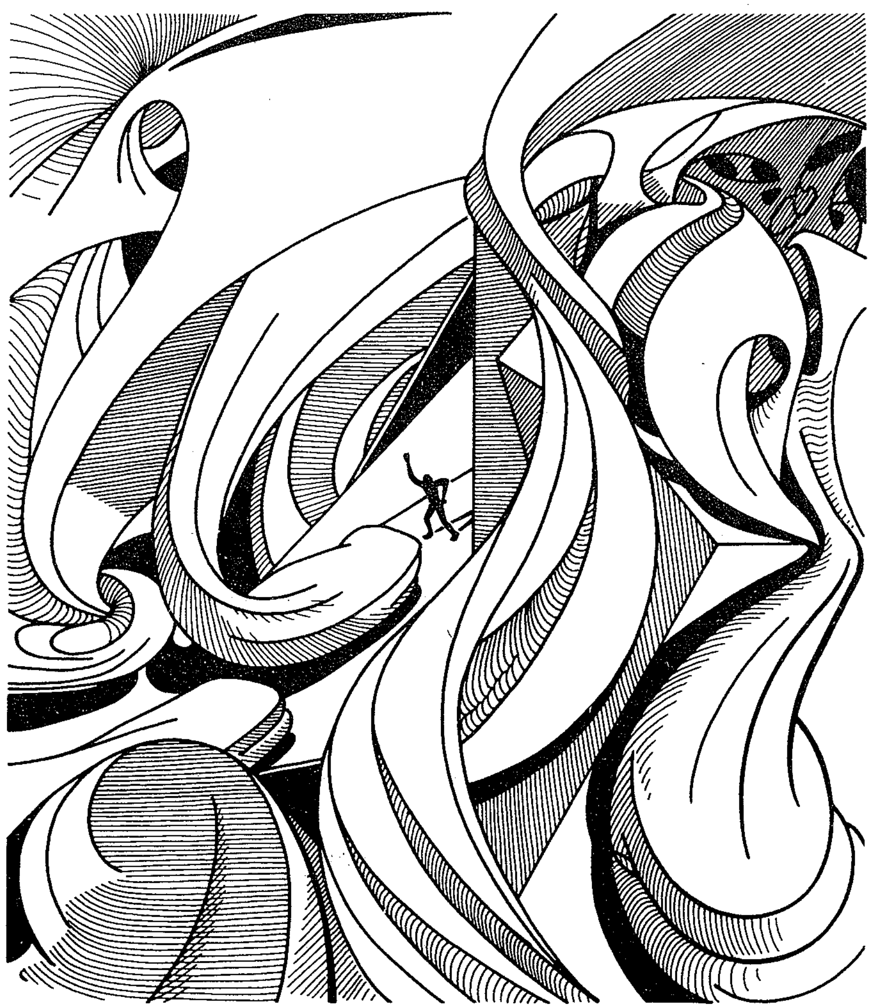
$$\bar{f}(x) = \begin{cases} f(x) & \text{if } f(x) \notin B_{3/6} \text{ or } x \notin e^p, \\ f_1(x) & \text{if } f(x) \in B_{2/6} \text{ or } x \in e^p, \\ f_{3-6\varphi(x)}(x) & \text{otherwise.} \end{cases}$$

Here $\varphi(x)$ is the distance between the point $f(x)$ and the centre of the ball ε^q. (It is defined only if $f(x) \in \varepsilon^q$.)

Clearly $\bar{f}$ is continuous, homotopic to f, and coincides with f outside of e^p as well as outside of $f^{-1}(\varepsilon^q)$. The image of e^p meets $B_{1/6}$ only in finitely many p-dimensional planes. Thus it does not fill ε^q. By the Borsuk theorem, the homotopy between f and $\bar{f}$ can be continued on the whole K.

We have reduced case (ii) to case (i).

Now we are ready to prove the theorem inductively, applying the above construction. If K is a finite complex, there is nothing to be added. The case of infinite complexes still requires some accuracy. Instead of going into details, we leave this part to the reader. Note: if K has infinitely many cells of the same dimension, then the best is to apply the construction to all these cells simultaneously. Further, if K has cells of arbitrarily large dimensions, axiom (W) has to be referred to.



This completes the proof of the cellular approximation theorem.

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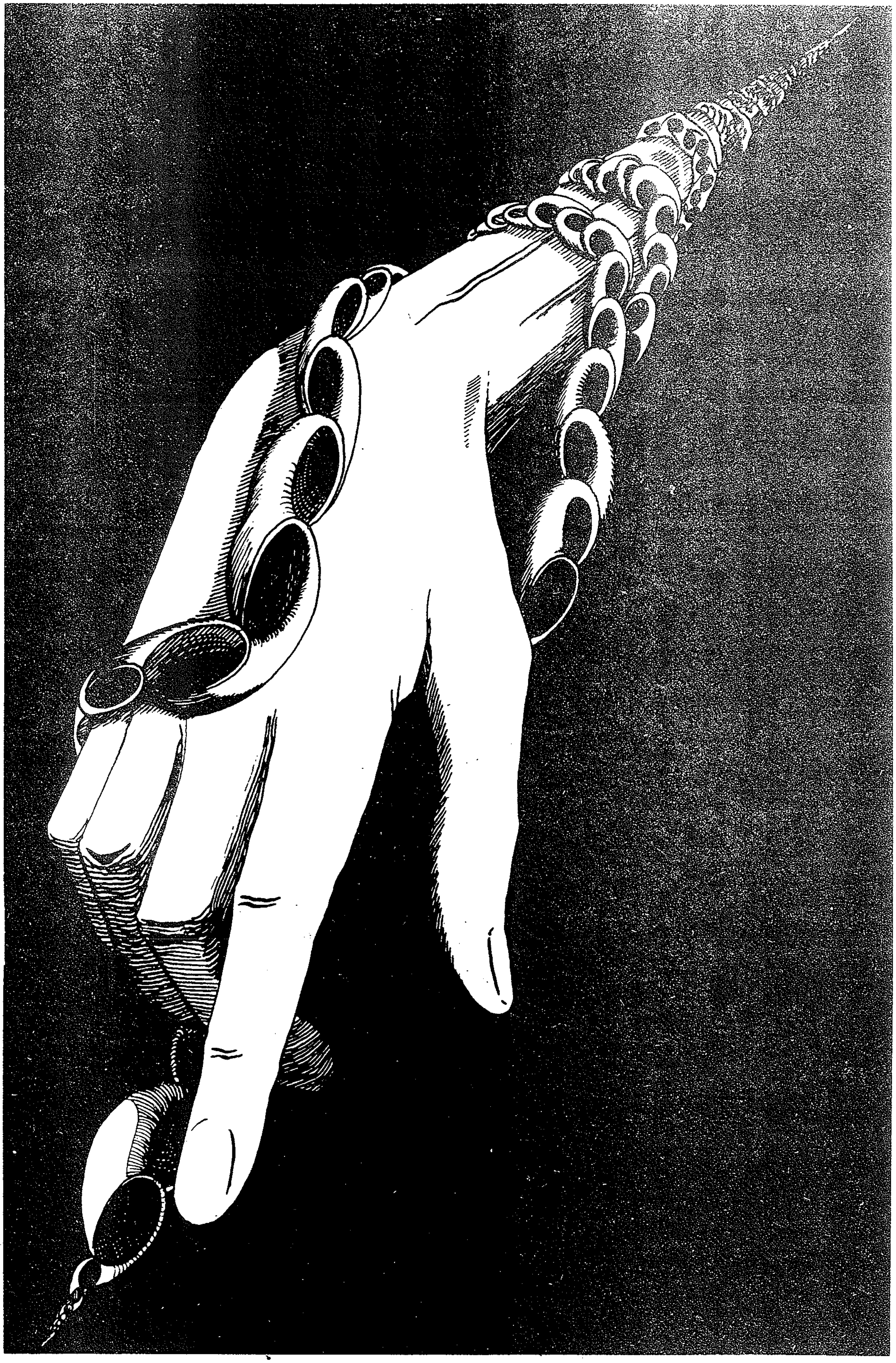


Some consequences of the cellular approximation theorem

1. Let X and Y be CW complexes and assume that X has a single vertex and no other cells up to dimension q while dim $Y<q$. Then any mapping $Y \to X$ is homotopic to the constant (i. e. the mapping that carries Y into a single point).

This immediately follows from the theorem. Indeed, if $f\colon Y \to X$ is cellular, it carries the $(q-1)$-skeleton of Y, equal to Y, into the $(q-1)$-skeleton of X, which is the vertex.

In particular $\pi(S^m, S^q) = \pi_b(S^m, S^q) = 0$ for $m<q$ (i. e. it consists of a single element).

2. A space X is called n-connected if $\pi(S^q, X)$ contains a single element for $q \leq n$ (i. e. all the mappings $S^q \to X$ are homotopic).

★*Exercise.* Prove that the following conditions are equivalent to n-connectivity:

(a) $\pi_b(S^q, X)$ contains a single element for $q \leq n$;

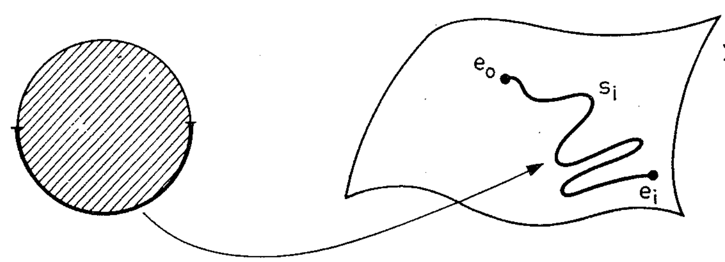
(b) any continuous mapping $S^q \to X$ extends to a continuous mapping $D^{q+1} \to X$.

★*Exercise.* A space is 0-connected if and only if it is path-connected. (We recall that S^0 consists of a pair of points.)

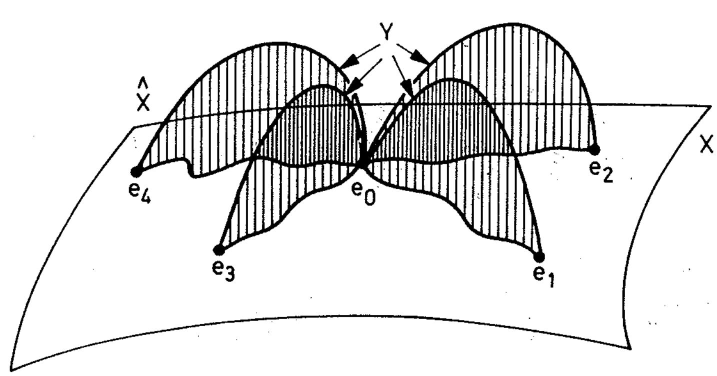
The term "1-connected" is more often used in the form "simply connected".

Theorem. Any n-connected CW complex is homotopy equivalent to a CW complex that contains a single vertex and no other cells in dimensions 1 through n.

Proof. Let us choose a vertex e_0 of the complex X and connect it with the remaining vertices with paths. This is possible since the complex is n-connected and, in particular, path-connected. (The paths may intersect.) By the cellular approximation theorem they may be assumed to belong to the one-dimensional skeleton of X. Let s_i be the path connecting e_0 with e_i. We attach to X the two-dimensional disk along the mapping of



the lower half-circle by means of the path s_i. By carrying out this procedure with each s_i we obtain a new complex $\hat{X}$ which contains X as well as cells e_i^1, e_i^2 (the upper half-circle resp. the interior of the attached disk).



The boundaries of the two-dimensional cells e_i^2 belong to the 1-skeleton because the same is true for the paths s_i.

Now X is clearly a deformation retract in $\hat{X}$, as each disk can be deformed onto the lower half-circle.

Let Y be the union of the closures of the cells e_i^1.

Clearly Y is contractible. Thus $\hat{X}/Y \sim \hat{X} \sim X$. On the other hand, Y has only a single vertex.

The next step is similar. Assume that $X \sim X'$ and has a single vertex and no cells in dimensions $1, \ldots, k-1$, where $k<n$. In that case every k-dimensional cell is a k-dimensional sphere. Since X is n-connected as well as X', the imbedding of the sphere into X' can be continued on the $(k+1)$-dimensional ball whose image, in turn, may be considered as belonging to the $(k+1)$-skeleton, in view of the cellular approximation theorem. We attach the ball D^{k+2} to X' along the mapping, thus adding one $(k+1)$-dimensional and one $(k+2)$-dimensional cell to X'. The complex $\hat{X}'$ obtained is homotopy equivalent to X' and contains a contractible subcomplex Y (the union of closures of the newly added $(k+1)$-dimensional cells) that contains all the k-dimensional cells. We have $\hat{X}'/Y \sim \hat{X}' \sim X' \sim X$. Now $\hat{X}'/Y$ has a single vertex and no cells in dimensions $1, \ldots, k$. Q.e.d.

Corollary. If X is a k-connected CW complex and Y is a k-dimensional CW complex then $\pi(Y, X)$ consists of a single element. The same is true for $\pi_b(Y, X)$ if X and Y have vertices for basepoints.

Exercise. Prove that an arbitrary one-dimensional CW complex is homotopy equivalent to a union of circles.

§4. THE FUNDAMENTAL GROUP $\pi_1(X)$

The one-dimensional homotopy group $\pi_1(X)$ is also called the fundamental group of X. The definition given for $\pi_n(X)$ in §2 was a very general one so it is worth repeating it in terms of the particular case $n=1$.

Let us consider all possible loops passing through a fixed point $x_0 \in X$, i. e. continuous mappings $\varphi\colon I \to X$ such that $\varphi(0) = \varphi(1) = x_0$. Two loops are said to be homotopic if there exists a homotopy $\varphi_t\colon I \to X$ such that $\varphi_0 = \varphi$, $\varphi_1 = \psi$ and $\varphi_t(0) = \varphi_t(1) = x_0$ $(0 \le t \le 1)$. The product loop of φ and ψ is given by

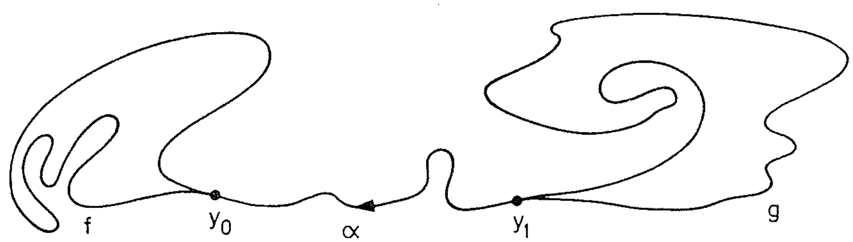
$$\chi(t) = \begin{cases} \varphi(2t) & \text{for } 0 \le t \le \frac{1}{2}, \\ \psi(2t-1) & \text{for } \frac{1}{2} \le t \le 1. \end{cases}$$

In other words, the product loop is obtained by passing φ and ψ successively, first φ and then ψ. It is easy to verify that this multiplication is compatible with homotopy, so it gives at the same time a multiplication in the set of homotopy classes of loops, and in result a group which will be denoted by $\pi_1(X, x_0)$. The homotopy class containing the loop φ: $I \to X$ is clearly the inverse element of the class of φ': $I \to X$ defined by $\varphi'(t) = \varphi(1-t)$.

Any mapping f: $X \to Y$ induces a homomorphism f_*: $\pi_1(X, x_0) \to \pi_1(Y, y_0)$ where $y_0 = f(x_0)$. If f', f'' : $X \to Y$ are homotopic mappings of pointed spaces, $'_*$ and f''_* clearly coincide.

Theorem 1. If Y is path-connected and y_0, y_1 is an arbitrary pair of points, then there is an isomorphism $\pi_1(Y, y_1)$.

Proof. Because Y is path-connected, there exists a path $\alpha: I \to Y$ with $\alpha(0) = y_0$ and $\alpha(1) = y_1$.



We construct a mapping $\alpha_\# : \pi_1(Y, y_0) \to \pi_1(Y, y_1)$. Let $\bar{f} \in \pi_1(Y, y_0)$, $\bar{g} \in \pi_1(Y, y_1)$ and $f \in \bar{f}$, $g \in \bar{g}$. We put $\alpha_\# (f) = \alpha \cdot f \cdot \alpha^{-1}$. We obtain a loop with its beginning and end at y_1. By replacing f, g and α by homotopic paths we only change $\alpha_\# (f)$ for a homotopic loop, so $\alpha_\#$ defines a mapping of homotopy classes $\pi_1(Y, y_0) \to \pi_1(Y, y_1)$. The inverse mapping $\alpha_\#^{-1}$ is constructed analoguously by $\alpha_\#^{-1} (g) = \alpha^{-1} \cdot g \cdot \alpha$.

It is easy to see that $\alpha_\#$ is a group isomorphism between $\pi_1(Y, y_0)$ and $\pi_1(Y, y_1)$.

This isomorphism depends on the choice of the path α. By changing α for another path β which is not homotopic to α we usually get a different isomorphism. In short, the isomorphism is not canonical. It will be emphasized however that $\alpha_\#$ and $\beta_\#$ may coincide even if α and β are not homotopic.

Exercise. The isomorphisms $\alpha_\#$ coincide for all α if and only if $\pi_1(Y, y_0)$ is commutative.

In view of the theorem the group $\pi_1(Y, y_0)$ may be regarded as independent of y_0. This justifies the notation $\pi_1(Y)$ and the name, fundamental group of the space X, accepted for $\pi_1(Y, y_0)$.

★ *Exercise.* For homotopy equivalent spaces Y_1 and Y_2, $\pi_1(Y_1) = \pi_1(Y_2)$.

Computation of fundamental groups

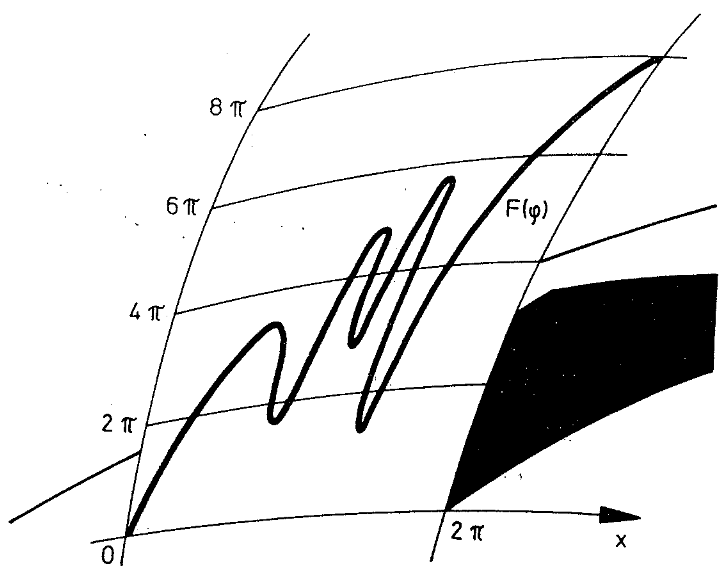
Theorem 2. The fundamental group of the circle is isomorphic to the additive group of integers: $\pi_1(S^1) = \mathbf{Z}$.

Proof. We are going to construct the so-called universal covering space over the circle. This notion will be expounded in the next section. The reader will be advised to return at each step of the general construction to the corresponding part of the present proof.

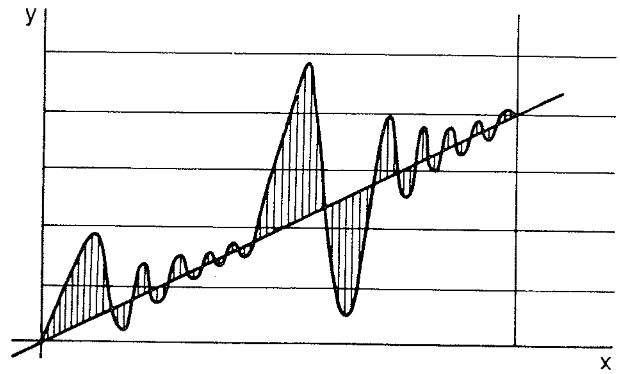
The points of the circle are assumed to be parametrized by real numbers defined up to a summand $2k\pi$. The base point is 0. We recall that the elements of $\pi_1(S^1)$ are homotopy classes of base-point preserving mappings $S^1 \to S^1$. Any such mapping may be given as a multivalued continuous function defined on $[0, 2\pi]$ whose value at each point is given up to additive terms $2k\pi$, satisfying $f(0) = f(2\pi)$.

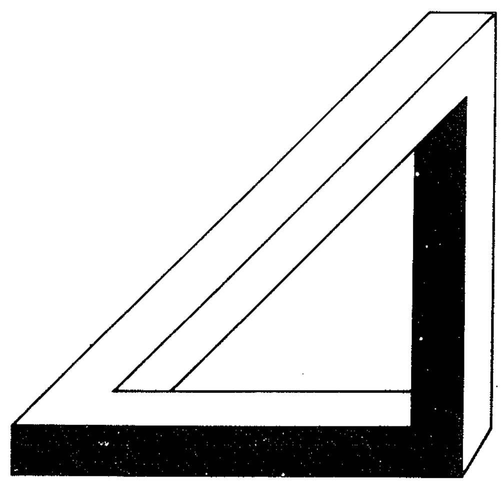
We are not going into the details, what is meant by the notion of continuous multivalued function and how to prove, by using the usual $\varepsilon-\delta$-technics that such a function has a single-valued branch, i. e. a function defined and continuous on $[0, 2\pi]$ whose value at each point coincides with one of the values of f. Let $f^{\#}$ be such as a function on $[0, 2\pi]$ with $f^{\#}_{(0)} = 0$. It is uniquely determined by f, moreover any homotopy f_t $(0 \le t \le 1)$ will define a homotopy of the functions $f_t^{\#}$ $(0 \le t \le 1)$ as mappings of the interval to the line.

Conversely, any continuous function F defined on $[0, 2\pi]$ and such that $F(0)=0$, $F(2\pi) = 2k\pi$, where k is an integer, is of the form $f^{\#}$ with a suitable f. To finish the proof we only have to make three very simple remarks. First, the number k (in $f^{\#}(2\pi) = 2k\pi$) will not change during a homotopy as the range of the admitted values of $f^{\#}(2\pi)$ is discrete. Thus it only depends on the element of $\pi_1(S^1)$ represented by f.



Second, if for any pair of mappings f_1, f_2 we have $f_1^{\#}(2\pi)=f_2^{\#}(2\pi)$, then certainly $f_1 \sim f_2$. In particular if $f^{\#}(2\pi)=2k\pi$ then $f \sim h_k$. Here $h_k^{\#}(x)=2k\pi x$, as shown on the figure. Finally, we have $h_k \cdot h_l = h_{k+l}$. Q. e. d.





Theorem 3. Let $B_A^1 = \vee_{\alpha \in A} S_\alpha^1$ be the union of circles S_α^1. Then $\pi_1(B_A^1)$ is a free group whose generators correspond to the elements of A.

Proof. The proof will be carried out in two steps. The second step will actually be postponed until §5. Let B_A^1 be a union of circles (whose common point is considered as the base point of the space). We denote by i_α the α-th standard imbedding of the circle S^1 into B_A^1 (assumed to preserve the base points), and by $\eta_\alpha \in \pi_1(B_A^1)$ the class of i_α. We show that (1) any element of $\pi_1(B_A^1)$ may be written as a finite product of elements η_α and η_α^{-1}; (2) such representation is unique up to reduction by pairs of adjacent factors η_α and η_α^{-1}. The two statements put together are equivalent to the theorem.

Now (1) actually follows from the simplicial approximation theorem. Consider a mapping $f: S^1 \to B_A^1$. The spaces S^1 and B_A will be divided into simplexes in the obvious way by each circle S^1, S_α^1 being divided into three one-dimensional simplexes P, Q, R resp. $P_\alpha, Q_\alpha, R_\alpha$. By the simplicial approximation theorem f is homotopic to a simplicial mapping between suitable subdivisions of S^1 and B_A^1. (It is left to the reader to make this argument more precise, taking into account that the simplicial approximation theorem does not involve base points. Actually the constructions of the simplicial approximation theorem in the original proof may automatically give a simplicial mapping that preserves the base points.) Next the mapping is multiplied from the right by a homotopy of B_A^1 into itself connecting the identity mapping with a mapping which maps all the simplexes P_α, R_α onto the base point, and stretches each simplex Q_α on the whole circle S_α^1. The result is a mapping $\tilde{f}: S^1 \to B_\alpha^1$ homotopic to f and of the following structure. The circle is divided into several arcs, each of which is either mapped onto the base point or is stretched over one of the circles of the union. By the definition of multiplication in the fundamental group, the class of this mapping in $\pi_1(B_\alpha^1)$ is a product of elements η_α, η_α^{-1} and the unit element (which is the class of the constant mapping).

To prove (2) it suffices to show that any product $\eta_{\alpha_1}^{\varepsilon_1} \ldots \eta_{\alpha_k}^{\varepsilon_k}$ $(\varepsilon_k = \pm 1)$ is unequal to the unit element unless it contains elements η_α and η_α^{-1} in succession. This will be obtained as a corollary of the main theorem in §5.

Let X be a space with base point x_0 and φ be a mapping which sends the base point of S^1 into $x_1 \in X$. Let us be given a path s connecting x_0 and x_1. The loop $s\varphi s^{-1}$ defines an element f of $\pi_1(X, x_0)$. If s is replaced by another path s', we get gfg^{-1} instead of f, where g is the class of the loop $s's^{-1}$. Thus any mapping of the circle defines an element of the fundamental group up to conjugacy.

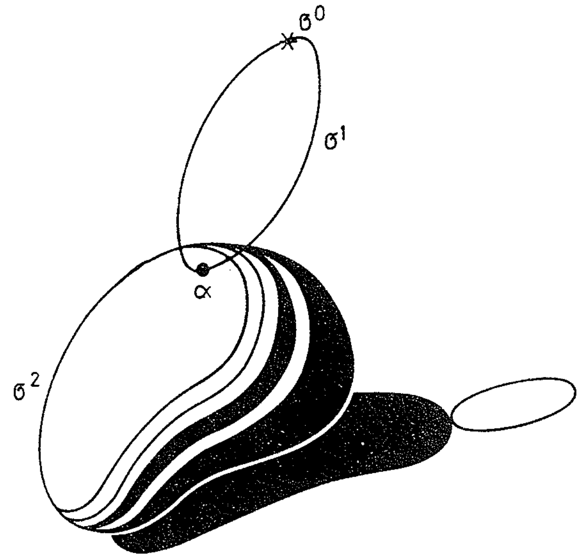
Theorem 4. Let K be a CW complex having a single vertex, one-dimensional cells $e_i^1 (i \in I)$, two-dimensional cells e_j^2 $(j \in J)$ and characteristic mappings f_i^1: $B^1 \to K$, f_j^2: $B^2 \to K$. The mappings obtained by restricting f_j^2 to $\dot{B}^2 = S^1$ determine, up to conjugacy, the elements $\beta_j \in \pi_1(K^1)$. Then $\pi_1(K)$ is the group generated by the set of generators I with relations $\beta_j = 1$, $j \in J$ (see theorem 3).

Proof. Let us compute $\pi_1(K) = \pi_1(K, *)$ for a CW complex that consists of a single vertex $*$, 1-dimensional cells e_i^1 $(i \in I)$ (with characteristic mappings $f_i^1 : B^1 = I \to K$) and 2-dimensional cells e_j^2 $(j \in J)$ (with characteristic mappings $f_j^2 : B^2 \to K$). Every element of the group is represented by a base-point preserving mapping $\varphi: S^1 \to K$. If the circle is regarded as a CW complex with a single 0-dimensional and a single 1-dimensional cell, φ is already cellular on the 0-skeleton, consequently it is homotopic to a cellular mapping that is constant on the 0-skeleton. In other words, every element of $\pi_1(K)$ is represented by some mapping $\varphi: S^1 \to K^1 \subset K$.

The characteristic mappings f_i^1 of the cells e_i^1 represent certain elements $\Theta_i \in \pi_1(K)$. By theorem 3 any element of $\pi_1(K)$ can be written as a product $\Theta_{i_1}^{\varepsilon_1} \ldots \Theta_{i_k}^{\varepsilon_k}$ $(\varepsilon_s = \pm 1)$.

Finally we have to find the products equal to 1. Let $j \in J$, g_j: $S^1 \to K^1 \subset K$ be the restriction of the characteristic mapping f_j^2. By the paragraph preceding theorem 4, it defines, up to a conjugacy class an element in $\pi_1(K^1)$ whose image is clearly the unit element in $\pi_1(K)$ (as g_j extends to a mapping of the disk B^2). Moreover the conjugates of such elements will be equal to the unit element as well as their products.

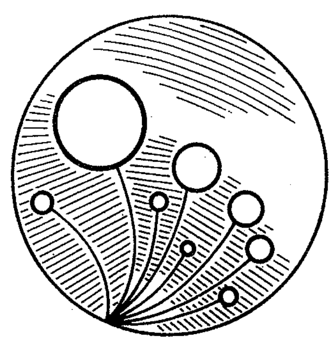
Remark. It seems as if we might have assumed that all the mappings in question send the respective base points into each other. Neither is it so for an arbitrary CW complex nor would it make the proof any easier.



Let $\Theta \in \pi_1(K^1)$ be an element sent to the unit element by the homomorphism induced by the inclusion $K^1 \subset K$. That is, any representative $f\colon S^1 \to K^1 \subset K$ may be extended to a mapping $F: B^2 \to K$. By the cellular approximation theorem the image of F may be assumed to be in K^2. By the same procedure as the one in the proof of the cellular approximation theorem we may ensure that F is simplicial on the pre-image of a small disk surrounding the centre of each 2-dimensional cell of K. Let these disks be further diminished until they contain only images of points of open 2-dimensional simplexes (by the respective simplicial mappings).

Nowwe have the following situation: each 2-dimensional cell contains at its centre a small disk whose pre-image consists of similar disks that belong to B^2 and are linearly mapped onto the corresponding disks. (Nothing is assumed about orientation!)

Next the complex K is deformed in itself so that the one-dimensional skeleton is fixed, in each cell the disk is stretched to cover the whole cell, and its complement is



squeezed out to the one-dimensional skeleton. By the Borsuk theorem this deformation extends to a deformation $K \to K$, so $\tilde{F}$ may be defined as the composite of F with it. Then $\tilde{F}$ will have the following description. The complement in B^2 of a number of disks is mapped onto the 1-skeleton, the disks themselves are mapped onto corresponding 2-

dimensional cells and each of these mappings either coincides with the characteristic mapping or differs from it in an axial reflexion.

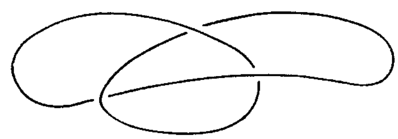
Now let each disk be connected with the base point of S^1 by a line. The complement to the union of the disks and lines is an open disk mapped by $\tilde{F}$ onto K^1. Its boundary is also mapped onto K^1 and represents the identity element of $\pi_1(K)$, because it is the restriction of a mapping of the disk. On the other hand, it is equal to a product of Θ (represented by the boundary of B^2) and certain classes which are conjugate to elements represented by the characteristic mappings of 2-dimensional cells. Thus Θ is equal to such a product. Q.e.d.

★ *Exercise.* Every contractible space is simply connected.

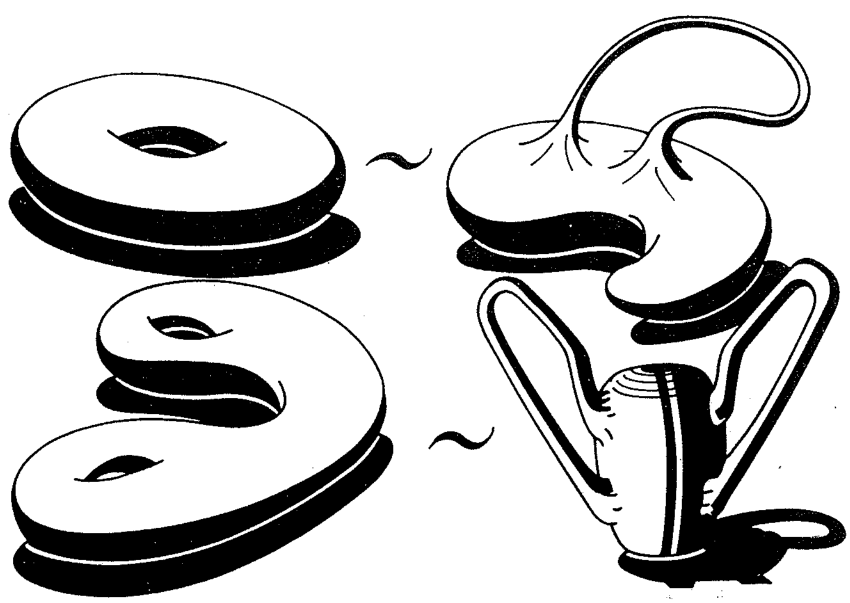
Exercise. The fundamental group of any H-space is commutative.

Exercise. On the 2-dimensional sphere S^2 there are given two continuous odd functions (i. e. functions such that $f(x) = -f(\tau x)$ for $x \in S^2$, where τ is the antipodal mapping of S^2). Then they have a common zero.

Exercise. A trifolium is the simplest knot in the 3-dimensional space.



Find the fundamental group of its complement. Deduce from this that the trifolium cannot be "undone" i. e. there is no homeomorphism of the space E^3 into itself that transforms the trifolium into the standard circle. Can any group be isomorphic to the fundamental group of the complement of a knot? (A knot is a closed polygonal non-self-intersecting line in the three-dimensional space.)

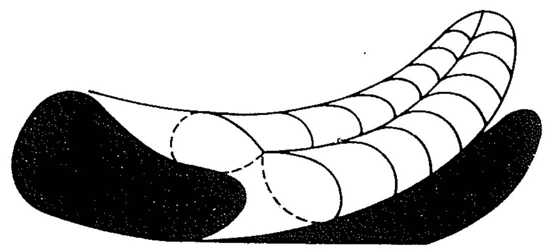


It is well known that every closed two-dimensional manifold may be obtained from the sphere by attaching to it handles ("orientable surface") and Möbius bands (cf. Milnor, Morse theory, Princeton Univ. Press, 1963).

For example, a torus is a sphere with a single handle.

Attaching of a möbius band cannot be carried out within the three-dimensional space without causing self-intersection. For the sake of better visualization we recall that the boundary of the Möbius band is nothing else than an ordinary circle. Thus it is possible to attach along the boundary of the Möbius band a sphere with a hole.

Exercise. Prove that we get the same thing by attaching three Möbius bands as by attaching a Möbius band and a handle.



★ *Exercise.* Compute the fundamental group of an arbitrary two-dimensional surface. What are the fundamental groups of the sphere, the torus, the projective plane, the Klein bottle? Which of the surfaces have commutative fundamental groups?

Exercise. Prove the existence of a group that can be the fundamental group of no closed 3-dimensional manifold.

Let G be an arbitrary group with finitely many generators and relations.

Exercise. Construct a closed manifold the fundamental group of which is G.

The problem will be more difficult if we add the strongest possible condition on the dimension:

Exercise. Construct a closed 4-dimensional manifold the fundamental group of which is G.

§5. COVERINGS

A path-connected space T is called a covering space over the path-connected space X if there is given a mapping $\pi: T \to X$ such that for every $x \in X$ there exists an open neighbourhood $U(x) \subset X$ for which $\pi^{-1}(U)$ is homeomorphic to $U \times D$ where D is a discrete set, and the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \approx & U \times D \quad \text{(homeomorphism)} \\ & \searrow \quad \swarrow & \text{projection} \\ & U & \end{array}$$

is commutative. The mapping $\pi: T \to X$ is called a covering projection or simply a covering.

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Examples.

1) $T = \mathbf{R}$ is the real line and $X = S^1$ is the circle $S^1 = \{z \in \mathbf{C};\ |z| = 1\}$. The projection $\pi\colon \mathbf{R} \to S^1$ is given by the formula $\pi(t) = e^{2\pi i t}$.

2) $\pi\colon S^1 \to S^1$ is given by the formula $\pi(z) = z^k$ where $k \neq 0$ is an integer.

3) $T = S^n$ is the n-dimensional unit sphere in $\mathbf{R}^{n+1}$, $X = \mathbf{RP}^n$ is the real n-dimensional projective space. The mapping $\pi\colon S^n \to \mathbf{RP}^n$ is given as assigning to any point of S^n the line connecting it with the centre of the sphere.

4) $T = \mathbf{R}^2$ is the plane, $X = S^1 \times S^1$ is the torus, and π is the projection of $T = \mathbf{R} \times \mathbf{P}$ onto the quotient group $X = (\mathbf{R} + \mathbf{R})/(\mathbf{Z} + \mathbf{Z})$.

The covering homotopy theorem

Theorem 1. Let $\pi\colon T \to X$ be a covering. Let us be given an arbitrary space Z, a mapping $f\colon Z \to T$ and a homotopy $\Phi\colon Z \times I \to X$ such that $\pi \circ f = \Phi|_{Z \times \{0\}}$. Then there exists a unique homotopy $F\colon Z \times I \to T$ such that $F|_{Z \times \{0\}} = f$ and $\pi \circ F = \Phi$.

Lemma. Let $\pi\colon T \to X$ be a covering, $x \in X$, $t \in \pi^{-1}(x)$ and let $f\colon I \to X$ be a path for which $f(0) = x$. Then there exists a unique path $g\colon I \to T$ such that $g(0) = t$ and $\pi \circ g = f$.

Proof of the lemma. The neighbourhoods $U(x)$ with the property required by the definition of covering will be called elementary.

Every path is compact, being the continuous image of a compact set. For every point $f(\tau) \in X$ there exists an elementary neighbourhood. Their system contains a finite subsystem covering the path. For the sake of convenience we order the elements so that a neighbourhood precedes another one if it contains a point of the path whose parameter is smaller than the parameter of any point belonging to the latter. Let us consider the first neighbourhood. Its pre-image by π is homeomorphic to a discrete union of similar neighbourhoods. Only one of them contains the point t. In this neighbourhood we consider the inverse pre-image of the path f. This is the only way to "lift" the part of the path contained in $U(f(0))$ to T.

Now the second neighbourhood clearly meets the first one. Thus it contains some point $f(\tau)$ that has already been lifted, and we see the previous situation repeated, etc. The process is finite. The lifted path is unique, as has been at each step.

Proof of Theorem 1. Let $z \in Z$. Them $\varphi|_{z \times I}\colon I \to X$ defines a path in X. The function φ is continuous in $\tau \in I$, therefore the path has a unique "lifting" in T, where the starting point of the path is given by f. By making z to run through Z we obtain the mapping $F\colon Z \times I \to T$. The reader will easily show it to be continuous and unique.

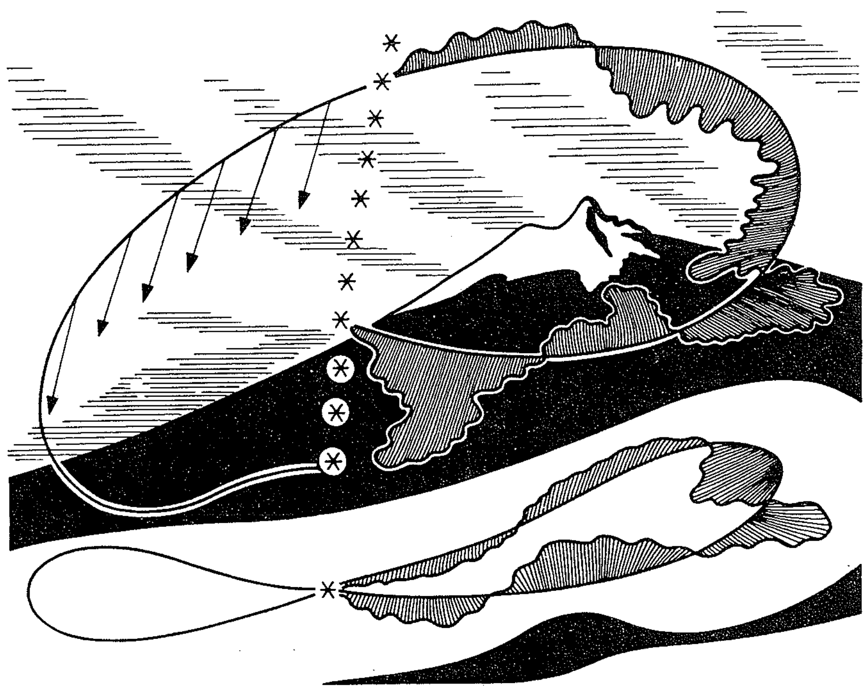
Theorem 2. The mapping $\pi_*\colon \pi_1(T)$ is a monomorphism. (Here $\pi_1(X)$ stands for $\pi_1(X, x_0)$ where x_0 is an arbitrary fixed point of X.)

Theorem 3. The pre-image $\pi^{-1}(*) = D$ of an arbitrary point $*$ is in a one-to-one correspondence with the cosets of $\pi_1(X)$ by the subgroup $\pi_*(\pi_1(T))$.

Proof of Theorem 2. Suppose that a loop α of T is projected on a loop $\pi(\alpha)$ homotopic to zero. It has to be shown to be zero homotopic, too.

Now a loop is a mapping $\alpha = F: S^1 \to T$. By assumption $\pi \circ F = f_0: S^1 \to X$ is zero homotopic, i. e. there exists a homotopy $f_t: S^1 \to X$ such that $f_1(S^1) = *$. By the homotopy covering theorem there exists a homotopy $F_t: S^1 \to T$ such that $\pi F_t = f_t$ and $F_0 = F$. As the pre-image of the point $*$ is a discrete subset of T, F_1 is a mapping onto a single point. Thus the loop α is contractible into a single point, too. Q.e.d.

Proof of Theorem 3. (Actually we have more of a construction than a theorem, i. e. something between a definition and a theorem.) The correspondence is established as follows. Let us consider a loop in X. It can be lifted to T. We assign to it the endpoint of the corresponding path in T. (The starting point coincides with $*$ while the endpoint is only known to belong to $\pi^{-1}(*)$.)



The following facts are to be verified:

(i) The definition is correct, i. e. if two loops belong to the same class in $\pi_1(X)/\pi_1(T)$, the same point is assigned to them.

(ii) If two loops belong to different classes in $\pi_1(X)/\pi_1(T)$ the points assigned to them are different.

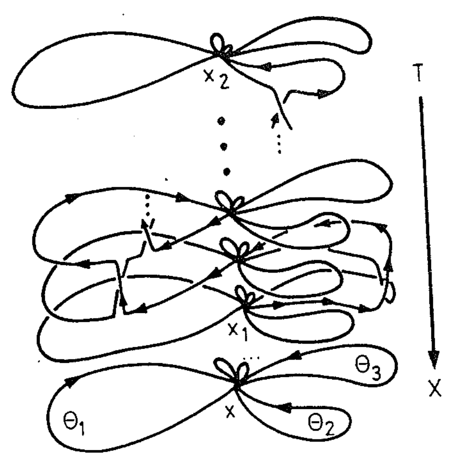
(iii) Every point of $\pi^{-1}(*)$ is assigned to some loop.

(i) Let $\alpha, \beta \in \pi_1(X)/\pi_1(T)$. Then $\pi_*^{-1}(\beta^{-1}\alpha)$ is a loop in T, i. e. the path $\pi_*^{-1}(\beta)$ starts at the endpoint of $\pi_*^{-1}(\alpha)$, hence $\pi_*^{-1}(\beta)$ ends at the same point as $\pi_*^{-1}(\alpha)$, as claimed.

(ii) Indeed, if a point corresponds to both α and β, the pre-image of the loop $\beta^{-1}\alpha$ is also a loop in T, i. e. α and β belong to the same class.

(iii) Consider a point $t \in T$ for which $\pi(t) = * \in X$. As T is path-connected, t and $* \in T$ can be connected by a path, whose image is necessarily a loop in X. Its homotopy class corresponds to t.

Theorem 2 implies that if $\pi: T \to X$ is a covering, $x \in X$, $x_1, x_2 \in T$ and $\pi(x_1) = \pi(x_2) = x$, further s is an arbitrary path connecting x_1 and x_2, then $\pi(s)$ is a loop, with its vertex in x, which is not homotopic to zero. This fact enables us to fill up the gap in the proof of theorem 3 in §4. We have to prove that no loop, represented in the form of a product $\Theta = \Theta_{i_1}^{\varepsilon_1} \dots \Theta_{i_k}^{\varepsilon_k}$, is zero homotopic unless it contains a factor Θ_i immediately followed by Θ_i^{-1}. Here $\Theta_{i_s}^{\varepsilon_s}$, $\varepsilon_s = \pm 1$ denotes a loop along the i_s-th circle of the union, that is, we are supposed to walk along the circle. The direction depends on the sign of ε_s. Let k be the number of letters in the word Θ. Let us consider $k+1$ copies of the union as shown on the picture.



$\Theta = \Theta_3 \cdot \Theta_1^2 \cdot \ldots \cdot \Theta_2$

At first we take the first letter of the word. It corresponds to some circle in the union. We cut out a small section of this circle in the first and second copies. Then we unite the free ends crosswise. The projection is defined on the modified space in the obvious way. Next we connect the second and third copies similarly, this time by using the second letter in the word. We go on with this procedure until we get a connected space with a projection on the union. If two identical letters follows each other, we cut off two segments from the same circle. It is then necessary that the first segment should precede the second one if the letters in question are on the first power and should follow it in the opposite case. (All circles in question are oriented, otherwise it would make no sense to speak about powers.)

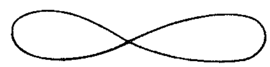
As it will easily be verified by the reader, the result is a $(k+1)$-fold covering of the union of circles, moreover the loop in point is covered by a path that starts at the lowest among the points projected onto the base point and ends at the highest one. Hence it is not homotopic to zero.

A covering is *regular* if the image of $\pi_1(T)$ is a normal subgroup of $\pi_1(X)$.

Exercise. Prove that a covering space is regular if and only if for any path in X, the paths above it in T are either all closed or are not closed.

★*Exercise.* Any covering space over the torus $T^2 = S^1 \times S^1$ is regular, Describe the covering spaces of the torus.

Exercise. Find all the covering spaces of the figure 8 space.



A covering space is *universal* if $\pi_1(T) = 0$.

In examples 1, 3 and 4 we had universal covering spaces.

★*Exercise.* Prove that a universal covering space over X is a covering space of any covering space over X too.

★*Exercise.* Prove that for $n \geq 2$, $\pi_n(T) = \pi_n(X)$ (the proof can be found in §7).

Classification of the covering spaces over a given base

In the sequel X will be assumed locally simply-connected. That is, for any $x \in X$ there exists a path-connected neighbourhood $U(x)$ such that for any pair $x_1, x_2 \in U(x)$, all paths which connect them within $U(x)$ are homotopic in X.

Existence of covering spaces

Theorem. For any subgroup $G \subset \pi_1(X)$ there exists a covering $\pi: T \to X$ such that for a suitable point $\sigma^0 \in \pi^{-1}(x_0)$, $\mathrm{Im}\ \pi_*(\pi_1(T, \sigma^0)) = G$.

Construction. Consider the path space of X. Two paths with identical endpoints α and β will be identified if the class of $\beta^{-1}\alpha$ belongs to G. Let the space of equivalence classes of paths be chosen for T. For the projection we take the mapping that assigns the second endpoint to each path.

The result is a covering space satisfying the condition $\mathrm{Im}\ \pi_*(\pi_1(T)) = G$. (Prove it! As you will find the proof requires the use of local simply-connectedness.)

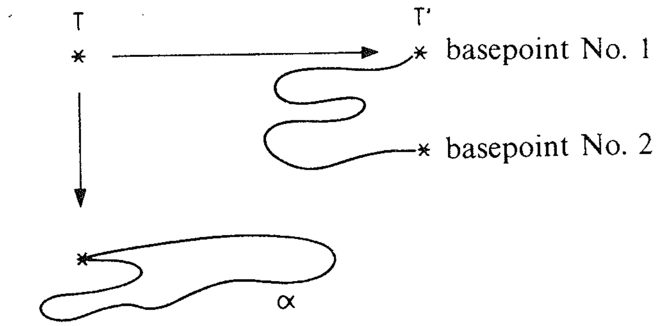
Two covering spaces $\pi: T \to X$ and $\pi': T' \to X$ are said to be *equivalent* if there exists a homeomorphism $T \approx T'$ such that the diagram

$$\begin{array}{ccc} T & \approx & T' \\ & {\scriptstyle \pi}\searrow \ \swarrow{\scriptstyle \pi'} & \\ & X & \end{array}$$

is commutative.

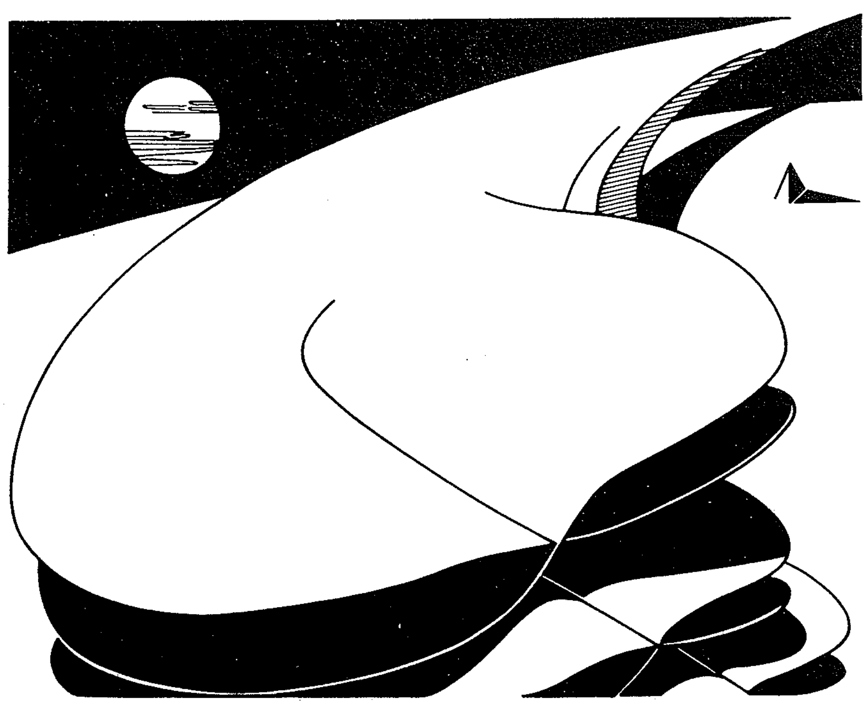
Theorem. The covering spaces $\pi: T \to X$ and $\pi': T' \to X$ are equivalent if and only if $\pi_*(\pi_1(T))$ and $\pi'_*(\pi_1(T))$ are conjugate subgroups of $\pi_1(T)$.

Proof. If the covering spaces are equivalent, the subgroups under consideration are clearly conjugate. To prove the converse statement we construct the required homeomorphism. First of all we notice that by choosing the base point $* \in T'$ properly we can obtain that the conjugate subgroups simply coincide.



Next we assign to $t \in T$ a point of T' according to the following rule: Let f be an arbitrary path in T which starts at $*$. We lower it into X and then again lift it to T'. The endpoint of T' will be associated with t. We leave it to the reader to show that this does not depend on the choice of the path, the obtained mapping is a homeomorphism and the diagram is commutative.

Exercise. Construct a pair of non-equivalent, homeomorphic covering spaces over the torus.

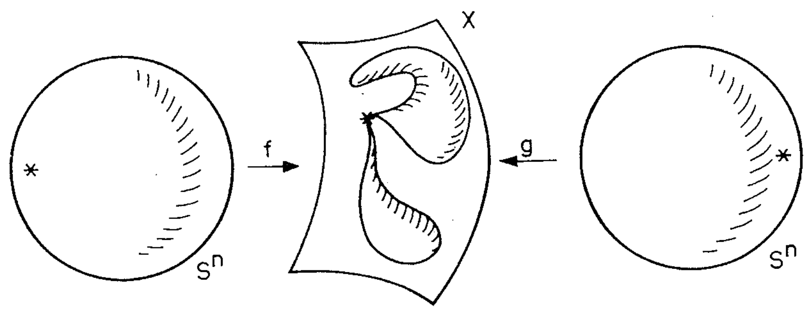


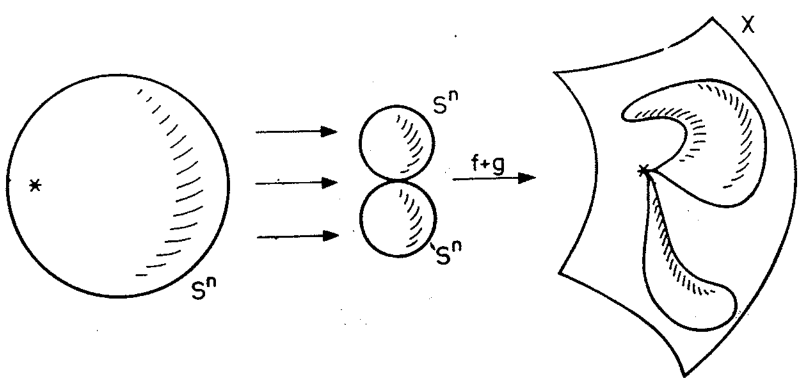
§6. HOMOTOPY GROUPS

The homotopy groups $\pi_n(X, x_0)$ of a pointed topological space X were already defined in §2 as a special case of a general covariant homotopy invariant. The extreme importance of the notion justifies its thorough study.

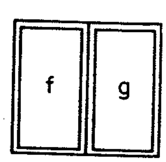
We recall that the set $\pi_n(X, x_0)$ is defined as the set of all homotopy classes of mappings $S^n \to X$ which send the base point of S^n into x_0. Such mappings are called *spheroids*. In a slightly different way a spheroid may be defined as a mapping of the n-dimensional cube I^n into X that sends the boundary ∂I^n of the cube into the single point x_0.

The sum of two spheroids $f, g: S^n \to X$ is the spheroid $f+g: S^n \to X$ defined as follows: first the equator of S^n (containing the base point) is contracted to a single point so that the sphere becomes a union of two spheres. Then the two spheres of the union are mapped into X by means of f and g, respectively. Let the spheroids $f, g: I^n \to X$ be given

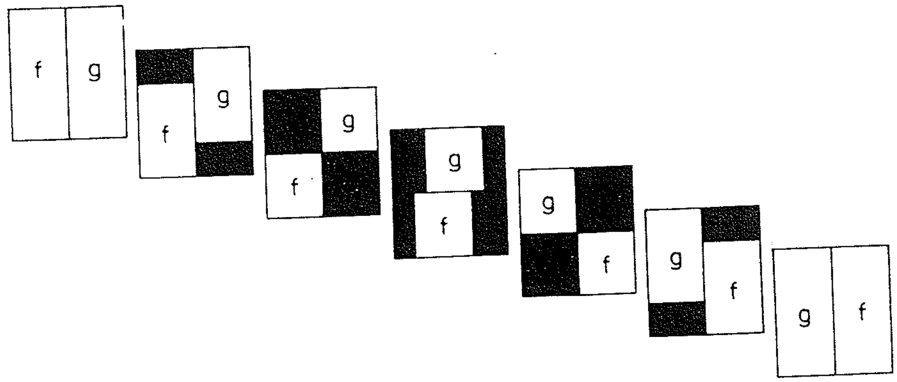




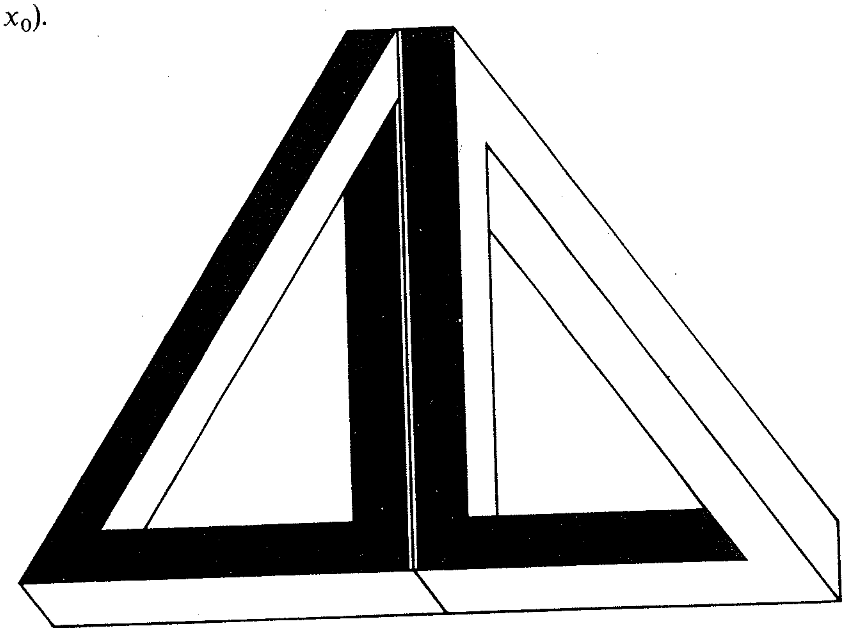
in terms of cubes. Then the sum $f+g$ is defined to coincide on the left half-cube with the composite of f and the contraction of I^n to the left half of the cube and on the right half-cube with the composite of g and the analogous contraction.



Though the addition of spheroids is not a group operation, it is invariant to homotopy (i. e. $f \sim f'$, $g \sim g'$ implies $f+g \sim f'+g'$) and induces a group operation on $\pi_n(X, x_0)$. Associativity and the existence of a unit element are directly verified (cf. §2).



For $n \geq 2$ the operation is commutative as well. The homotopy between the spheroids $f+g$ and $g+f$ may be carried out as shown on the picture (the shaded part is mapped to x_0).



As in the case of fundamental groups, the natural question arises about the dependence of $\pi_n(X, x_0)$ on the base point, x_0.

If X is path-connected, it can be shown that $\pi_n(X, x_0)$ and $\pi_n(X, x_1)$ are isomorphic for any x_0, $x_1 \in X$ by using argumentation analogous to that in §4. Again the isomorphisms coming from homotopic paths will coincide. The question when is this isomorphism independent of the choice of the path between x_0 and x_1 is an interesting one. Loops define automorphism of $\pi_n(X, x_0)$. This defines a group action of $\pi_1(X, x_0)$ on $\pi_n(X, x_n)$.

A space X is n-simple if the isomorphisms coincide for any pair of paths connecting x_0 and x_1. Equivalently X is n-simple if the action of $\pi_1(X)$ is trivial. Thus, as shown in §4, a space X is 1-simple if and only if $\pi_1(X, x_0)$ is commutative. If $\pi_1(X, x_0)=0$, then X is clearly n-simple for any n.

Exercise. Prove that any H-space is n-simple for arbitrary n.

Homotopy groups and covering

Theorem. For any covering $\pi\colon T\to X$ the mapping $\pi_*\colon \pi_n(T)\to\pi_n(X)$ is an isomorphism for $n\geq 2$.

This immediately follows from the next lemma.

Lemma. For any covering $\pi\colon T\to X$, simply-connected (and locally simply-connected) space Y and continuous mapping $f\colon Y\to X$ such that $\pi(t_0) = f(y_0) = x_0$, where t_0, y_0, x_0 are the base points of the respective spaces, there exists a unique natural mapping $F\colon Y\to T$ such that $F(y_0) = t_0$ and $\pi\circ F = f$.

Proof of the lemma. Let $y\in Y$ and let s be a path connecting y_0 with y. Its image $f(s)$ is a path connecting with x_0 in $f(y)$. Now there exists a path $\tilde{s}$ in T starting at t_0 and covering $f(s)$. Let the endpoint of $\tilde{s}$ be denoted by $F(y)$. It does not depend on the choice of s because Y is simply connected and so any path between y_0 and y is homotopic to s. A mapping $F\colon Y\to T$ for which $F(y_0) = t_0$, $\pi\circ F = f_0$ arises. It is left to the reader to show that F is continuous (hint: use the local simply-connectedness of Y). Unicity of F clearly follows from that of the covering path.

Proof of the theorem. As the sphere S^n is simply-connected for $n\geq 2$, for any spheroid $f\colon S^n\to X$ there exists a unique $F\colon S^n\to T$ with $\pi\circ F=f$. Thus $\pi_*\colon \pi_n(T)\to\pi_n(X)$ is an epimorphism.

Now $S^n\times I$ is again simply-connected, so the homotopy $\varphi\colon S^n\times I\to X$ is covered by a unique $\Phi\colon S^n\times I\to T$, i. e. by a homotopy connecting $\Phi|_{S^n\times\{0\}}$ with $\Phi|_{S^n\times\{1\}}$ which are the spheroids homotopically unique by the lemma that cover $\varphi|_{S^n\times\{0\}}$ and $\varphi|_{S^n\times\{1\}}$ respectively. We obtain that spheroids covering homotopic spheroids are homotopic, too, i. e. $\pi_*\colon \pi_n(T)\to\pi_n(X)$ is a monomorphism. Q.e.d.

The theorem may immediately be applied to compute the homotopy groups of some spaces. For example,

$$\pi_n(S^1) = \begin{cases} \mathbf{Z} & \text{for } n=1, \\ 0 & \text{for } n>1. \end{cases}$$

The first statement was proved in §4; the rest follows from $\pi_n(S^1) = \pi_n(\mathbf{R})$ for $n\geq 2$, and from the contractibility of $\mathbf{R}$.

Exercise. Prove that if X is a graph, then $\pi_n(X) = 0$ for $n\geq 2$.

Exercise. Find the homotopy groups of a surface of genus $g\geq 1$ (a sphere with g handles).

★ *Exercise.* Prove that $\pi_n(X\times Y) = \pi_n(X)+\pi_n(Y)$.

★ *Exercise.* Prove that if CW complex X has no cells of dimension $1, \ldots, n$, then $\pi_i(X)=0$ for $i\le n$. In particular, $\pi_i(S^n)=0$ for $i<n$.

Hint. This follows from the cellular approximation theorem of §3.

§7. FIBRATIONS

In §5 we studied the covering spaces which are locally constructed as direct products of an open and a discrete set. They are particular cases of a broader notion, the so-called locally-trivial fibration.

~~*Definition.* We say that (E, B, F, p), where E, B, F are spaces and p is a mapping of E into B is a locally trivial fibration if for every $x\in B$ there exists a neighbourhood $U\subset B$ and homeomorphism φ such that $p^{-1}(U)\overset{\varphi}{\approx} U\times F$ and the diagram~~

Definition. We say that (E, B, F, p) were E, B, F are spaces and p is a mapping of E into B, is a locally trivial fibration if for every $x\in B$ there exists a neighbourhood $U\subset B$ and homeomorphism φ such that $p^{-1}(U)\overset{\varphi}{\approx} U\times F$ and the diagram

$$\begin{array}{ccc} p^{-1}(U) & \overset{\varphi}{\approx} & U\times F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \text{the natural projection}} & \\ & U & \end{array}$$

is commutative.

We say that p, F, B and E are the *projection*, the *fibre*, the *base space* and the *total space* of the fibration, respectively. The term "fibre space" is also used for E.

A fibration is *trivial* if $E\approx B\times F$ and

$$\begin{array}{ccc} E & \approx & B\times F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \text{the natural projection}} & \\ & B & \end{array}$$

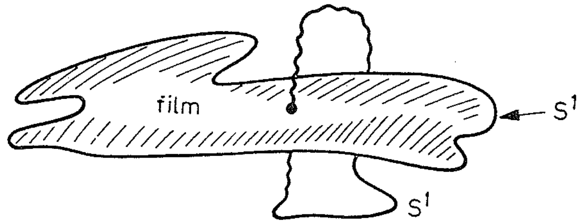
is commutative.

Examples for fibrations

1. Coverings.

2. (E, B, F, p) where E is the Möbius band, B the circle (middle line) of the Möbius band, p the natural projection and F a line segment. This is probably the most popular among the examples of nontrivial fibrations. (Prove it!)

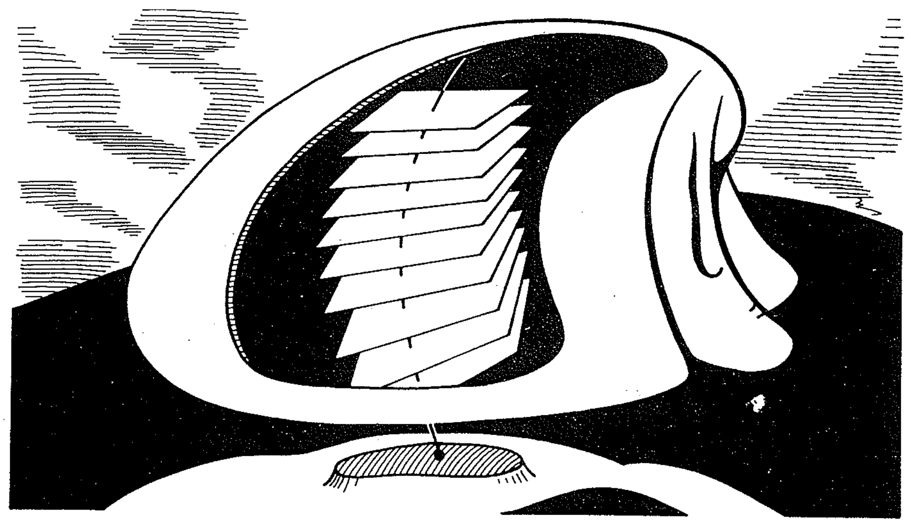
3. p: $S^3\to S^2$ where $S^3=\{(z_1,z_2) \mid z_1\bar z_1+z_2\bar z_2=1\}\subset \mathbf{C}^2$ and $S^2=\mathbf{CP}^1$ is the complex projective line, $p:(z_1,z_2)\to z_1z_2^{-1}$. The base space and the fibre are S^2 and S^1, respectively. Prove that for any pair $x_0\neq x_1\in S^2$, $p^{-1}(x_0)$ and $p^{-1}(x_1)$ are linked in S^3, i. e. a film spanned on one of the fibres will necessarily intersect the other fibre.



4. Let G and H be a Lie group and its closed subgroup. The homogeneous space G/H is defined as the set of right cosets of G mod H. The projection $G \to G/H$ gives a locally-trivial fibration.

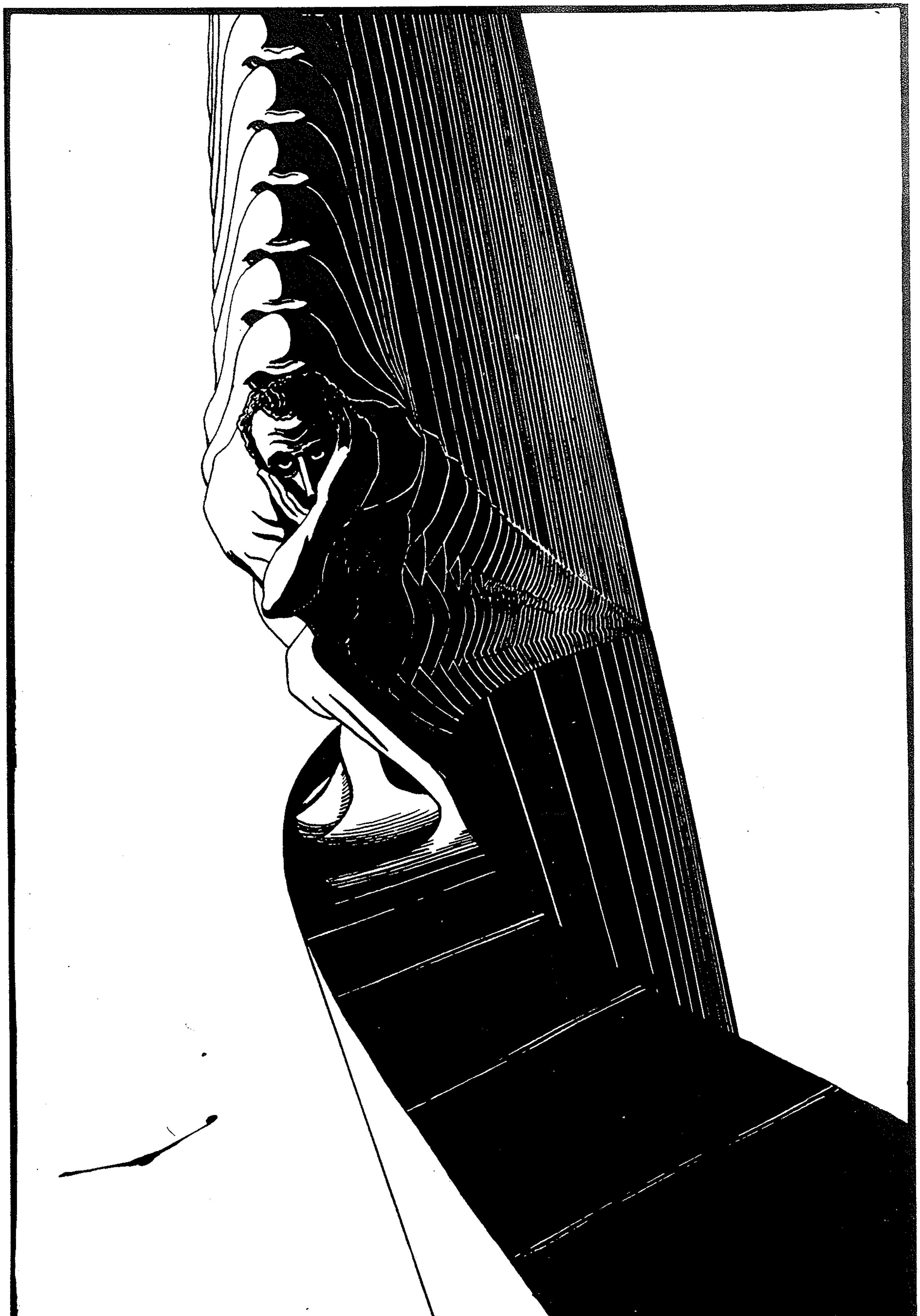
5. Let f: $M^n \to M^k$ be a regular mapping between compact smooth manifolds, i. e. a mapping whose differential is a monomorphism on each tangent space. It is a locally trivial fibration whose fibre $F = L^{n-k}$ is a compact smooth manifold of dimension $(n-k)$.

The proof of this fact may be carried out in two steps: (i) $f^{-1}(y)$ is a smooth manifold of dimension $(n-k)$ (where $y \in M^k$); (ii) for any y_1, y_2, $f^{-1}(y_1) \approx f^{-1}(y_2)$.



The pre-image of each point is a smooth manifold by the implicit function theorem. Planes which are normal to a fibre can intersect each other only at some distance from the fibre, as follows from the smoothness of the mapping.

6. Let E be the space of the unit tangent vectors to the sphere S^{2k}, $B = S^{2k}$, and $p: E \to S^{2k}$ the natural projection. Were this fibration trivial, there would exist a non-zero section, i. e. a continuous mapping $\varphi: S^{2k} \to E$ such that $p \circ \varphi = 1_B$ and $\varphi(b) \neq b$ for any $b \in B$. Thus there would exist on S^{2k} a continuous vector field that does not vanish any where on S^{2k}. It is well known that no such vector field exists on even-dimensional spheres.



Covering homotopy

It turns out that, like covering projections, any locally trivial fibrations have the covering homotopy property except the uniqueness.

Theorem 1. (Covering homotopy theorem.) Let (E, B, F, p) be a locally trivial fibration and Z be a CW complex. For any mapping $f\colon Z \to E$ and homotopy $\Phi\colon Z \times I \to B$ such that $p \circ f = \Phi|_{Z\times\{0\}}$ there exists a homotopy $F\colon Z \times I \to E$ with $F|_{Z\times\{0\}} = f$ and $p \circ F = \Phi$. Moreover, if such a homotopy is already given on a subcomplex $Z' \subset Z$, it can be extended onto Z.

Definition. Let $p\colon E \to B$ be a locally-trivial fibration, $B' \subset B$, and $E' = p^{-1}(B')$. Then the restriction $p\colon E' \to B'$ of p is evidently a locally trivial fibration with the same fibre. It is called the *restriction* of the fibration $p\colon E \to B$ to the subspace B'. It is a particular case of a more general notion.

Definition. Let $p\colon E \to B$ be a locally-trivial fibration and $f\colon B_1 \to B$ a mapping of some space B_1 into the base space. A fibration $p_1\colon E_1 \to B_1$ is said to be induced from p by f if there exists a mapping $\hat{f}\colon E_1 \to E$ such that the fibre over each point $x \in B_1$ is sent into the fibre of p over $f(x) \in B$ and the mappings between the fibres are homeomorphisms.

Lemma. For any locally-trivial fibration $p\colon R \to B$ and mapping $f\colon B_1 \to B$ there exists an induced fibration.

Proof. Let E_1 be the subspace of $E \times B$ defined by $E_1 = \{(e, b) | f(b) = p(e)\}$ and let $\hat{f}\colon E_1 \to E$ and $p_1\colon E_1 \to B_1$ be the restrictions to E_1 of the respective natural projections of the product space. Then $p_1\colon E_1 \to B_1$ is easily shown to be a locally-trivial fibration induced from $p\colon E \to B$ by f.

Lemma. (Feldbau's theorem). Every locally-trivial fibration $p\colon E \to I^q$ over the q-dimensional cube I^q is trivial.

Proof. At first we show that if $p\colon E \to I^q$ has trivial restrictions on the half-cubes

$$I_1^q = \left\{(x_1, \ldots, x_q) \,|\, 0 \le x_i \le 1,\ i = 1, 2, \ldots, q-1;\ 0 \le x_q \le \frac{1}{2}\right\}$$

and

$$I_2^q = \left\{(x_1, \ldots, x_q) \,|\, 0 \le x_i \le 1,\ i = 1, 2, \ldots, q-1;\ \frac{1}{2} \le x_q \le 1\right\}$$

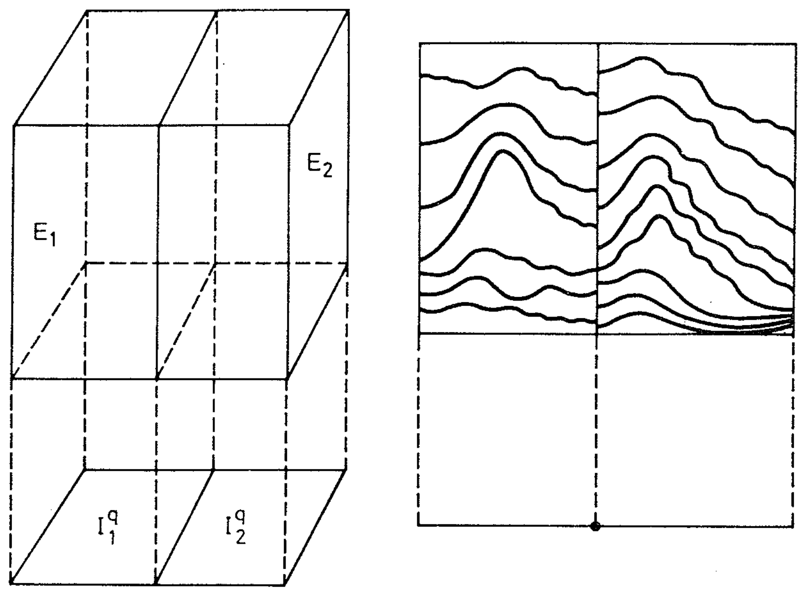
then it is equivalent to a trivial fibration.

Indeed, let $p_1\colon E_1 \to I_1^q$ and $p_2\colon E_2 \to I_2^q$ be the restrictions. We have $E_1 = I_1^q \times P$ and $E_2 = I_2^q \times P$. The points of E_1 and E_2 are given by coordinates (x, y) and $[x, y]$, where $y \in P$ and $x \in I_1^q$ or $x \in I_2^q$, respectively. Let $x \in I^{q-1} = I_1^q \cap I_2^q$. Each point of E with coordinates (x, y) has also coordinates $[x, y']$. The correspondence $y \to y'$ defines a function $f_x\colon P \to P$. Let $\pi\colon I_2^q \to I$ be given by the formula

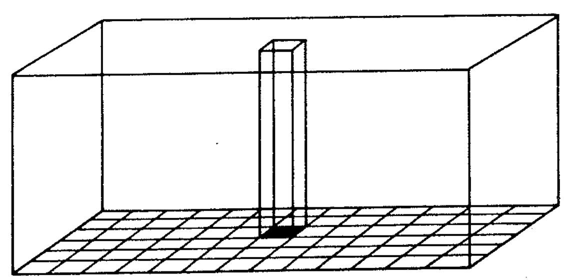
$$\pi(x_1, \ldots, x_q) = \left(x_1, \ldots, x_{q-1}, \frac{1}{2}\right)$$

and consider a mapping $\varphi\colon E \to E$ which is the identity on E_1, and sends $[x, y] \in E_2$

into $(x, f_{\pi(x)}(y))$. As it can easily be seen, the existence of φ implies the equivalence of the fibrations while the fibration constructed is trivial.



Now the fibration being locally trivial, there exists a sufficiently fine division of I^q into cubicles over which it is trivial. We begin with any of the cubicles and prove the triviality of the fibration joining the other cubicles one after another and applying the above argument at each step.



Proof of the covering homotopy theorem. 1. Suppose first that the fibration p is trivial. Then the statement reduces to the Borsuk theorem. Indeed, if $E = B \times F$, then $\psi\colon Z \to F$ and homotopies $\varphi'_t\colon Z \to B$, $\psi'_t\colon Z' \to F$, where by assumption $\varphi = f_0$ and $\varphi'_t = f_t|_{Z'}$. Now by the Borsuk theorem there exists a homotopy ψ_t such that $\psi_0 = \psi$ and $\psi_t|_{Z'} = \psi'_t$. We put $F_t(z) = (f_t(z), \psi_t(z))$.

2. In the general case we use induction. For 0-dimensional cells of Z the theorem is obvious. Suppose the homotopy is given on $Z_1 = Z' \cup Z^{k-1} \cup_{i=1}^{s-1} e_i^k$. It will be extended to a homotopy $F_1\colon Z_2 \to E$ where $Z_2 = Z_1 \cup e_s^k$. Consider the characteristic mapping $f_s^k\colon B^k \to Z$ for the cell e_s^k. The fibration $p'\colon E' \to B^k \times I$ induced by the composite

$$B^k \times I \xrightarrow{f_s^k \times I} Z \times I \xrightarrow{f_t} B$$

is trivial by the Feldbau theorem. We define $\Phi(\xi) \in E' \subset B^k \times I \times E$ and $\varphi_t(\xi) \in B \times I$ for $\xi \in B^k$ by $\Phi(\xi) = (\xi, 0, Ff_s^k(\xi))$ and $\varphi_t(\xi) = (\xi, t)$. Then we define $\Phi_t'(\xi) \in E' \subset B^k \times I \times E$ for $\xi \in \partial B^k$ by $\Phi_t'(\xi) = (\xi, t, F_t f_s^k(\xi))$.

We recall that $\xi \in \partial B^k$ implies $f_s^k(\xi) \in Z^{k-1}$, i. e. for such points F_t is already defined.

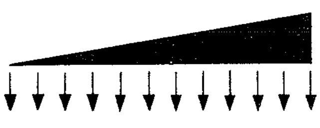
We have obtained mappings Φ: $B^k \to E'$, φ_t: $B^k \to B^k \times I$ and Φ_t': $\partial B^k \to E'$ such that $\Phi|_{\partial B^k} \sim \Phi_0'$, $p' \circ \Phi = \varphi_0$, $p' \circ \Phi_t = \varphi_t$. As the theorem holds for trivial fibrations, there exists a homotopy Φ_t: $B^k \to E$ with $\Phi_t|_{\partial B^k} = \Phi_t'$, $p' \circ \Phi_t = \varphi_t$, $\Phi_0 = \Phi$. For points ξ of the cell e_s^k we set $F_t(\xi) = \psi \Phi_t (f_s^k)^{-1}(\xi)$ where ψ: $E' \to E$ is the restriction to E' of the projection of $B^k \times I \times E$ on the last factor. It is an extension of F_t on Z_2 as required by the step of induction. Q. e. d.

Serre fibrations

The covering homotopy property gives rise to a new class of fibrations.

A *Serre fibration* is a triple (E, B, p) of spaces E and B (the latter is assumed to be path-wise connected) and a mapping p: $E \to B$ having the covering homotopy property (CHP) for arbitrary CW complexes, i. e. if Z is a CW complex and f: $Z \to E$ a mapping, then for any homotopy Φ: $Z \times I \to B$ for which $p \circ f = \Phi|_{Z \times \{0\}}$ there exists F: $Z \times I \to E$ such that $F|_{Z \times \{0\}}$ and $p \circ F = \Phi$.

A Serre fibration is not necessarily a locally trivial fibration. A simple example:



We remark that unicity of the covering homotopy has not been required.

Examples for Serre fibrations

1. Any locally trivial fibration (by the theorem above).

2. Mapping space fibrations. Let Y be an arbitrary space, X and A a CW complex and its subcomplex. We recall that $H(X, Y)$ denotes the space of all continuous mappings $X \to Y$. We take $E = H(X, Y)$, $B = H(A, Y)$ and define p: $E \to B$ the natural mapping given by restricting f to A:

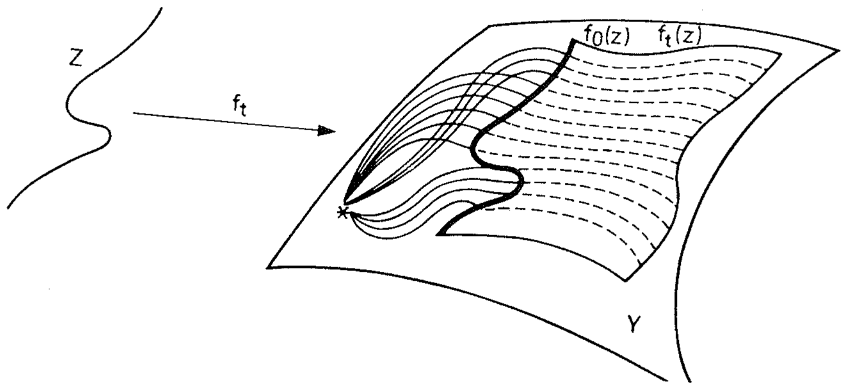
$$p(f) = f|_A.$$

It turns out that we obtain a Serre fibration—prove it. If X, S and Y are pointed, we get a Serre fibration $H_b(X, Y) \to H_b(A, Y)$ too.

3. Take in the above example the unit interval (0, 1) for X with 0 as the base point, and the set $\{0, 1\}$ with the base point 0 for A. We obtain a fibration whose base space is $Y = H_b(A, Y)$, the fibre over $y_0 \in Y$ is the set of all paths connecting the base point of Y with y_0, and the projection assigns to the paths their endpoints.

Covering homotopies can be given by the formula

$$[F_t(z)](\tau) = \begin{cases} [F(z)](\tau(1+t)) & \text{for } \tau(1+t) \leq 1, \\ f_{\tau(1+t)-1}(z) & \text{for } \tau(1+t) \geq 1. \end{cases}$$



Note: the covering homotopy property (CHP) holds here not only for CW complexes but also for arbitrary spaces. Fibrations with this strong CHP are called *Hurewicz fibrations.*

Fibres

As shown on the very first example of a Serre fibration: the fibres (i. e. pre-images of points) are not necessarily homeomorphic. Nevertheless it turns out that in a sense, like locally trivial fibrations, any Serre fibration has a standard fibre over each point.

A space X is said to be *weakly homotopy equivalent* to Y if for any CW complex Z, there is a natural isomorphism $\pi(Z, X) = \pi(Z, Y)$. That is, for any CW complex Z there exists a one-to-one mapping $\varphi_Z: \pi(Z, X) \to \pi(Z, Y)$ such that for any Z' and f: $Z \to Z'$ the diagram

$$\begin{array}{ll} \varphi_Z: & \pi(Z, X) \to \pi(Z, Y) \\ & \quad \uparrow f^* \qquad\quad \uparrow f^* \\ \varphi_{Z'}: & \pi(Z', X) \to \pi(Z', Y) \end{array}$$

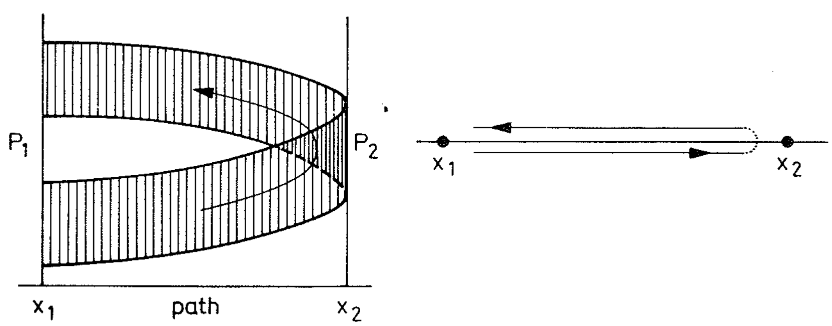
is commutative. Evidently homotopy equivalence implies weak homotopy equivalence. Compare this definition to definition 3 of homotopy equivalence in §1.

Theorem. If $p\colon E \to B$ is a Serre fibration and $x_1, x_2 \in B$, then $p^{-1}(x_1)$ is weakly homotopy equivalent to $p^{-1}(x_2)$.

Remark. If the covering homotopy property holds for any topological space Z and not only for CW complexes, i. e. if we have a Hurewicz fibration, the fibres are homotopy equivalent in the ordinary sense.

Proof. Let $P_1 = p^{-1}(x_1)$, $P_2 = p^{-1}(x_2)$. We have to show that $\pi(Z, P_1) = \pi(Z, P_2)$ for an arbitrary CW complex Z. Let be given a mapping $F\colon Z \to P_1 \subset E$. Clearly $p \circ F\colon Z \to x_1$. Let x_1 and x_2 be connected by a path ψ. We define a homotopy $f_t\colon Z \to B$ by $f_t(Z) = \psi(t)$ (each f_t sending Z into a single point). Clearly $f_0(Z) = x_1, f_1(Z) = x_2$, moreover $p \circ F = f_0$. By the covering homotopy property there exists a $F_t\colon Z \to E$ with $p \circ F_t = f_t$, implying $p \circ F_1(Z) = f_1(Z) = x_2$ i. e. $F_1\colon Z \to p^{-1}(x_2) = P_2$. Let the mapping F_1 be assigned to F. If two mappings are homotopic, so are the mappings assigned to them. Clearly the correspondence between $\pi(Z, P_1)$ and $\pi(Z, P_2)$ is natural. It remains to show that it is one-to-one.

To this end we define the inverse mapping in a similar way. The only difference is that the path ψ connecting x_1 and x_2 is now to be passed in the opposite direction. We obtain $g\colon Z \times [0, 2] \to E$ such that $p \circ g$ is the path ψ twice: there and back. This two-fold path is contractible to x_1, thus $p \circ f$ is homotopic to $Z \times [0, 2] \to x_2$. By lifting this homotopy to E we get a homotopy between the original $F\colon Z \to P_1$ and the mapping which is obtained by "driving" F into F_1 and then again turning it into a mapping $Z \to P_1$. This proves that the correspondence between $\pi(Z, P_1)$ and $\pi(Z, P_2)$ is one-to-one.



The theorem implies that the fibres of a Serre fibration over its different points are weakly homotopy equivalent. That is, if x_1, x_2, x_3, x_4 are points of a path-connected space X, the space of all paths connecting x_1 and x_2 is weakly homotopy equivalent to the space of the paths connecting x_3 and x_4. In fact these spaces are homotopy equivalent in the usual sense, too, for the fibrations involved are Hurewicz.

Any mapping is homotopy equivalent to a Serre fibration

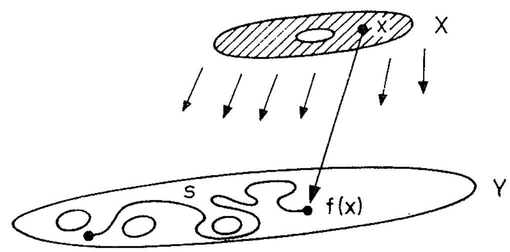
Let the two mappings $f\colon X \to Y$, $g\colon X_1 \to Y_1$ be given. We say that f and g are homotopy equivalent if there exist homotopy equivalences $\varphi\colon X \to X_1$ and $\psi\colon Y \to Y_1$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \varphi\downarrow & & \downarrow\psi \\ X_1 & \xrightarrow{g} & Y_1 \end{array}$$

is commutative.

Theorem. Let Y be path-connected. Then for an arbitrary mapping $f\colon X \to Y$ there exists an equivalent Serre fibration $p\colon X' \to Y$.

Proof. We construct a space $X' \to X$ in the following way.



The points of X' are the pairs (x, s) such that $x \in X$ and s is a path in Y beginning at $f(x)$. Clearly $X \sim X'$.

We define $f'\colon X' \to Y$ as assigning to (x, s) the endpoint of s.

Clearly f' is homotopic to f and it is easy to see that it is a Serre fibration. Q. e. d.

Moreover, the fibration p constructed above is a Hurewicz fibration. In particular, its fibres are homotopy equivalent in the usual sense.

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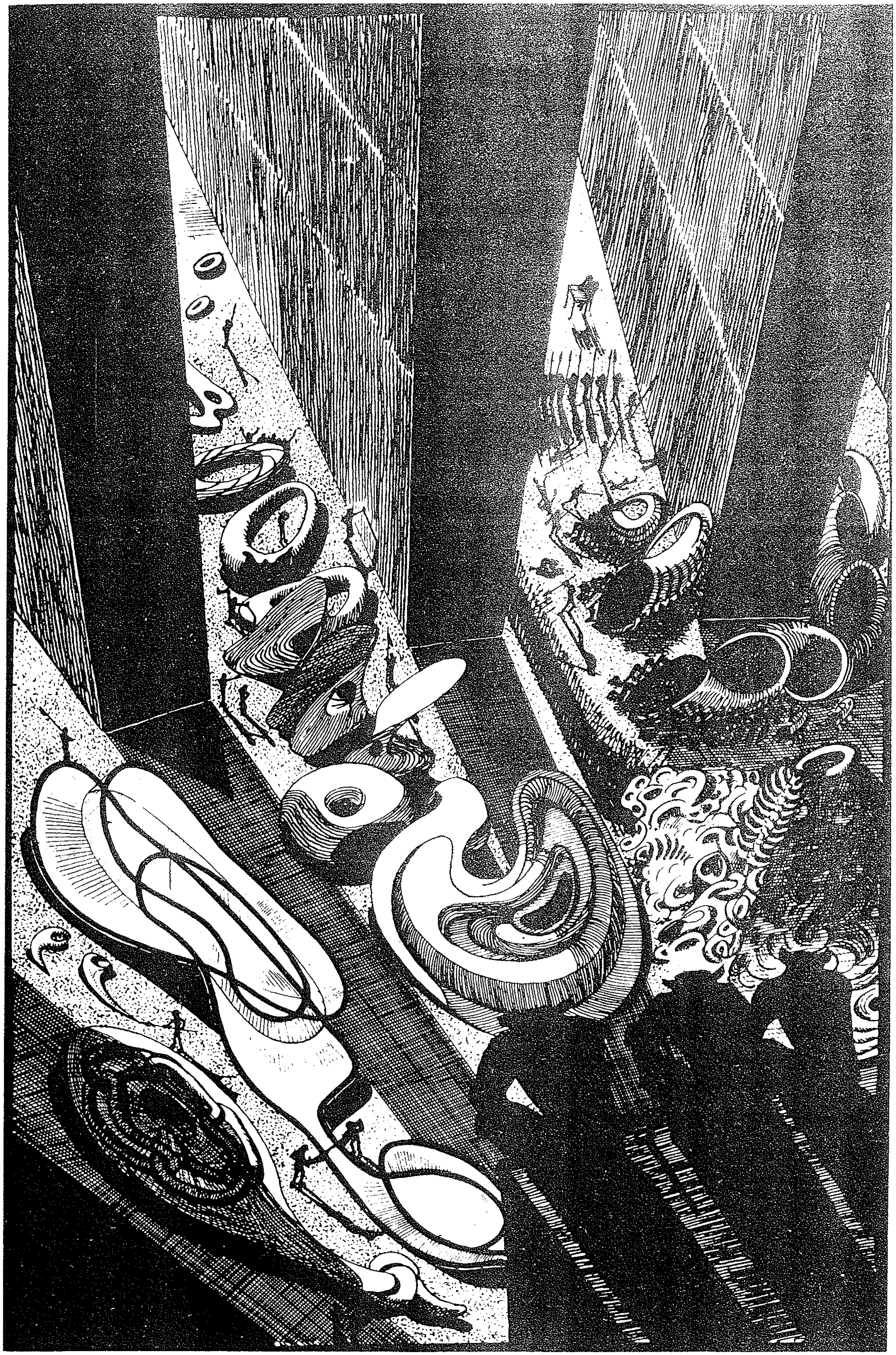
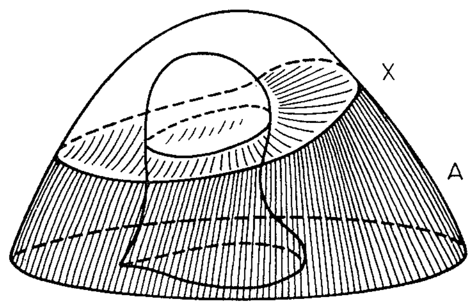


§8. RELATIVE HOMOTOPY GROUPS AND THE HOMOTOPY SEQUENCE OF A FIBRATION

Similarly to the case of pointed spaces, homotopy groups can also be assigned to pairs of pointed spaces, i. e. triples (X, A, x_0) that consists of a space X, subspace A and base-point $x_0 \in A$.

Let the cube I^n be represented in the form $I^{n-1} \times [0, 1]$. A relative n-dimensional spheroid of a pair (X, A) with base-point x_0 is a mapping $f\colon I^n \to X$ for which $f(I^{n-1} \times \{0\}) \subset A$ and $f(\partial I^n \setminus (I^{n-1} \times \{0\}) = x_0$.



Relative spheroids $f, g\colon I^n \to X$ are *homotopic* if the mappings f and g are homotopic in the class of relative spheroids. The set of homotopy classes of n-dimensional relative spheroids of (X, A) with base-point x_0 is denoted by $\pi_n(X, A, x_0)$.

The *sum* of relative spheroids $f, g\colon I \to X$ is defined by

$$h(x, t) = \begin{cases} f(x, 2t) & \text{for } 0 \le t \le \frac{1}{2}, \\ f(x, 2t-1) & \text{for } \frac{1}{2} \le t \le 1 \end{cases}$$

(here $x \in I^{n-1}$). We advise the reader to examine carefully this formula, in particular, to verify that the addition of relative spheroids is defined *only for* $n > 1$.

Addition of relative spheroids is a homotopy invariant operation.

That is, if $f \sim f'$ and $g \sim g'$, then $h \sim h'$.

It gives rise to an operation in $\pi_n(X, A, x_0)$, also called addition. The set $\pi_n(X, A, x_0)$ is a *group* with respect to the addition (for $n > 1$).

Associativity is proved by explicit construction of the homotopy $(f_1 + f_2) + f_3 \sim f_1 + (f_2 + f_3)$: the plane is deformed along the axis t_1 so that the segments $\left[0, \frac{1}{4}\right]\left[\frac{1}{4}, \frac{1}{2}\right]$ and $\left[\frac{1}{2}, 1\right]$ are transferred into $\left[0, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{3}{4}\right]$ and $\left[\frac{3}{4}, 1\right]$, respectively.

The neutral element of the group is the homotopy class of the constant mapping

$f_0(I^n) = x_0$. For an arbitrary mapping f, the homotopy $f_0 + f \sim f$ can be constructed by deforming the axis t_1 so that $\left[0, \frac{1}{2}\right]$ is contracted to the point 0, while $\left[\frac{1}{2}, 1\right]$ is stretched on the whole segment [0, 1].

All the mappings f that transfer the whole cube into A are clearly null homotopic as relative spheroids.

The spheroid defined by $\bar{f}: I^n \to X$, $\bar{f}(x, t) = f(x, 1-t)$ represents the inverse of the class of $f \in \pi_n(X, A, x_0)$. (Prove it!)

The group $\pi_n(X, A, x_0)$ is commutative if $n \geq 3$. This can be proved by directly constructing a homotopy connecting the spheroids $f+g$ and $g+f$ as well as by deducing it from the analoguous property of absolute (i. e. ordinary) homotopy groups. Indeed, we can use the following statement.

Lemma. If a Serre fibration $\eta = A' \to X$ is homotopy equivalent to the inclusion i: $A \to \to X$ (cf. the construction at the end of §6), then $\pi_n(X, A, x_0) = \pi_{n-1}(F)$, where F is the fibre of the fibration. (It was pointed out that all fibres of η are ~~strongly~~ weakly homotopy equivalent.)

We shall return to the proof later on.

We have defined $\pi_n(X, A, x_0)$ for any pair (X, A) and base point x_0. It is a set with a distinguished element (zero) for $n \geq 1$, a group for $n \geq 2$, and Abelian group for $n \geq 3$. If $A = x_0$, then $\pi_n(X, A, x_0) = \pi_n(X, x_0)$. Any mapping f: $(X, A) \to (Y, B)$ induces a homomorphism f_*: $\pi_n(X, A, x_0) \to \pi_n(Y, B, f(x_0))$. For path-connected A, $\pi_n(X, A, x_0)$ is independent of x_0 in the sense that $\pi_n(X, A, x_0)$ and $\pi_n(X, A, x_1)$ are isomorphic, and with the homotopy class of the path between x_0 and x_1 fixed, the isomorphism is canonical.

The homomorphism ∂

We define a homomorphism ∂: $\pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0)$ as follows. Let $\alpha \in \pi_n(X, A, x_0)$ be represented by a mapping f. Consider the restriction of $f|_{I^{n-1}}$ to the face I^{n-1} of I^n. The boundary I^{n-1} is again mapped onto x_0. Any homotopy between mappings b, g from (I^n, I^{n-1}, J^{n-1}) to (X, A, x_0) defines a homotopy between the restrictions.

Thus the correspondence $f \to \partial f$ gives rise to a mapping of homotopy classes $\alpha \to \partial\alpha$. Clearly $\partial(f_1 + f_2) = \partial f_1 + \partial f_2$.

The homotopy sequence of a pair

The sequence

$$\dots \xrightarrow{\partial} \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \to \dots$$

$$\dots \xrightarrow{j_*} \pi_2(X, A, x_0) \xrightarrow{\partial} \pi_1(A, x_0) \xrightarrow{i_*} \pi_1(X, x_0) \to \pi_1(X, A, x_0)$$

where i_* and j_* are the homomorphisms induced by the inclusions $i: A \subset X$ and $j: (X, x_0, x_0) \to (X, A, x_0)$, is called the homotopy sequence of the pair and has a remarkable property: it is *exact:* at each term the image of the left-hand homomorphism coincides with the kernel of the right-hand homomorphism. (We remind the reader that $\pi_1(X, A, x_0)$ is not necessarily a group. The kernel at this term is understood to be the pre-image of the class represented by $f: S^1 \to x_0$.)

That is, we have

(i) $\operatorname{Im} \partial = \operatorname{Ker} i_*$;
(ii) $\operatorname{Im} i_* = \operatorname{Ker} j_*$;
(iii) $\operatorname{Im} j_* = \operatorname{Ker} \partial$.

The proof is left to the reader.

Let us mention a further important property of this sequence. If $h: (X, A, x_0) \to$ $\to (Y, B, y_0)$ is a mapping, then the diagram

$$\begin{array}{ccccccccc}
\dots \xrightarrow{\partial} & \pi_n(A, x_0) & \xrightarrow{i_*} & \pi_n(X, x_0) & \xrightarrow{j_*} & \pi_n(X, A, x_0) & \xrightarrow{\partial} & \pi_{n-1}(A, x_0) & \to \dots \\
& \downarrow h_* & & \downarrow h_* & & \downarrow h_* & & \downarrow h_* & \\
\dots \xrightarrow{\partial} & \pi_n(B, y_0) & \xrightarrow{i_*} & \pi_n(Y, y_0) & \xrightarrow{j_*} & \pi_n(Y, B, y_0) & \xrightarrow{\partial} & \pi_{n-1}(B, y_0) & \to \dots
\end{array}$$

is commutative.

An algebraic insertion: exact sequences

★*Exercise 1.* The sequence $0 \to A \to 0$ is exact if and only if $A = 0$.

★*Exercise 2.* The sequence $0 \to A \xrightarrow{\varphi} B \to 0$ is exact if and only if A and B are isomorphic with each other and $\varphi: A \to B$ is an isomorphism.

★*Exercise 3.* The sequence $0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0$ is exact if and only if A is isomorphic to a subgroup of B, $i: A \to B$ is the inclusion $C = B/A$ and $\pi: B \to C =$ $= B/A$ is the natural projection.

★*Exercise 4.* If $0 \to A_1 \to \dots \to A_n \to 0$ is an exact sequence, then

$$\sum_{i=1}^{n} (-1)^i (\operatorname{rank} A_i) = 0.$$

★*Exercise 5* (the "five lemma"). Assume that in the following diagram

$$\begin{array}{ccccccccc} A_1 & \to & A_2 & \to & A_3 & \to & A_4 & \to & A_5 \\ \downarrow\varphi_1 & & \downarrow\varphi_2 & & \downarrow\varphi_3 & & \downarrow\varphi_4 & & \downarrow\varphi_5 \\ B_1 & \to & B_2 & \to & B_3 & \to & B_4 & \to & B_5 \end{array}$$

the horizontal lines are exact sequences, φ_2 and φ_4 are isomorphisms, φ_1 is an epimorphism and φ_5 is a monomorphism. Then φ_3 is an isomorphism.

The lemma is indispensable in topology and wherever exact sequences are used. It is highly advisale to prove it for the reader.

First applications of the exactness of sequences of pairs

Exercise 6. If a mapping $f\colon (X, A)\to(Y, B)$ gives rise to isomorphisms

$$\pi_q(X) \xrightarrow{(=)} \pi_q(Y) \text{ and } \pi_q(A) \xrightarrow{(=)} \pi_q(B)$$

then $\pi_q(X, A)\to\pi_q(Y, B)$ will also be isomorphisms for all q.

Exercise 7. Suppose that A is a deformation retract of X. Then for $n\geq 1$ and for any $x_0\in A$ the inclusion homomorphism $i_*\colon \pi_n(A, x_0)\to\pi_n(X, x_0)$ is a monomorphism. For $n\geq 2$ it yields a direct sum decomposition.

$$\pi_n(X, x_0) \cong \pi_n(A, x_0)+\pi_n(X, A, x_0).$$

Exercise 8. If A is contractible in X to a point $x_0\in A$, then for $n\geq 1$ the homomorphism $i_*\colon \pi_n(A, x_0)\to\pi_n(X, x_0)$ is trivial. Moreover for $n\geq 3$ we have the decomposition

$$\pi_n(X, A, x_0) \cong \pi_n(X, x_0)+\pi_{n-1}(A, x_0).$$

The homotopy sequence of a fibration

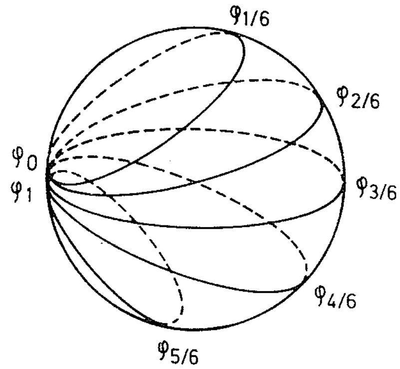
Let (E, B, F, p) be a Serre fibration. We can write out the homotopy sequence of the pair (E, F), $F=p^{-1}(p(x_0))$ with base-point x_0:

$$\dots \xrightarrow{\partial} \pi_n(F, x_0) \xrightarrow{i_*} \pi_n(E, x_0) \xrightarrow{j_*} \pi_n(E, F, x_0) \xrightarrow{\partial} \pi_{n-1}(F, x_0) \to \dots$$

$$\dots \xrightarrow{j_*} \pi_2(E, F, x_0) \xrightarrow{\partial} \pi_1(F, x_0) \xrightarrow{i_*} \pi_1(E, x_0) \xrightarrow{j_*} \pi_1(E, F, x_0).$$

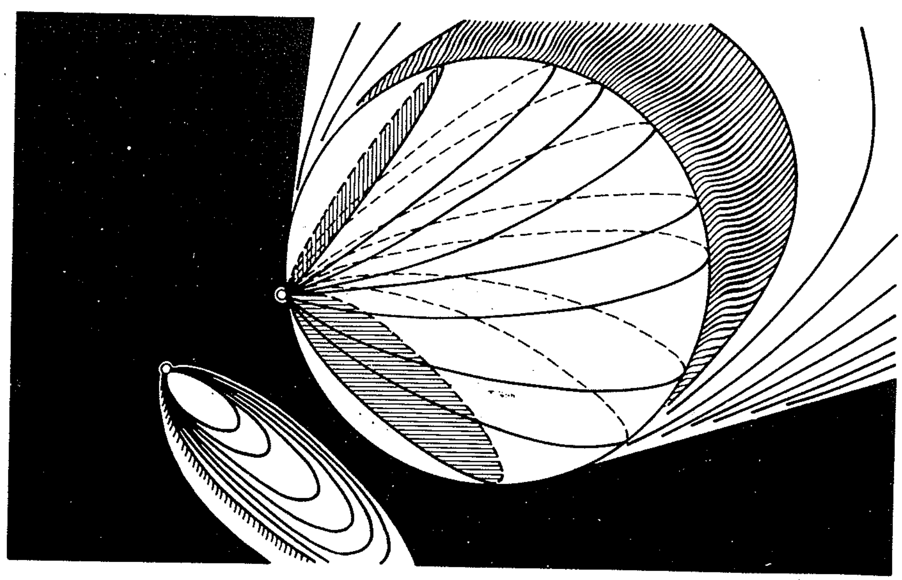
Now the remarkable fact is that it can be written by using only absolute groups. This follows from the isomorphism $\pi_n(E, F) \approx \pi_n(B)$, which can be proved quite simply. Indeed, the mapping $\pi_n(E, F)\to\pi_n(B,*)$ is induced by the projection of the fibration. It is a monomorphism by CHP (the covering homotopy property) and an epimorphism because of the following.

Let $\alpha \in \pi_n(B, *)$, and let $f: S^n \to B$ be a representative of α. Let us denote by $\varphi_t: S^{n-1} \to \to S^n$ the homotopy which is shown on the picture for $n = 2$ and it is analogously defined for arbitrary n.



Let us denote by Z the $(n-1)$-dimensional sphere and put $f_t = f \circ \varphi_t$, $F(Z) = x_0$. In view of (CHP) there exists a homotopy $F_t: Z \to E$ such that $F_0 = F$ and $p \circ F_t = f_t$, in particular $F_1(Z) \subset F$. Clearly $\bigcup_{0 \leqq t \leqq 1} F_t(Z)$ makes a relative spheroid in (E, F), which is projected onto f.

One can also immediately construct $\pi_n(B) \to \pi_{n-1}(F)$ without applying relative groups. We advise the reader to do it in the way suggested by the following picture:



The obtained sequence which contains only absolute groups is called the exact sequence of the fibration. Its final form is as follows:

$$\ldots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \ldots \to \pi_1(B).$$

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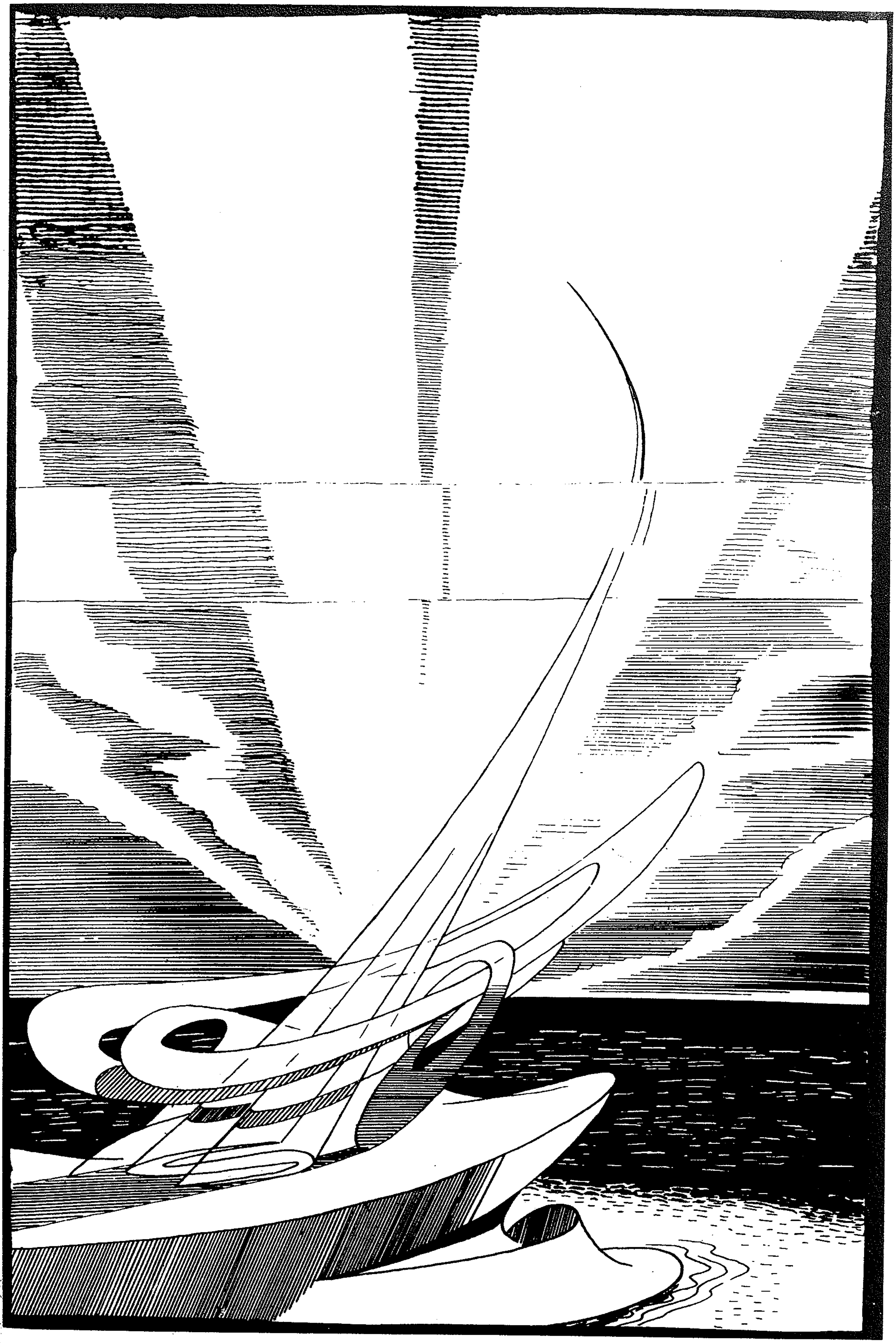


Applications of exact sequences of fibrations

★*Exercise*. Deduce all possible consequences from the exact sequence of the Hopf fibration. (Here the equality $\pi_2(S^2) = \mathbf{Z}$, which follows from the sequence, is important by various reasons: first, it proves that the two-dimensional sphere is not contractible, second, the general procedure to compute $\pi_n(S^n)$ cannot be applied to this case, thus $\pi_2(S^2) = \mathbf{Z}$ will be necessarily the starting step of any induction.) The equality $\pi_3(S^2) = \pi_3(S^3)$, also following from this exact sequence, was one of the main sensations of the early thirties.

★*Exercise*. Find the homotopy groups of the infinite-dimensional complex projective space (by using the fibration $S^\infty \to \mathbf{CP}^\infty$ with fibre S^1).

★*Exercise*. If the base (or fibre) of a fibration is contractible, the homotopy groups of the total space are isomorphic to the homotopy groups of the fibre (resp. base).

★*Exercise*. If all homotopy groups of the base as well as those of the fibre are finite, so are the homotopy groups of the total space, and their orders do not exceed the product of the orders of the homotopy groups of the same dimension of the base and the fibre.

★*Exercise*. If the base and the fibre have finitely generated homotopy groups, then the total space of fibration has the same property. Moreover, the rank of the q-th homotopy group of the space is not larger than the sum of the ranks of the q-th homotopy groups of the base and the fibre.

★*Exercise*. Prove that for any pathwise connected X and an arbitrary $x_0 \in X$ we have the isomorphism

$$\pi_q(X, x_0) \approx \pi_{q-1}(\Omega_{x_0}X, \omega_{x_0})$$

where ω_{x_0} is the constant loop in the point x_0.

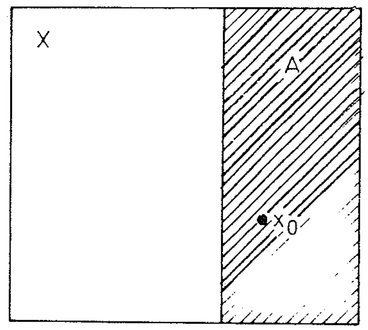
★*Exercise*. Consider a pair (X, A) with path-connected X. Denote by Λ the space of all paths in X which begin at a fixed point x_0 and end in A. The $\pi_n(X, A, a) = \pi_{q-1}(\Lambda, \lambda a)$ where λa is an arbitrary path beginning at x_0 and ending at $a \in A$.

Exercise. Suppose that the fibration $p\colon E \to B$ admits a section $\chi\colon B \to E$, where $e_0 = \chi(b_0)$. Then for $n \geq 1$, p_* is an epimorphism, and for $n \geq 2$ yields a direct sum decomposition $\pi_n(E, e_0) = \pi_n(B, b_0) + \pi_n(F, e_0)$.

Let (X, A) be an arbitrary pair. We already know that the inclusion $A \to X$ may be turned into a Serre fibration by substituting A by a homotopy equivalent A'. Let us consider the exact sequence of the pair (X, A) and the fibration $p\colon A' \to X$ and construct a mapping

$$\begin{array}{ccccccccc}
\ldots \to & \pi_n(X) & \to & \pi_n(X, A) & \to & \pi_{n-1}(A) & \to & \pi_{n-1}(X) & \to \ldots \\
 & \| & & \| & & \| & & & \\
\ldots \to & \pi_n(X) & \to & \pi_{n-1}(F) & \to & \pi_{n-1}(A') & \to & \pi_{n-1}(X) & \to \ldots
\end{array}$$

(where F denotes the fibre of the fibration p). Here $\pi_{n-1}(A')\to\pi_{n-1}(A)$ is the homomorphism induced by the projection $A'\to A$ (cf. §5). We define $\pi_{n-1}(F)\to\pi_n(X, A)$ the following way. A point of F is a path in X that starts at x_0 and ends somewhere in A. If $f\colon I^{n-1}\to F$ is a spheroid, then the mapping $F\colon I^n=I^{n-1}\times I\to X$, given by $F(x,t)=$ $=(f(x))(1-t)$, is a relative spheroid of the pair (X, A).



By assigning F to f we get a homomorphism $\pi_{n-1}(F)\to\pi_n(X, A)$ and the arising diagram is commutative. By the five lemma, $\pi_{n-1}(F)\to\pi_n(X, A)$ is an isomorphism.

We could have got the same conclusion without applying the five lemma by only noticing that the correspondence in question between the spheroids is one-to-one. We preferred the longer way because it makes the nature of the exact sequences of fibrations clear.

§9. THE SUSPENSION HOMOMORPHISM

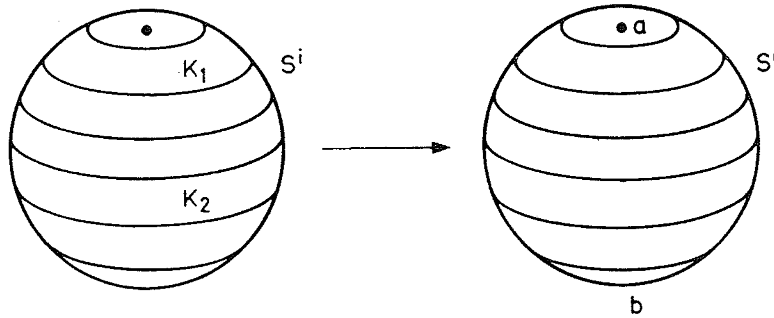
The suspension over a spheroid $f\colon S^n\to X$ is obviously a spheroid, too: $\Sigma f\colon S^{n+1}=$ $=\Sigma S^n\to\Sigma X$. (For the definition of suspension see §2.) If $f, g\colon S^n\to X$ are homotopic, then so are Σf and Σg. As it can easily be seen, the spheroid $\Sigma(f+g)$ is homotopic to $\Sigma f+\Sigma g$. Hence, by assigning Σf to f, we obtain a homomorphism $\pi_n(X)\to\pi_{n+1}(\Sigma X)$ that will be called the *suspension homomorphism* and denoted by Σ.

Theorem (Freudenthal). The homomorphism $\pi_{i-1}(S^{n-1})\to\pi_i(S^n)$ is an epimorphism for $i\le 2n-2$ and isomorphism for $i<2n-2$.

This was called the "easy part" of the Freudenthal theorem. The "difficult part" will be given further on in the present §. The following generalization of the Freudenthal theorem cannot be proved until Chapter III and may be considered as an exercise to this chapter.

If K is a CW complex and $\pi_i(K)=0$ for $i<n-1$ then $\Sigma\colon \pi_{i-1}(K)\to\pi_i(\Sigma K)$ is an isomorphism for $i<2n-2$ and epimorphism for $i=2n-2$.

Proof of the Freudenthal theorem. Let $f\colon S^i\to S^n$, $i<2n-1$. We must prove that there exists a $h\colon S^{i-1}\to S^{n-1}$ such that f is homotopic to $\Sigma h\colon \Sigma S^{i-1}=S^i\to S^n$.



Let the spheres S^i and $S^n = \Sigma S^{n-1}$ be triangulated. Let a and b denote the "poles" of the sphere S^n. The triangulation of S^n will be done so that a and b will be inner points of n-dimensional simplexes, and f will be supposed to be simplicial.

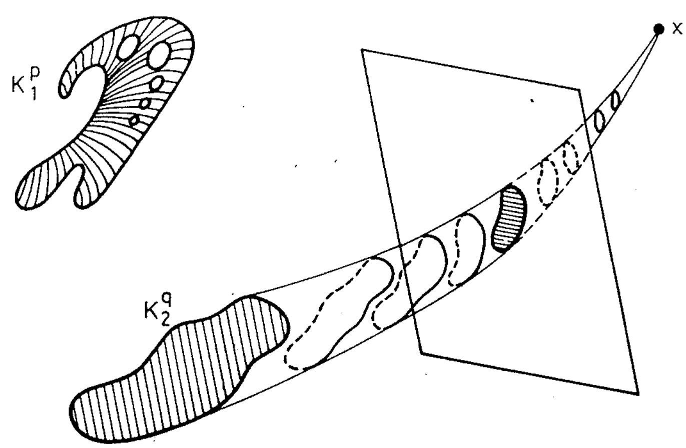
Let $K_1 = f^{-1}(a)$ and $K_2 = f^{-1}(b)$. Consisting of convex polyhedra they are not simplicial complexes in general, nevertheless they can easily be triangulated.

Next we are going to state some obvious geometric facts concerning the situation of linear simplicial complexes in a Euclidean space.

1. Let K_1^p and K_2^p be complexes in E^n. If $p+q<n$ then for any $\varepsilon>0$, we may make them disjoint by an ε'-shift of each, where $\varepsilon'>\varepsilon$.

2. K_1^p and K_2^p are said to be unlinked if there exists an isotopy (i. e. a homotopy consisting of isomorphism) of $\operatorname{id} E^n$ transforming them into complexes which are separated by an $(n-1)$-dimensional hyperplane. If $p+q<n-1$, then K_1^p, K_2^p are always unlinked.

Let us explain the second statement. Let us choose a hyperplane such that K_1^p is on one side of it. Let x be a point on the opposite side. We construct the cone L^{q+1} over K_2^q with its summit in x. Because $p+(q+1)<n$ we may assume L^{q+1} and K_1^p not to meet each other. Next we pull K_2^q through the cone into the other side of the hyperplane. Outside of a small neighbourhood of L^{q+1} the isotopy may be forced to be stationary.

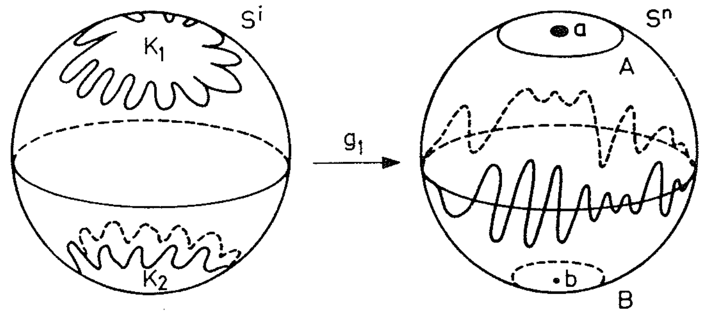


Let us now consider the sphere S^i and the complexes K_1 and K_2.

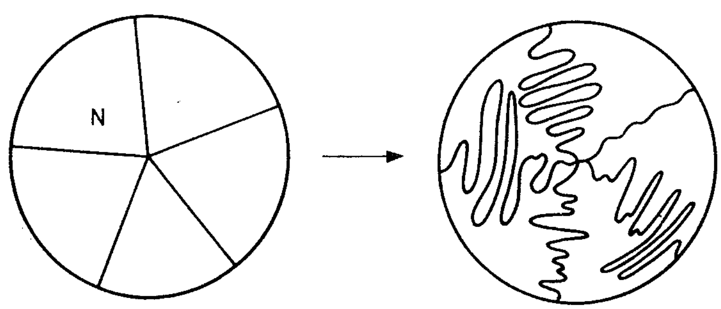
As a and b are inner points of n-dimensional simplexes, we have $\dim K_j \leq i-n$. If $(i-n)+(i-n)<i-1$, i. e. $i<2n-1$, then K_1 and K_2 are unlinked, i. e. there is an isotopy $\varphi_\perp$ of S^i into itself which carries K_1 and K_2 into different hemispheres (K_1 to "North" and K_2 to "South"), i. e. there exists an isotopy φ_t: $S^i \to S^i$ such that φ_0 is identity, $\varphi_1(K_1)$ and $\varphi_1(K_2)$ belong to different hemispheres and every φ_t is homeomorphism.

Let us consider the homotopy $g_t: S^i \xrightarrow{\varphi_1^{-1}} S^i \xrightarrow{f} S^n$. Then $g_1^{-1}(a)$ and $g_1^{-1}(b)$ are in different hemispheres while the image of the equator of S^1 does not contain either a or b. Moreover there exist neighbourhoods A and B of a and b respectively, not containing any point of the equator.

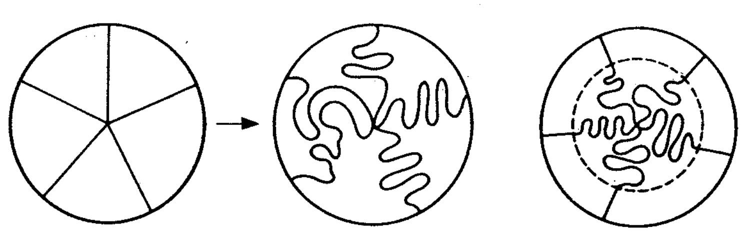
There exists a homotopy $S^n \to S^n$ that stretches A and B to the northern and southern hemisphere, respectively, and squeezes the remainder onto the equator. By composing it with $g_1: S^i \to S^n$ we obtain a homotopy whose final state is a fairly good mapping $S^i \to S^n$. It sends the equator as well as the two hemispheres into themselves. Let us

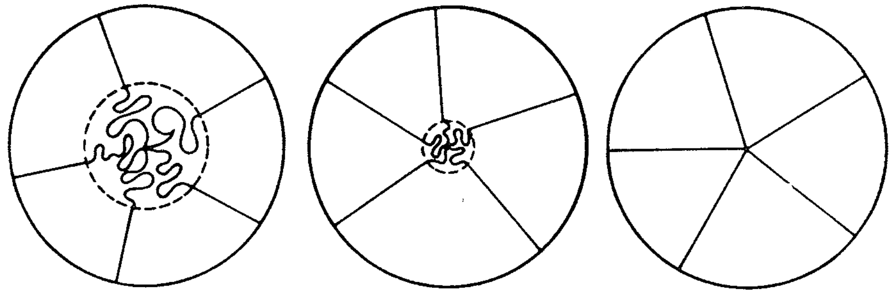


look at S^i and S^n from North. So we only see the northern hemisphere. We draw all possible meridians and follow where they are carried by the mapping.



A further homotopy may be constructed which finally turns the mapping into the suspension mapping. The construction is as seen on the picture:





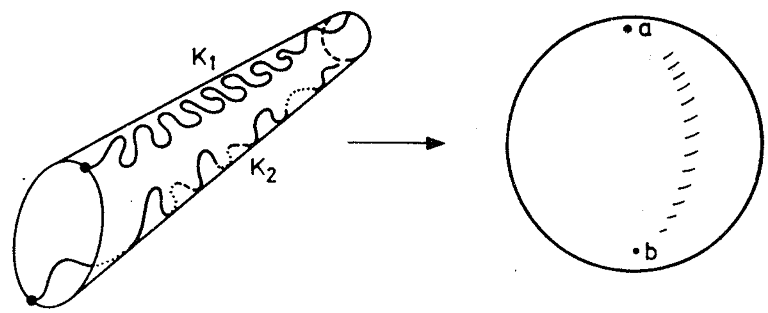
i. e. the image of each radius is pulled to the point, an increasing part of it being replaced by radii. This proves that the suspension homomorphism is really an epimorphism. (This smart homotopy was invented by J. Alexander a long time ago.)

To finish the proof it must be shown to be a monomorphism for $i<2n-2$.

Let $f_1 = \Sigma h_1\colon S^i \to S^n$ and $F_2 = \Sigma h_2\colon S^i \to S^n$ be spheroids. We show that if Σh_1 and Σh_2 are homotopic, then so are h_1 and h_2.

The homotopy f_t connecting f_1 and f_2 has to be altered so that each f_t would be a suspension spheroid.

Consider the homotopy f_t which is actually a mapping $S^i \times I \to S^n$.



Again we examine the sets $K_1 = f^{-1}(a)$, $K_2 = f^{-1}(b)$. We have $\dim K_j \leq (i+1)-n$ and $\dim (S^i \times I) = i+1$, so K_1 and K_2 may be deformed through $S^i \times I$ so that they are separated. This can be done whenever $(i+1-n)+(i+1-n)<i+1-1$, i. e. $i<2n-2$.

The remaining arguments are analogous to those applied in proving the epimorphism property. Q.e.d.

Theorem (Hopf). $\pi_n(S^n) = \mathbf{Z}$.

For $n=1$, 2 it was proved in §§4 and 8. For $n\geq 3$ the equality follows from $\pi_{n-1}(S^{n-1}) = \mathbf{Z}$ in view of the Freudenthal theorem.

Corollary. No sphere S^n is contractible.

So far we have not been able to make this kind of statements. Neither could we answer the question whether a given space with nontrivial n-th homotopy group really exists.

A further corollary. $\pi_3(S^2) = \mathbf{Z}$.

Indeed, in view of the exact sequence of the Hopf fibration $S^3 \to S^2$ with fibre S^1,

$$\begin{array}{ccccccc} \pi_3(S^1) & \to & \pi_3(S^3) & \to & \pi_3(S^2) & \to & \pi_2(S^1) \\ \| & & & & & & \| \\ 0 & & & & & & 0 \end{array}$$

we have $\pi_3(S^2) = \pi_3(S^3)$.

Further analysis of the same exact sequence would show that the generator of $\pi_3(S^2)$ is represented by the Hopf fibration $S^3 \to S^2$.

Let us now recollect our informations on the homotopy groups of spheres.

$\pi_i(S^n)$	$i=1$	2	3	4	5	6
$n=1$	**Z**	0	0	0	0	0
2	0	**Z**	**Z**	?	?	?
			$\|$	$\|$	$\|$	$\|$
3	0	0	**Z**	?	?	?
4	0	0	0	**Z**	?	?

(Each group $\pi_i(S^n)$ is mapped by Σ to $\pi_{i+1}(S^{n+1})$.)

We see that the set of homotopy groups of spheres is decomposed into series $\{\pi_{n+i}(S^n)\}_{n=1}^{\infty}$ which stabilize as n increases. Later at the end of this book we shall obtain a procedure to determine the first stable groups (and we shall actually compute the stable groups $\pi_{n+i}(S^n)$ for $i \leq 13$). So far it would be too difficult a task. We are only able to say that the stable groups $\pi_{n+i}(S^n)$ are zero for $i<0$, **Z** for $i=0$ and cyclic for $i=1$.

The product $S^m \times S^n$ is a CW complex with four cells e^0, e^n, e^m and e^{n+m}. The restriction of the characteristic mapping $f\colon B^{n+m} \to S^m \times S^n$ of the cell e^{n+m} to the sphere $S^{n+m-1} \subset B^{n+m}$ is a mapping $S^{n+m-1} \to S^n \vee S^m$. Let it be denoted by $W(m, n)$ or simply by W.

Definition. Let $\alpha \in \pi_n(X)$ and $\beta \in \pi_m(X)$. The element of $\pi_{n+m-1}(X)$ represented by the spheroid

$$S^{n+m-1} \xrightarrow{W} S^n \vee S^m \to X \qquad (f \in \alpha \text{ on } S^n,\ g \in \beta \text{ on } S^m)$$

will be called the Whitehead product of α, β. It will be denoted by $[\alpha, \beta]$.

Exercise. $[\alpha, \beta] = (-1)^{\dim\alpha\cdot\dim\beta}[\beta, \alpha]$.

Exercise. If $\alpha, \beta \in \pi_1(X)$, then $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1} \in \pi_1(X)$.

Exercise. If X is n-simple, $\alpha \in \pi_n(X)$ and $\beta \in \pi_n(X)$, then $[\alpha, \beta]=0$. If $\alpha \in \pi_1(X)$ and $\beta \in \pi_n(X)$ with $n>1$, then $[\alpha, \beta] = T_\alpha\beta - \beta$ where T_α denotes the action of α on $\pi_n(X)$, see §6.

Exercise.

$$(-1)^{\dim\alpha\dim\gamma}[[\alpha, \beta], \gamma] + (-1)^{\dim\beta\dim\alpha}[[\beta, \gamma], \alpha] +$$
$$+ (-1)^{\dim\gamma\dim\beta}[[\gamma, \alpha], \beta] = 0.$$

Exercise. If X is a H-space, $\alpha \in \pi_n(X)$ and $\beta \in \pi_m(X)$, then $[\alpha, \beta]=0$.

Exercise. For any $\alpha \in \pi_n(X)$ and $\beta \in \pi_m(X)$ the element $\Sigma([\alpha, \beta]) \in \pi_{m+n}(X)$ is zero. (Hint. The diagram

$$\begin{array}{ccccc}
S^{m+n} = \Sigma(S^{m+n-1}) & \xrightarrow{\Sigma W(n,m)} & \Sigma(S^n \vee S^m) & \xrightarrow{\Sigma(f\vee g)} & \Sigma X \\
\swarrow & & & & \| \\
S^{m+n+1} & \xrightarrow{W(n+1,m+1)} & S^{n+1} \vee S^{m+1} = (\Sigma S^n) \vee (\Sigma S^m) & \xrightarrow{\Sigma f \vee \Sigma g} & \Sigma X
\end{array}$$

may be completed to a commutative one by choosing a suitable mapping $S^{m+n} \to \to S^{m+n+1}$. Now any mapping $S^{m+n} \to S^{m+n+1}$ is homotopy trivial, which implies the statement.)

Exercise. For arbitrary m, n the space $\Sigma(S^n \times S^m)$ is homotopy equivalent to $S^{n+1} \vee S^{m+1} \vee S^{m+n+1}$. (Hint: the problem is equivalent to the preceding one.)

Exercise. The element $[i_2, i_3]$ of $\pi_3(S^2)$, where i_2 is the canonical generator of $\pi_2(S^2)$, is equal to the doubled generator of $\pi_3(S^2) = \mathbf{Z}$.

Exercise. (The difficult part of the Freudenthal theorem). The kernel of the epimorphism $\Sigma\colon\ \pi_{4n-1}(S^{2n}) \to \pi_{4n}(S^{2n+1})$ is generated by the single element $[i_{2n}, i_{2n}] \in \pi_{4n-1}(S^{2n})$ where i_{2n} is the canonical generator of the group $\pi_{2n}(S^{2n})$.

The last two exercises imply that $\pi_4(S^3) = \mathbf{Z}_2$. Thus $\pi_{n+1}(S^n) = \mathbf{Z}_2$ for $n \geq 4$.

§10. HOMOTOPY GROUPS AND CW COMPLEXES

Attaching cell theorem. Let X be a space and $f\colon S^{n-1} \to X$ be a mapping. The homomorphism $g_*\colon\ \pi_i(X) \to \pi_i(X \cup_f e^n)$ is isomorphism for $i<n-1$ and epimorphism for $i=n-1$. The kernel in the latter case is generated by $[f]$ and the elements of the form $T_\gamma[f]$ where $\gamma \in \pi_1(X)$.

We recall that the group $\pi_1(X)$ acts on $\pi_n(X)$ in the following way. Let $\alpha \in \pi_1(X)$ and let the loop s represent α. Then s is a path connecting the base point $x_0 \in X$ with itself. It induces an isomorphism $\pi_n(X, x_0) \to \pi_n(X, x_0)$ which sends $\beta \in \pi_n(X, x_0)$ into an element that will be denoted by $T_\alpha\beta$. We also recall that $T_\alpha\beta = \beta + [\alpha, \beta]$, and at last, that there exists an alternative definition of n-simplicity in terms of this action: X is n-simple if $T_\alpha\beta=\beta$ for arbitrary $\alpha\in\pi_1(X)$, $\beta\in\pi_n(X)$.

Proof of the attaching cell theorem. Let $i<n$ and let us be given an arbitrary mapping φ: $S^i\to X\cup_f e^n$. Analogously to the cellular approximation theorem, we show that there exists a mapping ψ: $S^i\to X\cup_f e^n$ homotopic to φ whose image does not cover the whole ball e^n. Then this image may be pulled to the boundary, i. e. φ is homotopic to a mapping of S^i into X. If $i<n-1$, the same argumentation holds for $S^i\times I\to X\cup_f e^n$ as we have an extra dimension, i. e. any homotopy connecting two spheroids of such dimensions may be made as not meeting e^n.

We obtain that the theorem is valid for $i<n-1$ and the homomorphism $\pi_{n-1}(X)\to\to\pi_{n-1}(X\cup_f e^n)$ is an epimorphism for $i=n-1$. It remains to describe the kernel. It clearly contains any element $T_\gamma[f]$ with $\gamma\in\pi_1(X)$ as well as the linear combinations of such elements. The statement that every element of the kernel has this form is less obvious, it may be proved similarly to the second part of theorem 4 in §4. It is left to the reader.

Corollary. If Y is a subcomplex of X and the difference $X\setminus Y$ contains no cells of dimension $\leq p$, then the homomorphism $\pi_i(Y)\to\pi_i(X)$ induced by the inclusion is an isomorphism for $i<p$ and epimorphism for $i=p$.

Corollary of the Corollary. For any CW complex X, $\pi_i(X)=\pi_i(X^{i+1})$, where X^{i+1} is the $(i+1)$-skeleton of X.

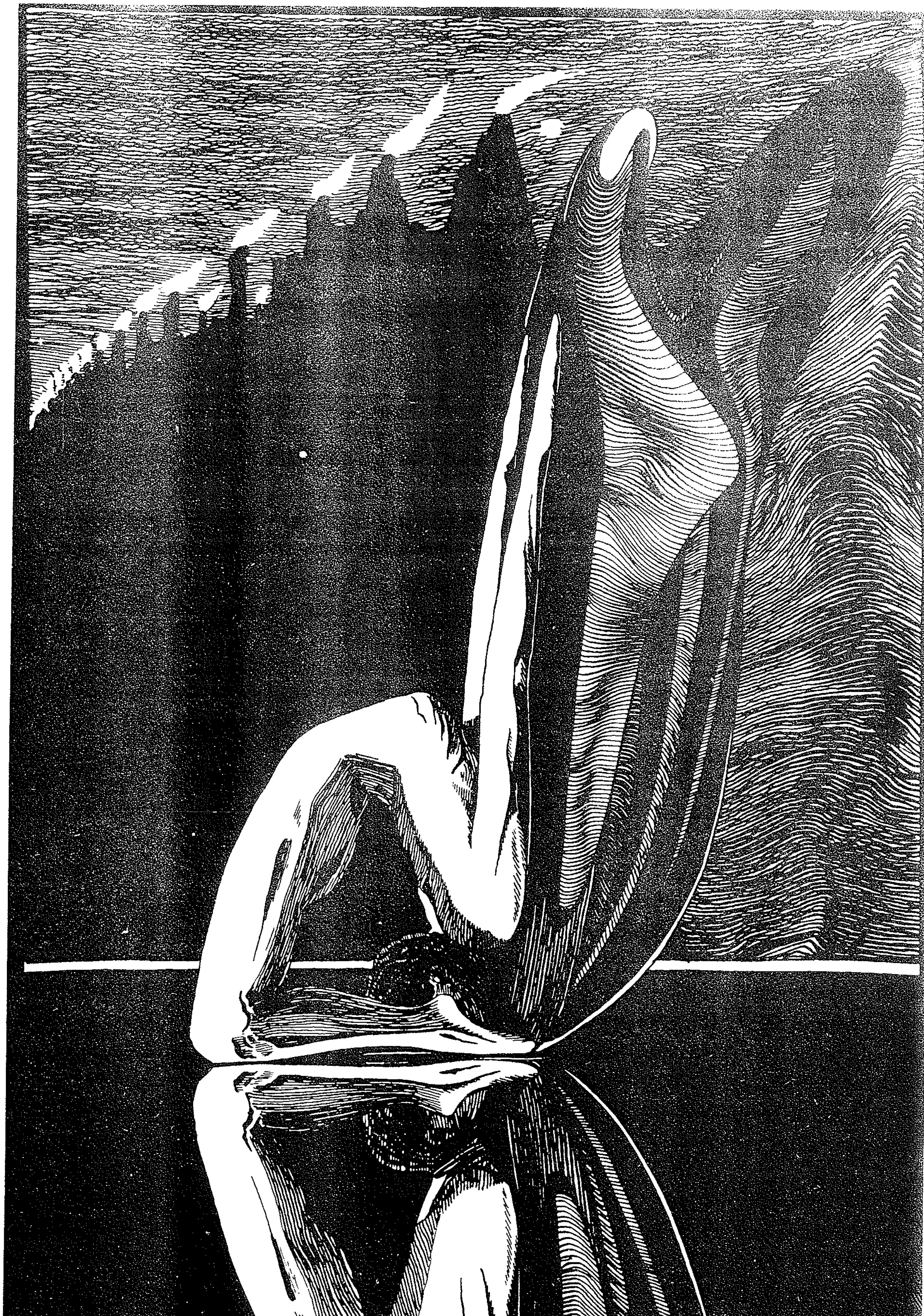
Theorem. If the CW complexes X and Y are p- and q-connected, respectively, then

a. $\pi_i(X\vee Y)=\pi_i(X\times Y)$ for $i<p+q-1$;

b. there exists an epimorphism $\pi_{p+q-1}(X\vee Y)\to\pi_{p+q-1}(X\times Y)$. In particular, $\pi_n(S^n\vee\ldots\vee S^n)=\mathbf{Z}+\ldots+\mathbf{Z}$ for $n>1$.

Proof. As proved in §3, there exist CW complexes X' and Y' which are homotopy equivalent to X and Y and have a single vertex each, and have no cells in dimensions $1,\ldots,p-1$ and $1,\ldots,q-1$, respectively. Now $X'\vee Y'$ is homotopy equivalent to $X\vee Y$ and is imbedded in $X'\times Y'$ which is homotopy equivalent to $X\times Y$; moreover, the difference $(X'\times Y')\setminus(X'\vee Y')$ is free of cells of dimension $<p+q$. According to the corollary to the attaching cell theorem $\pi_i(X'\vee Y')\to\pi_i(X'\times Y')=\pi_i(X')+\pi_i(Y')$ is an isomorphism if $i<p+q-1$.

Remark. It will be noted that not only the isomorphism between $\pi_i(X\vee Y)$ and $\pi_i(X)+\pi_i(Y)$ has been stated but also that it is induced by the imbedding $X\vee Y\to\to X\times Y$. In particular, the group $\pi_n(S^n\vee\ldots\vee S^n)=\mathbf{Z}+\ldots+\mathbf{Z}$ is generated by the classes of the natural imbeddings $S^n\to S^n\vee\ldots\vee S^n$.



Computing the first nontrivial homotopy group of a CW complex

Let $n>1$ and assume that for a connected CW complex K, $\pi_i(K)=0$ for $i<n$. There exists a CW complex K' having a single vertex and no other cell in dimension $<n$. Suppose K' to have n-dimensional cells σ_i^n, $i\in I$ and $(n+1)$-dimensional cells σ_j^{n+1}, $j\in J$.

We denote by $f_j^{n+1}: B^{n+1}\to K$ the characteristic mapping for σ_j^{n+1}. The mappings $\varphi_j=f_j^{n+1}|_{S^n}: S^n\to K^n=\bigvee_{i\in I} S^n$ represent elements of the group $\pi_n(K^n)=\bigoplus_{i\in I}\mathbf{Z}$.

Theorem. The group $\pi_n(K)$ is the quotient group of $\pi_n(K^n)=\bigoplus_{i\in I}\mathbf{Z}$ by the subgroup generated by the elements $\{\varphi_j\}\in\pi_n(K^n)$.

Or, in a formulation which, though not really adequate, is nevertheless more convenient to memorize: $\pi_n(K)$ is the Abelian group whose generators and relations correspond to the n-dimensional and $(n+1)$-dimensional cells, respectively.

The proof is similar to that of the theorem on the fundamental groups of CW complexes.

The main steps are the following:

(1) Every n-spheroid of K' being homotopic to a spheroid of K^n, we can choose for the generators of $\pi_n(K^n)$ the imbeddings α_i of the n-dimensional cells in K.

(2) The relations $\Sigma a_{ij}\alpha_i=0$ are obviously satisfied, as each φ_j: $S^n\to K$ extends to $f_j^{n+1}: B^{n+1}\to K$.

(3) Any relation reduces to $\Sigma a_{ij}\alpha_i=0$.

The Whitehead theorem

Theorem. Let X and Y be CW complexes. If the mapping f: $X\to Y$ induces isomorphism between the respective homotopy groups, then it is a homotopy equivalence.

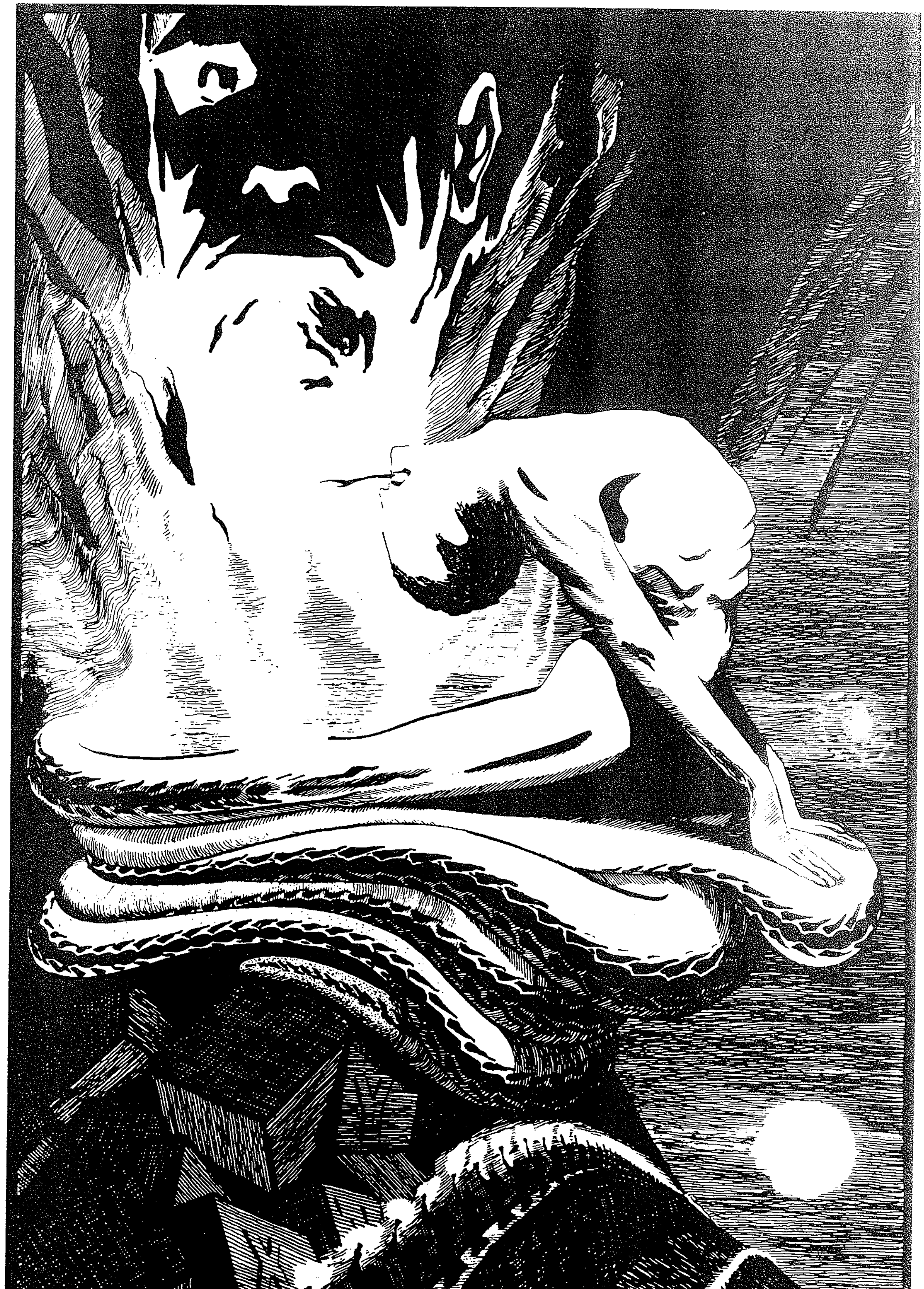
Equivalently: for CW complexes weak and ordinary homotopy equivalence are the same.

Exercise. Prove the equivalence of the two statements.

Lemma. For any CW pair (K, L) for which $\pi_i(K, L)=0$, for $i\le n$, there exists a homotopy equivalent pair (K', L') such that for all $i\le n$, the i-dimensional cells of K' belong to L'. Here n may be infinite; then it is claimed that $\pi_i(K, L)=0$ for every i implies $K\sim L$. The equality $\pi_i(K, L)=0$ is meant to say that $\pi_1(K, L)$ consists of a single element.

Deduction of the theorem from the lemma. Let us choose the cylinder of f for K and Y for L. Then f as well as the imbedding $L\to K$ induce isomorphisms between the homotopy groups. Taking into account the exact sequence of the pair (K, L) we obtain $\pi_i(K, L)=0$ for every i. Then the lemma, applied for $n=\infty$, implies $K\sim L$.

Proof of the lemma. The reader is advised to prove it, following the line of the "absolute" theorem as given in §3. Nevertheless we present it here.

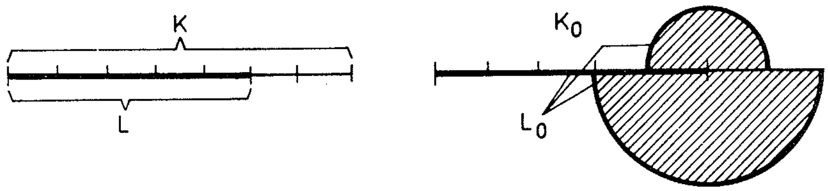


We construct step-by-step imbeddings

$$\begin{array}{ccccccc} K \subset & K_0 \subset & K_1 \subset & \dots & \subset K_n \\ \cup & \cup & \cup & & \cup \\ L \subset & L_0 \subset & L_1 \subset & \dots & \subset L_n \end{array}$$

where the imbeddings $K \subset K_0 \subset K_1 \subset \dots$, $L \subset L_0 \subset L_1 \subset \dots$ are homotopy equivalences, the diagram is commutative and the differences $K_i \setminus L_i$ contain no cells with dimension at most i while $K_i \setminus K_{i-1}$ only consists of cells with dimensions $i+1$ and $i+2$. Hence we get the lemma for $n < \infty$. If $n = \infty$, we put $K_\infty = \cup_i K_i$ and $L_\infty = \cup_i L_i$. Clearly $K_\infty = L_\infty$. Now the imbeddings $K \to K_\infty$ and $L \to L_\infty$ are homotopy equivalences. Indeed, let f_i: $K_i \to K_{i-1}$ be a homotopy inverse mapping of the imbedding $K_{i-1} \to K_i$. By the Borsuk theorem it can be chosen so that it coincides with the identity on $K_{i-1} \subset K_i$. We define f: $K_\infty \to K$ as being equal to $f_0 \circ f_1 \circ \dots \circ f_i$ on K_i. It is correctly defined and it is a homotopy inverse of $K \to K_\infty$. The homotopy inverse of $L \to L_\infty$ is defined analogously.

Construction of the chain of imbeddings



Suppose K_i, L_i as well as the preceding spaces and mappings already defined. Then every $(i+1)$-dimensional cell $e^{i+1} \subset K_i$ which is not contained in L_i is a $(i+1)$-dimensional relative spheroid of (K_i, L_i) (not the cell itself, of course, but its characteristic mapping). Such a spheroid is homotopic to a spheroid belonging to L_i, moreover the homotopy is constant on the boundary of e^{i+1}, it takes place within the $(i+2)$-skeleton of K_i, and its final result belongs to the $(i+1)$-skeleton of L_i (by virtue of the cellular approximation theorem). This homotopy is a mapping $D^{i+2} \to K_i$ which can be used for attaching D^{i+3} to K_i (D^{i+2} is the lower hemisphere of the boundary sphere of D^{i+3}). There are two new cells attached to K_i: one of dimension $(i+2)$ and one of dimension $(i+3)$ (the interiors of the upper hemisphere and of D^{i+3}). This procedure is repeated for every $(i+1)$-dimensional cell of K_i which does not belong to L_i. The result is a complex K_{i+1}. For L_{i+1} we choose the union of L_i, the $(i+1)$-skeleton of K_i and all the new $(i+2)$-dimensional cells. The inclusion relations

$$\begin{array}{cc} K_i \subset & K_{i+1} \\ \cup & \cup \\ L_i \subset & L_{i+1} \end{array}$$

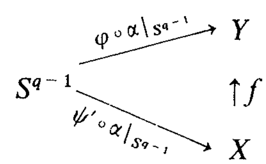
are obvious and the assumptions of the lemma are satisfied.

Remark. By the Whitehead theorem, the homotopy groups completely characterize in a way the homotopy type of a CW complex. Nevertheless this statement should not be taken literally: coincidence of the homotopy groups of two CW complexes does not necessarily imply homotopy equivalence. It is also required that the isomorphisms are established by a continuous mapping. For example, $\pi_i(S^3)=\pi_i(S^3\times \mathbf{CP}^\infty)$ for every i, however S^3 and $S^3\times\mathbf{CP}^\infty$ are not homotopy equivalent spaces. (Prove it.)

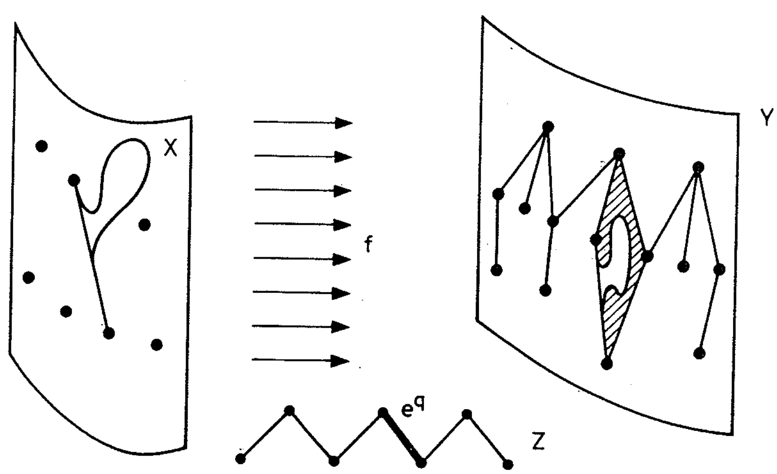
Refinement of the notion of weak homotopy equivalence

Theorem. If there exists a mapping between connected topological spaces X and Y that induces isomorphisms between the homotopy groups, then X and Y are weakly homotopy equivalent. Moreover, for any CW complex Z, the mapping $f_*\colon \pi(Z, X)\to$ $\to\pi(Z, Y)$ is one-to-one.

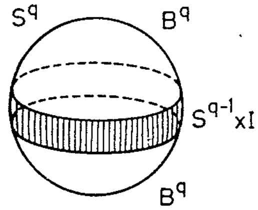
Proof. For $\dim Z = 0$ statement is obvious. Namely, $\pi(Z, X) = \pi(Z, Y) = *$ in this case. Assume the statement proved for any Z with $\dim Z < q$. Let us take q-dimensional complex Z. First of all we show that $f_*\colon \pi(Z, X)\to\pi(Z, Y)$ is an epimorphism, i. e. for arbitrary $\varphi\colon Z\to Y$ there exists $\psi\colon Z\to X$ such that $f\circ\psi\sim\varphi$. By induction there exists $\psi'\colon Z^{q-1}\to X$ with $f\circ\psi'\sim\varphi|_{Z^{q-1}}$ (as usual Z^{q-1} stands for the $(q-1)$-skeleton of Z). Let a homotopy $\Phi\colon Z^{q-1}\times I\to Y$ connecting $f\circ\psi'$ with $\varphi|_{Z^{q-1}}$ be fixed. Let $e^q\subset Z$ be a q-dimensional cell and $\alpha\colon B^q\to Z$ be its characteristic mapping. As the mapping $\varphi\circ\alpha|_{S^{q-1}}\colon S^{q-1}\to Y$ is null homotopic (it extends to $\varphi\circ\alpha\colon B^q\to Y$), so is $\psi'\circ\alpha|_{S^{q-1}}\colon$ $S^{q-1}\to X$ by the commutative diagram



(here $f_*\colon \pi_{q-1}(X)\to\pi_{q-1}(Y)$ is a monomorphism) and it can be extended to $\beta\colon B^q\to X$. The mapping β is not unique: it is determined up to some q-dimensional spheroid added to.



Let us now consider in Y the spheroid consisting of



$f \circ \beta: D^q \to Y,$

$\Phi \cdot (\alpha|_{S^{q-1}} \times I): S^{q-1} \times I \to Y$

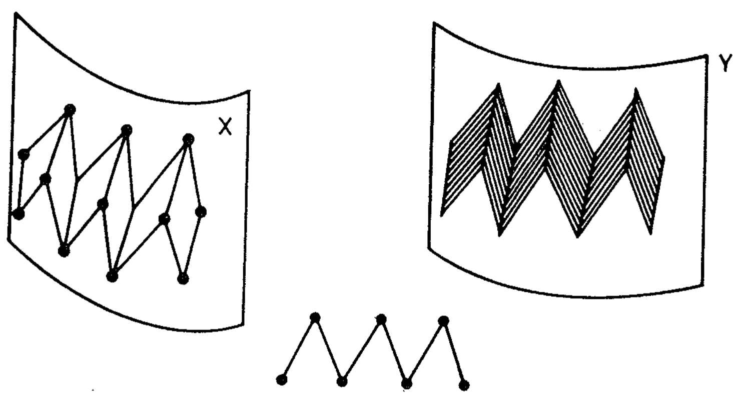
$\varphi \circ \varphi: D^q \to Y$

As $f_*: \pi_q(X) \to \pi_q(Y)$ is an epimorphism, the spheroid is homotopic to the image of some spheroid of X. By substituting it instead of β we can make the compound spheroid be null homotopic in Y. Then it can be extended to $\gamma: D^{q+1} \to Y$.

Mappings similar to $\beta: D^q \to X$ (we mean its improved variant) and $\gamma: D^{q+1} \to Y$ are next defined for all cells. They are then applied for extending $\psi': Z^{q-1} \to X$ to $\psi: Z \to X$. (On e^q, ψ is defined as the restriction to the interior of D^q of the mapping β corresponding to D^q.) Similarly $\Phi: Z^{q-1} \times I \to Y$ is extended to a homotopy $\Psi: Z \times I \to Y$ connecting $f \circ \psi$ to φ (on $e^q \times (0, 1)$ the mapping Ψ is given as the restriction to $\operatorname{Int} D^{q+1}$ of the corresponding γ).

Thus we have $\psi: Z \to X$ with $f \circ \psi \sim \varphi$, i. e. f_* is an epimorphism.

Now we show that f_* is a monomorphism, i. e. for any pair $\psi_1, \psi_2: Z \to X$, $f \circ \psi_1 \sim f \circ \psi_2$ implies $\psi_1 \sim \psi_2$. We have $\psi_1|_{Z^{q-1}} \sim \psi_2|_{Z^{q-1}}$ by the induction. Moreover, given



a homotopy $\Phi: Z \times I \to Y$ connecting $f \circ \psi_1$ with $f \circ \psi_2$, we can construct a mapping ψ': $((Z \times \{0\}) \cup (Z^{q-1} \times I) \cup (Z \times \{1\}) \to X$ that coincides with ψ_1 on $Z \times \{0\}$ and with ψ_2 on $Z \times \{1\}$ and is homotopic to the restriction of Φ on $(Z \times \{0\}) \cup (Z^{q-1} \times I) \cup (Z \times \{1\})$ (in the same way as above; we only have to realize that the construction was carried out independently on each q-dimensional cell, and had already been given a suitable mapping on a q-dimensional cell, there was no necessity to change it later). As $f_*: \pi_q(X) \to \pi_q(Y)$ is a monomorphism, ψ' can be extended to $\Psi: Z \times I \to X$. We have obtained a homotopy connecting ψ_1 with ψ_2.

Thus $f_*: \pi(Z, X) \to \pi(Z, Y)$ is a one-to-one correspondence for every finite-dimensional Z. Transition to the case of infinite-dimensional Z is made by the familiar induction on increasing skeletons.

Cellular approximation of topological spaces

Theorem. To any topological space there exists a weakly homotopy equivalent CW complex.

Proof. The space X will be assumed to be connected (otherwise the constructions are repeated for each component). Let K_0 be the single point space. Suppose that we already have the CW complexes $K_1, \dots, K_i$, imbeddings $K_0 \subset K_1 \subset \dots \subset K_i$ and a mapping $f_i\colon K_i \to X$ which induces isomorphisms between the homotopy groups of dimensions $<i$ and an epimorphism in dimension i. Let $\xi_\alpha \in \pi_i(K_i)$, $(\alpha \in A)$ denote the generators of Ker $(f_i)_*$, where $(f_i)_*\colon \pi_i(K_i) \to \pi_i(X)$, and let $\eta_\beta (\beta \in B)$ be generators of $\pi_{i+1}(X)$. The spheroids representing these elements will be denoted by $\tilde{\xi}_\alpha\colon S^i \to K_i$ and $\tilde{\eta}_\beta\colon S^{i+1} \to X$, respectively.

Along each $\tilde{\xi}_\alpha$ an $(i+1)$-dimensional ball will be attached to K_i. The union of the resulting space and a union of spheres indexed by elements of B will be denoted by K_{i+1}. Next we define $f_{i+1}\colon K_{i+1} \to X$ as coinciding with f_i on K_i, with $\tilde{\eta}_\beta$ on the β-th $(i+1)$-dimensional sphere of the union as well as on the ball attached to it along the mapping $\tilde{\xi}_\alpha$, and with an extension $\zeta_\alpha\colon D^{i+1} \to X$ of $f_i \circ \tilde{\xi}_\alpha\colon S^i \to X$ whose existence follows from the choice of ξ_α as an element of Ker $(f_i)_*$. Clearly f_{i+1} induces isomorphisms of the homotopy groups of dimensions $\leq i$ and epimorphism in dimension $i+1$.

By induction we have K_i for every i, inclusions $K_0 \subset K_1 \subset K_2 \subset \dots$ and mappings $f_i\colon K_i \to X$, each being an extension of the preceding one. Further f_i induces isomorphisms of the homotopy groups with dimensions less than i and an epimorphism in dimension i.

We write $K = \bigcup_i K_i$ and define $f\colon K \to X$ as coinciding with f_i on K_i. Then K is a CW complex with spaces K_i as its skeletons, and f induces isomorphisms of the homotopy groups. Thus K and X are weakly homotopy equivalent spaces.

Eilenberg–MacLane complexes

As it was announced in §2, for every natural number n and group Π (Abelian if $n>1$) there exists a space whose i-th homotopy group is zero if $i \neq n$ and Π if $i=n$. We are now ready to construct such CW complexes, for any Π and n.

Let $\{\alpha_i\}_{i \in J}$ be a system of generators in Π. We denote by K_n a union of n-dimensional spheres indexed by the elements of J, i. e. $K_n = \bigvee_{i \in J} S_i^n$, $S_i^n = S^n$. We have $\pi_i(K_n) = 0$ for $i<n$. Now $\pi_n(K_n)$ is a free Abelian group (if $n>1$ and a free group, if $n=1$) with generating system J. Let $\{\Sigma k_{ij}\alpha_i = 0\}_{j \in J}$ be a generating set of relations in Π (for $n>1$ we may presume that the elements α_i commute, so not to take into consideration relations of the form $\alpha_i + \alpha_j = \alpha_j + \alpha_i$; in the case $n=1$, J must be a complete set of generating relations). Let us denote by η_j the spheroid $S^n \to K_n$ equal to $\Sigma k_{ij} S_i^n$ (the notation applied here is not quite exact: S_i^n is taken as a spheroid in K_n). We attach to K_n

an $(n+1)$-dimensional ball along each mapping $\eta_i\colon S^n \to K_n$. By the attaching cell theorem, for the space K_{n+1} obtained, we have $\pi_i(K_{n+1})=0$ if $i<n$ and $\pi_n(K_n)=\Pi$ if $i=n$. Next we attach an $(n+2)$-dimensional ball to K_{n+1} along each $(n+1)$-dimensional spheroids representing any set of generators of $\pi_{n+1}(K_{n+1})$. We obtain a space K_{n+2} with $\pi_i(K_{n+2}) = 0$ for $i<n$, $i = n+1$ and $\pi_n(K_{n+2}) = \Pi$. Next we kill $\pi_{n+2}(K_{n+2})$ with balls of dimension $n+3$, etc. The limit space K will have the prescribed homotopy groups in all dimensions.

A space K with

$$\pi_i(K) = \begin{cases} 0 & \text{for} \quad i \neq n, \\ \Pi & \text{for} \quad i = n \end{cases}$$

is called an *Eilenberg–MacLane space* (or, if it is a complex, an Eilenberg–MacLane complex) or a *space* (complex) *of the type* $K(\Pi, n)$, or simply a $K(\Pi, n)$.

Exercise. Any two spaces of the type $K(\Pi, n)$ are weakly homotopy equivalent.

Comment. This statement will be proved and frequently referred to in Chapter II. It is the exception of the general rule formulated above: the weak homotopy equivalence of spaces follows from mere coincidence of the homotopy groups.

★*Exercise.* $K(\Pi', n) \times K(\Pi'', n) = K(\Pi' + \Pi'', n)$.

★*Exercise.* $\Omega K(\Pi, n) = K(\Pi, n-1)$.

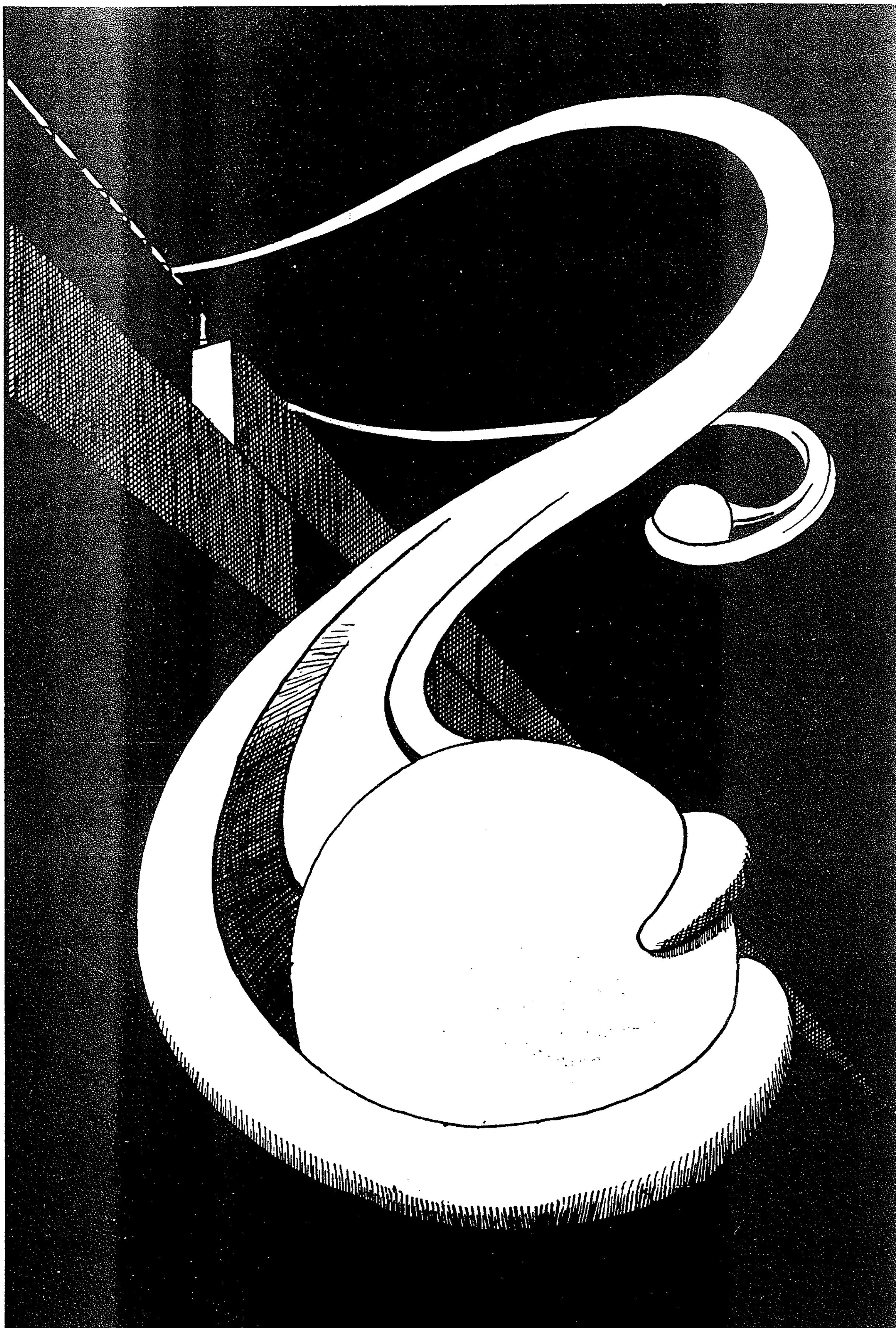
★*Exercise.* The circle is a space of the type $K(\mathbf{Z}, 1)$. The real infinite-dimensional projective space is a $K(\mathbf{Z}_2, 1)$ space. The complex infinite-dimensional projective space is a $K(\mathbf{Z}, 2)$ space. The lense space $L_m^\infty = S^\infty/\mathbf{Z}_m$ is of the type $K(\mathbf{Z}_m, 1)$. Here S^∞ is the set of infinite rows $(z_1, z_2, \ldots)$ with $\Sigma |z_i|^2 = 1$ where all but finitely many elements are equal to zero, and the generator of $\mathbf{Z}_m$ acts on S^∞ by the formula $(z_1, z_2, \ldots) \mapsto$ $\mapsto (z_1 e^{\frac{2\pi i}{m}}, z_2 e^{\frac{2\pi i}{m}}, \ldots)$.

Comment. The list of all "good" $K(\Pi, n)$ spaces with Abelian Π in fact exhausted by the examples of the previous exercise and their products.

★ *Exercise.* Any one-dimensional CW complex is a space of type $K(\Pi, 1)$, where Π is a free group.

Exercise. (V. I. Arnold). The set of all points $(z_1, \ldots, z_n) \in \mathbf{C}^n$ with distinct complex numbers $z_1, \ldots, z_n$ is a $K(\Pi, 1)$ space with some group Π.

Exercise. The supplement of a piecewise-smooth curve in S^3 is of the type $K(\Pi, 1)$ with some Π.



CHAPTER II

HOMOLOGY

After the homotopy groups now we turn to another type of series of groups associated with topological spaces—to homology and cohomology groups. In comparison with the homotopy groups they have some significant disadvantage—their correct definition involves a certain amount of formal difficulties—as well as significant advantages—they can be computed more easily (indeed they will immediately be found at once for the basic examples of spaces) and have more transparent geometric contents (in the case of homology groups we shall not be affronted by incomprehensible facts like $\pi_3(S^2) = \mathbf{Z}$). The information carried by the homology groups about a simply-connected space X, is approximately the same as the one contained in the homotopies.

The geometric idea of homology is the following. Spheroids are substituted by cycles. A k-dimensional cycle is a k-dimensional oriented surface (it may be either a sphere or something else; a torus, for example). The relation of homotopy is substituted by that of homology—a k-dimensional cycle is null homological if it forms the boundary of a piece of a $(k+1)$-dimensional surface.

How can we accurately define the notion of a cycle (and those "pieces", called chains, which can be bordered by these cycles)? We might try to define them as mappings of certain standard objects (spheres and still something else) but this would turn out to be very difficult. (Still in dimensions 1 and 2 it would do, but how to gon?) Actually it is easier to define cycles, as well as chains, as unions of standard elements ("bricks"). For this, we introduce the notion of singular simplexes.

§11. SINGULAR HOMOLOGY

Singular simplexes

We recall that the q-dimensional standard simplex Δ^q is the set of all points $(t_0,\ldots,t_q)\in E^{q+1}$ such that $t_0\geq 0,\ldots,t_q\geq 0$ and $t_0+t_1+\ldots+t_q=1$. Evidently, Δ^0 is a point, Δ^1 is a segment, Δ^2 is a triangle, Δ^3 is a tetrahedron. The q-simplex Δ^q has $q+1$ $(q-1)$-faces $\Delta_0^{q-1},\ldots,\Delta_q^{q-1}$; Δ_i^{q-1} is defined by the equation $t_i=0$.

Let X be a topological space.

A q-dimensional singular simplex of X is a continuous mapping of Δ^q into X.

Chains

A q-dimensional chain of the space X is by definition a (finite) formal linear combination with integral coefficients of singular simplexes of X.

The set of q-dimensional singular chains of X will be denoted by $C_q(X)$. The addition of chains as linear combinations makes $C_q(X)$ an Abelian group. Clearly $C_q(X)$ is a free Abelian group whose generators are the singular simplexes.

The boundary homomorphism

Next we define the homomorphism $\partial_q = C_q(X) \to C_{q-1}(X)$. Since $C_q(X)$ is free, it suffices to give ∂_q on every singular simplex f^q.

Put $\partial_q(f^q) = \sum_{i=0}^{q} (-1)^i f_i^{q-1}$, where $f_i^{q-1} = f^q|_{\Delta^{q-1}}$ is the restriction to the *i-th face* Δ_i^{q-1} of the standard simplex Δ^q. Δ_i^{q-1} is standardly identified with Δ^{q-1} so that $(t_0,\ldots,t_{i-1}, 0\ t_{i+1},\ldots,t_q) \in \Delta_i^{q-1} \subset \Delta^q$ corresponds to $(t_0,\ldots,t_{i-1}, t_{i+1},\ldots,t_q) \in^{q-1}$. As it can easily be seen, $\partial_q \circ \partial_{q+1} = 0$, i. e. Ker $\partial_{q+1} \subseteq$ Im ∂_q.

Homology

The group $H_q(X) = \operatorname{Ker} \partial_q / \operatorname{Im} \partial_{q+1}$ is called the q-dimensional homology group of X. The chains belonging to $\operatorname{Im} \partial_{q+1} \subset C_q(X)$ are called q-dimensional *boundaries.* The subgroup $\operatorname{Im} \partial_{q+1}$ of $C_q(X)$ is the *group of q-dimensional boundaries.* We shall denote it by $B_q(X)$.

Chains of $C_q(X)$ belonging to the subgroup $\operatorname{Ker} \partial_q$ will be called q-dimensional *cycles.* The subgroup $\operatorname{Ker} \partial_q$ of $C_q(X)$ is the *group of q-dimensional cycles.* We shall denote it by $Z_q(X)$. Thus $H_q(X) = Z_q(X)/B_q(X)$. A cycle is said to be null homological: $z_q \sim 0$, if $z_q \in B_q(X)$, i. e. if there exists a chain C_{q+1} such that $\partial C_{q+1} = Z_q$. Similarly, cycles z_q^1 and z_q^2 are said to be homological: $z_q^1 \sim z_q^2$ if the cycle $z_q^1 - z_q^2$ is null homological.

If $H_q(X)$ is finitely generated, it is well known to be of the following form: $H_q(X) = \mathbf{Z} \oplus \ldots \oplus \mathbf{Z} \oplus (\oplus \mathbf{Z}_{k_j})$ where $\mathbf{Z}_{k_j}$ is a cyclic group of order k_j, and k_j is divisible by k_m if $j < m$.

The number of terms $\mathbf{Z}$ in the decomposition of $H_q(X)$ is the q-dimensional Betti number of X; $k_1, k_2, \ldots$ are the q-dimensional torsion numbers of X.

Chain complexes

A *chain complex* is a sequence of Abelian groups and homomorphisms

$$\ldots \to C_k \xrightarrow{\partial_k} C_{k-1} \to \ldots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} \mathbf{Z}$$

such that $\partial_q \circ \partial_{q+1} = 0$, $\varepsilon \circ \partial_1 = 0$ and ε is an epimorphism. Clearly $\operatorname{Im}\partial_{k-1} \subset \operatorname{Ker}\delta_k$. We call $\operatorname{Ker}\partial_k/\operatorname{Im}\partial_{k-1}$ the q-th (or q-dimensional) homology group of the chain complex. We see also that ε defines an epimorphism of the null-dimensional homology group onto **Z**.

If X is an arbitrary space, $C_q(X)$ and the boundary homomorphism ∂_q, together with ε to be defined below, form a chain complex. It is called the *singular complex* of X and is denoted by $C(X)$. The homomorphism ε is defined in the following way: consider a null-dimensional chain $c_0 = \Sigma k_i f_i^0$. We put $\varepsilon(c_0) = \varepsilon(\Sigma k_i f_i^0) = \Sigma k_i \in \mathbf{Z}$. (The sum Σk_i is called the *index* of the null-dimensional chain.)

Exercise. Verify that $\varepsilon \circ \delta_1 = 0$. Moreover, if X is path-connected then $\operatorname{Ker}\varepsilon = \operatorname{Im}\partial_1$.

Chain mappings

Let us have two chain complexes C' and C''. We define a chain mapping of C' into C'' as a family of homomorphisms φ_k: $C'_k \to C''_k$ such that the diagram

$$\begin{array}{ccccccccc}
\to C'_2 & \xrightarrow{\partial'_2} & C'_1 & \xrightarrow{\partial'_1} & C'_0 & \xrightarrow{\varepsilon'} & \mathbf{Z} & \to & 0 \\
\downarrow \varphi_2 & & \downarrow \varphi_1 & & \downarrow \varphi_0 & & \downarrow \mathrm{id} & & \\
\to C''_2 & \xrightarrow{\partial''_2} & C''_1 & \xrightarrow{\partial''_1} & C'_0 & \xrightarrow{\varepsilon''} & \mathbf{Z} & \to & 0
\end{array}$$

is commutative, i. e. $\varphi_{k-1} \circ \partial'_k = \partial''_k \circ \varphi_k$ for every k and $\varepsilon'' \circ \varphi_0 = \varepsilon'$. Any chain mapping of C' into C'' induces a mapping of the corresponding homology groups.

Let X, Y be topological spaces and $g: X \to Y$ a continuous mapping. Then g induces (by means of composition) a family of homomorphisms $g_k: C_k(X) \to C_k(Y)$ and $\{g_k\}$ is a chain mapping of $C' = C(X)$ into $C''_0 = C(Y)$. By the above, g: $X \to Y$ induces mappings $g_* = g_{k*}: H_k(X) \to H_k(Y)$ between the homology groups of X and Y.

Note that

(a) if g: $X_1 \to X_2$ and h: $X_2 \to X_3$ then $(h \circ g)_* = h_* \circ g_*$;

(b) $(\mathrm{id}\, X)_{k*}$ $(e_x)_{k*}$: $H_k(X) \to H_k(X)$ is the identity for any k.

This immediately implies that homology groups are topological invariants, i. e. if the spaces X and Y are homeomorphic, then their homology groups are isomorphic.

The reader who is already acquainted with other homology theories will notice that in the singular theory, as presented here, the theorem on topological invariance is a direct consequence of the definition, while in some other theories it is the result of long and rather complicated investigations.

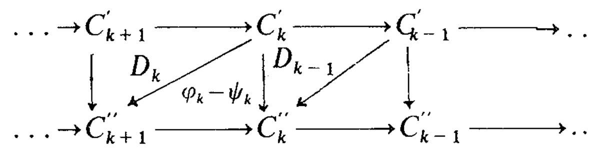
Chain homotopy

Assume again that C' and C'' are chain complexes, and let $\varphi=\{\varphi_k\}$: $C'\to C''$ and $\psi=\{\psi_k\}=C'\to C''$ be chain mappings.

A *chain homotopy* between φ and ψ is a family $\{D_k\}$ of homomorphisms $D_k: C'_k\to C''_{k+1}$ such that for any k,

$$D_{k-1}\circ\partial'_k+\partial''_{k+1}\circ D_k = \varphi_k-\psi_k,$$

(i. e. the diagram



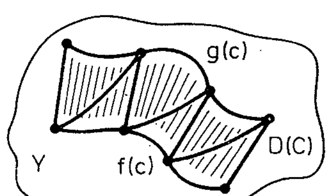
has a certain specific commutativity property).

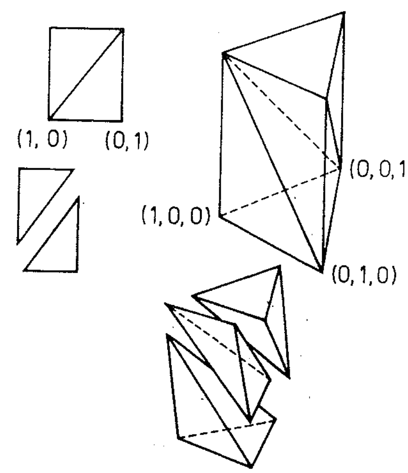
Chain mappings that can be connected with chain homotopy are said to be *homotopic.*

Homotopic chain mappings induce identical mappings of the homology groups: if $\alpha\in C'_k$ and $\partial'_k\alpha=0$, then $\varphi_k(\alpha)-\psi_k(\alpha)=D_{k-1}(\partial'_k\alpha)-\partial''_{k+1}(D_k\alpha)=-\partial''_{k+1}(D_k\alpha)\in Im\partial''_{k+1}$.

Let us explain where the name "chain homotopy" comes from. If f, $g: X\to Y$ are homotopic mappings between spaces, then the chain mappings induced by f and g are homotopic.

Indeed, let us fix a homotopy $F: X\times I\to Y$ connecting f with g. For any singular simplex $\varphi: \Delta^q\to X$, the mapping $F\circ(\varphi\times I): \Delta^q\times I\to Y$ is defined. The cylinder $\Delta^q\times I$





can be divided into simplexes Δ_i^{q+1} $(i=0, 1, \ldots, q)$. (Such a division is shown on the picture in the cases $q=2$, 3. In general it can be defined by $\Delta_i^{q+1}=\{(t_0, \ldots, t_q, \tau): t_0+ +\ldots+t_{i-1}\leq\tau\leq t_0+\ldots+t_i\}$). Evidently Δ_i^{q+1} is the simplex with the vertices $(v_0, 0)$, $\ldots, (v_i, 0), (v_i, 1), \ldots, (v_{q+1}, 1)$ where $(v_0, \ldots, v_{q+1})$ are the vertices of Δ^q. We identify Δ_i^{q+1} with the standard simplex Δ^{q+1} by means of an arbitrary orientation-preserving homeomorphism. Thus the mapping $F\circ(\varphi\times I)$ defines $q+1$ $(q+1)$-dimensional singular simplexes $\psi_0, \ldots, \psi_q$. We denote the sum $\Sigma\psi_i$ by $D_q(\varphi)$. Let $D_q(\Sigma k_i\varphi_i)= = \Sigma k_i D_q(\varphi_i)$. It is not difficult to verify that $\{D_q\}$ is a chain homotopy connecting f and g.

We conclude that homotopic mappings of spaces induce identical mappings of homology groups.

Corollary. Homotopy equivalent spaces have isomorphic homology groups. Moreover, homotopy equivalences induce homology isomorphisms.

Homology of a point

Suppose $X = *$. The only singular simplexes are those with the form $f^r = \Delta^r \to *$. Hence $C_r(*) = \mathbf{Z}$. Now $\partial f^r = \Sigma(-1)^i f_i^{r-1} = [\Sigma(-1)^i] f^{r-1}$,

$$\partial f^r = \begin{cases} 0 & \text{for } r = 2p+1 \text{ and } r=0; \\ f^{r-1} & \text{for } r = 2p. \end{cases}$$

Thus $H_0(*) = \mathbf{Z}$; $H_q(*) = 0$ for $k>0$. (If k is odd, then $B_k(*) = \mathbf{Z}_k(*) = \mathbf{Z}$; if k is even, then $B_k(*) = Z_k(*) = 0$.)

Null-dimensional homology

If X is pathwise connected, then $H_0(X) = \mathbf{Z}$. Moreover, ε: $H_0(X) \to \mathbf{Z}$ is an isomorphism. Indeed, any null-dimensional chain is a cycle, too: $C_0(X) = Z_0(X)$. Let us have an arbitrary null-dimensional chain $c = \Sigma k_i\varphi_i$. By adding to it a chain $d= = \Sigma k_i(\varphi_i - \varphi_0)$ with arbitrary 0-dimensional simplex φ_0 we obtain a chain concentrated to a single point. For X is path-connected, $\varphi_i - \varphi_0$ is a boundary and so is d. Moreover, all null-dimensional simplexes are homological, and so we obtain that $H_0(X) = \mathbf{Z}$. It remains to add that ε is an isomorphism, because it is an epimorphism.

Similarly, if I is the set of pathwise connected components of X, we have $H_0(X) = = \bigoplus_{i\in I} \mathbf{Z}$.

★*Exercise.* Let f: $X \to Y$ be a continuous mapping between pathwise connected spaces. Then f_*: $H_0(X) \to H_0(Y)$ is an isomorphism.

Relative homology

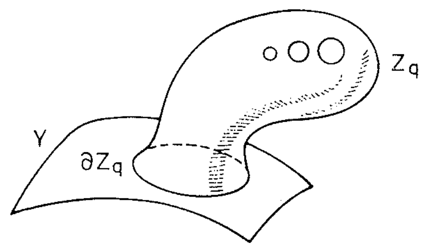
Let X, Y be a pair of topological spaces; $Y \subset X$. Then $C_q(Y) \subset C_q(X)$ and we may consider the quotient group $C_q(X, Y) = C_q(X)/C_q(Y)$.

We have compatible boundary operators ∂_q: $C_q(X) \to C_{q-1}(X)$ and ∂_q: $C_q(Y) \to$ $\to C_{q-1}(Y)$. So we have an operator $C_q(X, Y) \to C_{q-1}(X, Y)$ which will be denoted by the same letter ∂_q. Now $\operatorname{Ker} \partial = Z_q(X, Y) \supset \operatorname{Im} \partial = B_q(X, Y)$ so that we have a group $H_q(X, Y) = Z_q(X, Y)/B_q(X, Y)$ which is called the group of the relative q-dimensional homology classes of X modulo Y. For relative homology groups we also have the functorial property, topological and homotopical invariances.

The operator ∂ in the homology

Now we construct a new operator ∂_* that will map $H_q(X, Y)$ into $H_{q-1}(Y)$.

Let a relative cycle $z_q \in C_q(X, Y)$ be represented by $\tilde{z}_q \in C_q(X)$. From $\partial_q z_q = 0$ it follows that $\partial_q \tilde{z}_q \in C_{q-1}(Y)$. The homology class of the (absolute) cycle $\partial_q \tilde{z}_q$ is clearly independent of the choice of z_q. Thus there arises an operator ∂: $H_q(X, Y) \to H_{q-1}(Y)$.



The homology sequences of pairs and triples

Let i: $Y \subset X$ be the inclusion mapping. It induces i_*: $H_{q-1}(Y) \to H_{q-1}(X)$. Since every absolute cycle can be regarded as a relative one, we also have a mapping π: $H_{q-1}(X) \to H_{q-1}(X, Y)$.

Theorem. The following sequence is exact:

$$\ldots \to H_q(X, Y) \xrightarrow{\partial} H_{q-1}(Y) \xrightarrow{i_*} H_{q-1}(X) \xrightarrow{\pi} H_{q-1}(X, Y) \to \ldots$$

The proof reduces to the trivial verification of the fact that the corresponding kernels and images coincide. By the way we notice that for connected X and Y we have $H_0(X, Y) = 0$. In general, if every component of X contains a point of Y, then $H_0(X, Y) = 0$.

★*Exercise.* Prove that for any point x_0 of X we have $H_q(X) = H_q(X, x_0)$ for $q > 0$.

The *homology sequence of a triple* is a version of the exact sequence of a pair.

It is defined for a triple (X, Y, Z), $X \supset Y \supset Z$, and has the form

$$\ldots \to H_q(X, Y) \xrightarrow{\partial} H_{q-1}(Y, Z) \to H_{q-1}(X, Z) \to H_{q-1}(X, Y) \to \ldots$$

where ∂ is the composition of the former ∂ and the mapping $H_{q-1}(Y) \to H_{q-1}(Y, Z)$ similar to the former π, and the rest homomorphisms are induced by the inclusions of the pairs.

Exercise. Verify the exactness of the sequence obtained.

The connection between absolute and relative homology

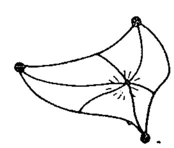
As it turns out, relative homology groups in a certain sense reduce to absolute ones. Namely, for any CW pair we have $H_q(X, Y) = H_q(X/Y)$ for $q \neq 0$.

The analogous formula is not valid in the case of homotopy groups. For example, $\pi_q(D^2/S^1) \neq \pi_q(D^2, S^1)$.

Denote by CY the cone over Y. It is obtained from $Y \times I$ by the upper face being contracted into a single point. Let us consider $X \cup CY$. If X and Y are CW complexes then $X \cup CY \approx X/Y$ so that the statement to be proved is $H_q(X, Y) = H_q(X \cup CY)$ for $q \neq 0$.

It suffices to prove $H_q(X, Y) = H_q(X \cup CY, *)$ for any complex X and subcomplex Y, where $*$ is the vertex of the cone. We recall that $H_q(X \cup CY, *) = \mathrm{H}_q(X \cup CY)$ for $q > 0$.

Let $\varphi\colon \Delta^q \to V$ be a singular simplex of the space V. Let us denote by $\beta\varphi$ the chain defined as the sum of the singular simplexes obtained by restricting φ to the q-dimensional simplexes of the barycentric subdivision of Δ^q. For any q-dimensional chain $c = \Sigma k_i \varphi_i \in C_q(X)$ we put $\beta c = \Sigma k_i \beta \varphi_i$. The correspondence $c \mapsto \beta c$ clearly defines a homomorphism $\beta_q\colon C_q(X) \to C_q(X)$.

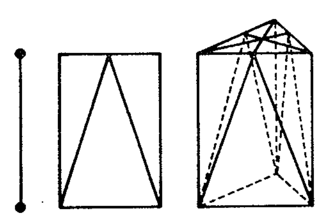


Lemma. The family $\{\beta_q\colon C_q(X) \to C_q(X)\}_q$ is a chain mapping of the complex $\{C_q(X)\}_q$ into itself. It is homotopic to the identity mapping.

The first statement is almost obvious, we leave the proof to the reader. The chain homotopy connecting $\{\beta_q\}$ with the identity is constructed as follows. For any $q \geq 0$ we define a triangulation of $\Delta^q \times I$ such that

(i) the base of the cylinder $\Delta^q \times I$ is triangulated as the standard simplex;

(ii) the product $\Delta_i^{q-1} \times I \supset \Delta^q \times I$, where Δ_i^{q-1} is a face, is a simplicial subcomplex of the complex $\Delta^q \times I$ which is triangulated as $\Delta^{q-1} \times I$;



q=0 q=1 q=2

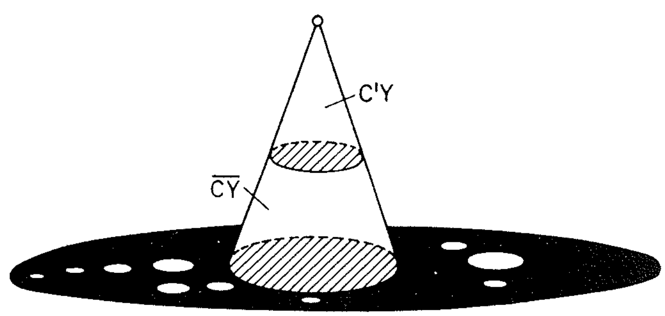
(iii) the upper face of the cylinder is triangulated as a barycentric subdivision of the standard simplex.

Such triangulation can be constructed by induction. For $q=0$ it is defined as shown on the last picture.
Assume that it is already defined for $q<k$. We triangulate $(\Delta^q \times 0) \cup (\partial\Delta^q \times I) \subset \Delta^q \times I$ in such a way that it is the standard triangulation of $\Delta^q = \Delta^q \times 0$ and coincides with the constructed one on $\partial\Delta^q \times I$. Then we divide $\Delta^q \times I$ into pyramides whose bases are the simplexes of the triangulation $(\Delta^q \times 0) \cup (\partial\Delta^q \times I)$ and whose common vertex is the centre of the upper face.

Let φ: $\Delta^q \to X$ be a singular simplex. We denote by $D(\varphi) \in C_{q+1}(X)$ the sum of all $(q+1)$-dimensional simplexes obtained by restricting Φ: $\Delta^q \times I \to X$ (where $\Phi(y, t) = \varphi(y)$) to the simplexes of the above triangulation of $\Delta^q \times I$. Then D: $C_q(X) \to C_{q+1}(X)$ is the homotopy which satisfies the lemma. (Verify this!)

Proof of the theorem. Let us consider the imbedding $(X, Y) \to (X \cup CY, CY)$. It induces a mapping $H_q(X, Y) \to H_q(X \cup CY, CY) = H_q(X \cup CY, *)$, since the cone CY is contractible to its vertex: $CY \approx *$.

We show that it is an epimorphism. Let $z \in Z_q(X \cup CY, CY)$ be a cycle. We have to find a cycle in $Z_q(X, Y)$ whose image is homologous with z. Let us cut CY into two pieces at the height $t = 1/2$. We obtain a cone $C'Y$ and a truncated cone $\overline{CY}$.



By the lemma, z may be substituted by a homologous cycle z' with simplexes so small that anyone intersecting $C'Y$, necessarily belongs to CY. Let us throw out from Z' the simplexes intersecting $C'Y$. This operation remains within CY, so we do not change the homology class of z mod CY. We get a relative cycle z'' mod CY. On the other hand, we have $H_q(X \cup \overline{CY}, \overline{CY}) = H_q(X, Y)$ by the homotopy invariance of homology groups. Thus there exists a relative cycle in X mod Y which is carried by the isomorphism into the relative cycle z'' in $X \cup \overline{CY}$ mod $\overline{CY}$. Thus $H_q(X, Y) \to H_q(X \cup CY, \text{CY})$ is an epimorphism, moreover it can be shown by a similar construction that it is a monomorphism too. We leave this to the reader.

§12. COMPUTATION OF THE HOMOLOGY GROUPS OF CW COMPLEXES

The homology groups of the 0-dimensional sphere, i. e. a pair of points, is already known:

$$H_i(S^0) = \begin{cases} \mathbf{Z} \oplus \mathbf{Z} & \text{for } i=0; \\ 0 & \text{for } i>0. \end{cases}$$

We show that for $n>0$

$$H_i(S^n) = \begin{cases} \mathbf{Z} & \text{for } i=0 \text{ and } i=n, \\ 0 & \text{for } i \neq 0, n. \end{cases}$$

Consider the homology sequence of the pair (B^1, S^0):

$$H_1(B^1) \to H_1(B^1, S^0) \to H_0(S^0) \to H_0(B^1) \to 0.$$

Since B^1 is homotopy equivalent to the single point space, we have $H_1(B^1)=0$, $H_0(B^1)=\mathbf{Z}$. Now $B^1/S^0=S^1$ implies $H_1(B^1, S^0)=H_1(S^1)$. We may write the sequence in the form $0 \to H_1(S^1) \to \mathbf{Z} \oplus \mathbf{Z} \to \mathbf{Z} \to 0$, implying $H_1(S^1)=\mathbf{Z}$ by the exactness. If $q>1$, then $H_q(B^1)$ and $H_{q-1}(S^0)$ are trivial in

$$H_q(B^1) \to H_q(B^1, S^0) \to H_{q-1}(S^0),$$

and $H_q(B^1, S^0)=0$. Hence $H_q(S^1)=0$ for $q>1$.

Assume now the statement to be valid for spheres of dimensions less than n. Consider the exact sequence of the pair (B^n, S^{n-1}):

$$H_q(B^n) \to H_q(B^n, S^{n-1}) \to H_{q-1}(S^{n-1}) \to H_{q-1}(B^n).$$

We make use of the formulas $H_q(B^n)=0$ for $q>0$ and $H_q(B^n, S^{n-1})=H_q(S^n)$ for $q>0$.

If $q>1$, we have

$$0 \to H_q(S^n) \to H_{q-1}(S^{n-1}) \to 0,$$

i. e. $H_q(S^n)=H^{n-1})$. For $q=1$ we get the exact sequence

$$H_1(B^n) \to H_1(B^n, S^{n-1}) \to H_0(S^{n-1}) \to H_0(B^n);$$

we already know $H_1(B^n)=0$, $H_0(S^{n-1})=H_0(B^n)=\mathbf{Z}$ and the last arrow is an isomorphism. Hence $H_1(S^n) = H_1(B^n, S^{n-1}) = 0$ and the statement is proved.

Remark. For the generator of $H_n(B^n, S^{n-1})$ one may choose the homology class of the singular chain $1 \cdot \varphi$, where $\varphi : \Delta^n \to B^n$ is a homeomorphism. The two possible selections of the generator in $H_n(B^n, S^{n-1}) = \mathbf{Z}$ clearly correspond to the two orientations of the ball B^n.

It is similarly easy to describe the generator of $H_n(S^n)$ (which we leave to the reader). Fixing the generator in $H_n(S^n) = \mathbf{Z}$ is equivalent to fixing the orientation of S^n.

Homology of the union of spheres

Let us be given the union of a family of n-dimensional spheres:

$$X = \vee_{i \in I} S_i^n$$

where I is a (finite or infinite) set of indexes.

If $n>0$ and $q>0$, there is a canonical isomorphism

$$H_q(X) = \oplus_{i \in I} H_q(S_i^n).$$

Thus $H_n(X)$ is a free Abelian group, i. e. a direct sum of groups which are in one-to-one correspondence with the spheres and are all isomorphic to **Z**. Moreover, if every sphere is oriented, the group has a system of generators whose elements are in a canonical one-to-one correspondence with the spheres in the union.

The easiest proof of this statement is made by induction, by applying the relation

$$(\vee_{i \in I} B_i^n)/(\vee_{i \in I} \partial B_i^n) = \vee_{i \in I} S_i^n.$$

Homology of a CW complex

Let K be a CW complex and Σ_r be the set of its r-dimensional cells. The orientation of each cell is assumed to be fixed.

By the above we have

$$H_q(K^r, K^{r-1}) = H_q(\vee_{i \in \Sigma_r} S_i^r) = \begin{cases} 0 & \text{for} \quad i \neq r, \\ \mathscr{C}_r(K) & \text{for} \quad i = r, \end{cases}$$

where $\mathscr{C}_r(K)$ is the free Abelian group whose generators are in one-to-one correspondence with Σ_r. The elements of this group may be identified with finite linear combinations $\Sigma k_i \sigma_i^r$, where σ_i^r are r-dimensional cells.

Because $\mathscr{C}_r(K) = H_r(K^r, K^{r-1})$ and $\mathscr{C}_{r-1}(K) = H_{r-1}(K^{r-1}, K^{r-2})$, we have a homomorphism $\partial_r: \mathscr{C}_r(K) \to \mathscr{C}_{r-1}(K)$ that comes from the exact sequence of the triple (K^r, K^{r-1}, K^{r-2}).

There arises then a chain complex

$$\ldots \to \mathscr{C}_r(K) \xrightarrow{\partial_r} \mathscr{C}_{r-1}(K) \to \ldots.$$

The next goal is to establish a canonical isomorphism between the homology groups of this complex and of the space.

The existence of such an isomorphism will prove to be the main tool in computing homology groups of specific spaces. One important corollary is already obvious: Any finite CW complex has finitely generated homology groups.

Lemma. $H_j(K) = H_j(K^{j+1}, K^{j-2})$.

Proof. Consider the exact sequence of the triple $(K^{j+1}, K^{j-2}, K^{j-3})$: $H_j(K^{j-2}, K^{j-3}) \to H_j(K^{j+1}, K^{j-3}) \to H_j(K^{j+1}, K^{j-2}) \to H_{j-1}(K^{j-2}, K^{j-3})$. The first and fourth terms are zero, so $H_j(K^{j+1}, K^{j-2}) = H_j(K^{j+1}, K^{j-3})$. By applying the same observation to the exact sequence of the triple $(K^{j+1}, K^{j-3}, K^{j-4})$ we get $H_j(K^{j+1}, K^{j-3}) = H_j(K^{j+1}, K^{j-4})$. Similarly $H_j(K^{j+1}, K^{j-3}) = H_j(K^{j+1}, K^{j-4}) = \ldots = H_j(K^{j+1}, K^0) = H_j(K^{j+1})$, i. e. $H_j(K^{j+1}) = H_j(K^{j+1}, K^{j-2})$.

We still have to prove that $H_j(K) = H_j(K^{j+1})$.

We prove that for any $q < j+1$ there is an isomorphism $H_q(K^{j+1}) = H_q(K^{j+2})$.

As seen from the exact sequence of the pair (K^{j+2}, K^{j+1}):

$$0 = H_{j+1}(K^{j+2}, K^{j+1}) \to H_j(K^{j+1}) \to H_j(K^{j+2}) \to H_j(K^{j+2}, K^{j+1}) = 0$$

we have $H_j(K^{j+1}) = H_j(K^{j+2})$. Similarly $H_j(K^{j+2}) = H_j(K^{j+3}) = \ldots = H_j(K)$. (If K is infinite, the equality $H_j(K) = H_j(K^N)$ for sufficiently large N follows from the compactness of Δ^j and axiom (W) in the definition of the CW complex. The details are left to the reader.)

Remark. In the proof of the lemma we have established a canonical isomorphism $H_j(K^{j+1}, K^{j-2}) = H_j(K)$.

Next we establish the isomorphism $\operatorname{Ker} \partial_j / \operatorname{Im} \partial_{j+1} = H_j(K^{j+1}, K^{j-2})$.

Consider the commutative diagram

$$\begin{array}{ccccccc}
 & & H_j(K^{j-1}, K^{j-2}) = 0 & & & & \\
 & & \downarrow & & & & \\
H_{j+1}(K^{j+1}, K^j) & \xrightarrow{\partial} & H_j(K^j, K^{j-2}) & \xrightarrow{\alpha_*} & H_j(K^{j+1}, K^{j-2}) & \to & H_j(K^{j+2}, K_j) = 0 \\
 & \searrow^{\partial_{j+1}} & \downarrow \beta_* & & & & \\
 & & H_j(K^j, K^{j-1}) & & & & \\
 & & \downarrow \partial_j & & & & \\
 & & H_{j-1}(K^{j-1}, K^{j-2}) & & & &
\end{array}$$

where the row is a segment of the exact sequence of the triple (K^{j+1}, K^j, K^{j-2}), the column is taken from the exact sequence of (K^j, K^{j-1}, K^{j-2}) and α, β are the respective inclusion mappings of pairs.

By definition, $H_j(K^j, K^{j-1}) = \mathscr{C}_j(K)$ and $H_{j-1}(K^{j-1}, K^{j-2}) = \mathscr{C}_{j-1}(K)$. Since $H_j(K^{j-1}, K^{j-2})$ and $H_j(K^{j+1}, K^j)$ are trivial, by the exactness of the sequences we have that β_* is a monomorphism and α_* is an epimorphism. Hence $H_j(K^{j+1}, K^{j-2}) = H_j(K^j, K^{j-2}) / \operatorname{Ker} \alpha_* = H_j(K^j, K^{j-2}) / \operatorname{Im} \partial$. As β_* is a monomorphism, $H_j(K^j, K^{j-2}) / \operatorname{Im} \partial = \beta_* H_j(K^j, K^{j-2}) / \beta_*(\operatorname{Im} \partial) = \operatorname{Im} \beta_* / \operatorname{Im}(j_* \circ \partial)$. By the commutativity of the diagram $\beta_* \circ \partial = \partial_{j+1}$, and $\operatorname{Im} \beta_* = \operatorname{Ker} \partial_j$ by the exactness, i. e. the last quotient group is equal to $\operatorname{Ker} \partial_j / \operatorname{Im} \partial_{j+1}$.

Then $H_j(K) = H_j(K^{j+1}, K^{j-2}) = \operatorname{Ker} \partial_j / \operatorname{Im} \partial_{j+1}$, which proves the theorem for $j \geq 2$. If $j = 1$, it is necessary to consider the diagram

$$\begin{array}{ccccccc}
 & & H_1(K^0)=0 & & & & \\
 & & \downarrow & & & & \\
H_2(K^2,K^1) & \xrightarrow{\partial} & H_1(K^1) & \to & H_1(K^2) & \to & H_1(K^2,K^1)=0 \\
 & \searrow^{\partial_2} & \downarrow j_* & & & & \\
 & & H_1(K^1,K^0) & & & & \\
 & & \partial_1 \downarrow & & & & \\
 & & H_0(K^0) & & & &
\end{array}$$

For $j=0$ we have $H_1(K^1, K^0) \xrightarrow{\partial_1} H_0(K^0) \xrightarrow{i_*} H_0(K^1) \to H_0(K^1, K^0) = 0$.

The geometric meaning of the operator ∂_q

Consider two cells σ^q and σ^{q-1} with fixed orientation and characteristic mappings $f: B^q \to X$, $g: B^{q-1} \to X$, which are compatible with the orientation.

Consider the composite $\partial B^q = S^{q-1} \xrightarrow{f|_{S^{q-1}}} X^{q-1}/X^{q-2}$. Here X^{q-1}/X^{q-2} is a union of $(q-1)$-dimensional spheres. The cell σ^{q-1} belongs to the $(q-1)$-skeleton and is projected by the factorization $X^{q-1} \to X^{q-1}/X^{q-2}$ onto a sphere S^{q-1}. Let the whole union be projected onto this sphere so that the other spheres be mapped onto the base point.

The composite mapping $\partial B^q = S^{q-1} \to X^{q-1}/X^{q-2} \to S^{q-1}$ gives a mapping $S^{q-1} \to \to S^{q-1}$ which represents some element of $\pi_{q-1}(S^{q-1}) = \mathbf{Z}$, i. e. an integer, the so-called *degree* of the mapping.

So we can assign an integer, called the *incidence coefficient* $[\sigma^q : \sigma^{q-1}]$ to every pair of cells. It does depend on the orientation of σ^q and σ^{q-1} but is independent of the particular choice of the characteristic mappings f, g, as it can easily be verified by the reader.

Theorem. $\partial \sigma^q = \Sigma [\sigma^q \sigma^{q-1}] \sigma^{q-1}$, where the sum is taken over all $(q-1)$-dimensional cells of the complex K.

Thus the boundary operator, despite the purely algebraic definition, has an obvious geometric meaning too.

The sum in the theorem is finite in virtue of axiom (C) in the definition of CW complexes, since $[\sigma^q : \sigma^{q-1}] \neq 0$ only for those σ^{q-1} which meet the closure of σ^q and their number is finite.

Proof. Consider the mapping of triples $(B^q, S^{q-1}, \emptyset) \to (K^q, K^{q-1}, K^{q-2})$ where $B^q \to K^q$ is the characteristic mapping of σ^q and $S^{q-1} \to K^{q-1}$ is its restriction. Because of the functorial property of exact homology sequences, we obtain a commutative diagram:

$$\begin{array}{ccccc}
 & & \mathbf{Z} & & \\
 & & \| & & \\
0 = H_q(B^q) & \rightarrow & H_q(B^q, S^{q-1}) & \xrightarrow{\partial} & H_{q-1}(S^{q-1}) = \mathbf{Z} \rightarrow 0 \\
 & & \downarrow \alpha & & \downarrow \\
 & & H_q(K^q, K^{q-1}) & \xrightarrow{\partial} & H_{q-1}(K^{q-1}, K^{q-2}) \\
 & & \| & & \| \\
 & & \mathscr{C}_q(K) & & \mathscr{C}_{q-1}(K)
\end{array}$$

$H_q(B^q, S^{q-1}) = H_q(S^q) = \mathbf{Z}$.

(Here $q > 1$, the case $q = 1$ is quite similar and is left to the reader.)

Consider the generator $1 \in H_q(B^q, S^{q-1}) = \mathbf{Z}$. First it is sent by α into the chain $1 \cdot \sigma^q \in \mathscr{C}_q(K)$ and then by ∂ into $\partial\sigma^q$.

Let us now follow the same generator on the other route. First it is sent into $1 \in \mathbf{Z} = H_{q-1}(S^{q-1})$ as it follows from the definition of the homology sequence.

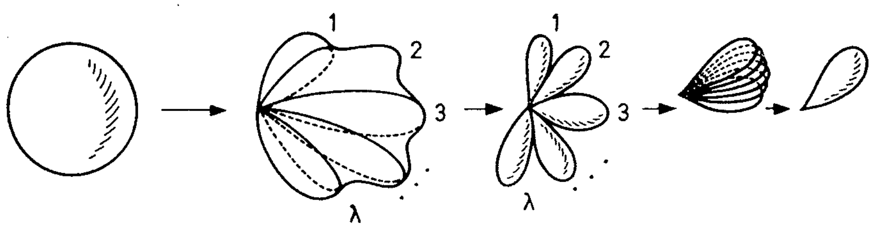
Further we have a mapping $(S^{q-1}, \emptyset) \rightarrow (K^{q-1}, K^{q-2})$ equivalent to $S^{q-1} \rightarrow \vee\, S^{q-1}$, which means $H_{q-1}(S^{q-1}) = \mathbf{Z} \rightarrow H_{q-1}(K^{q-1}, K^{q-2}) = \mathbf{Z} \oplus \mathbf{Z} \oplus \ldots \oplus \mathbf{Z}$.

Each generator of $\mathscr{C}_{q-1}(K)$ corresponds to a cell σ^{q-1}. We have to find their coefficients in the image of $1 \in H_{q-1}(S^{q-1})$.

Lemma. If $f\colon S^q \rightarrow S^q$ is of degree λ, then the endomorphism f_* of $H_q(S^q) = \mathbf{Z}$ sends 1 into λ.

So the coefficients are the very incidence numbers $[\sigma^q : \sigma^{q-1}]$, which means $\partial\sigma^q = \Sigma[\sigma^q : \sigma^{q-1}]\sigma^{q-1}$ as we are to prove.

Proof of the lemma. We recall how a mapping of degree λ is constructed:



The mapping of the sphere into a union of λ spheres takes the generator of $H_q(S^q)$ into the sum of all generators of the q-dimensional homology group of the union, next the mapping of the union into the sphere induces transition of each generator into the generator of $H_q(S^q)$, as claimed by the lemma. Q.e.d.

Computation of homology groups

(1) S^n. The sphere has a decomposition into two cells σ^0 and σ^n. Clearly $\mathscr{C}_0 = \mathscr{C}_n = \mathbf{Z}$ and $\partial \equiv 0$, hence $H_0 = \mathscr{C}_0 = H_n = \mathscr{C}_n = \mathbf{Z}$ (as we already know).

(2) The complex projective space $\mathbf{CP}^n$. The points of $\mathbf{CP}^n$ are sequences $(z_0 : z_1 : \ldots : z_n)$

of complex numbers such that for at least one of z_k does not vanish, considered up to multiplying by nonzero complex numbers. On $\mathbf{CP}_n$ we consider the following cell structure. The cell σ^{2q} (where $2q$ is real dimension), $0 \leq q \leq n$, is defined as the subset of all points of $\mathbf{CP}^n$ for which $z_q \neq 0$, $z_{q+1} = z_{q+2} = \ldots = z_n = 0$. (The characteristic mapping $B^{2q} \to \mathbf{CP}^n$ of the cell σ^{2q} is given by

$$(z_1, \ldots, z_q) \mapsto (z_1 : \ldots : z_q : \sqrt{1-|z_1|^2 - \ldots - |z_q|^2} : 0 : \ldots : 0).$$

We have

$$\mathscr{C}_i(\mathbf{CP}^n) = \begin{cases} 0 & \text{for} \quad i = 2k+1 \text{ or } i > 2n, \\ \mathbf{Z} & \text{for} \quad i = 2k. \end{cases}$$

All boundary operators are zero, so $H_i = \mathscr{C}_i$.

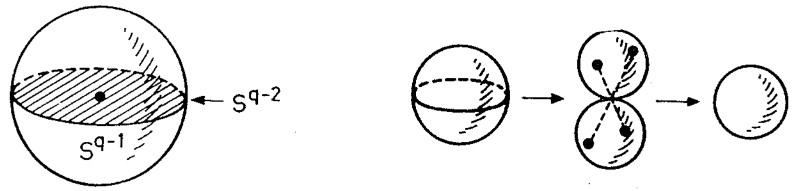
(3) The real projective space $\mathbf{RP}^n$. The points of this space are sequences of real numbers $(x_0 : x_1 : \ldots : x_n)$ such that $x_k \neq 0$ for at least one k, considered up to multiplication by nonzero real multiplier. The cell σ^q consists of all points for which $x_q \neq 0$ and $x_{q+1} = x_{q+2} = \ldots = x_n = 0$.

The difference between this and the previous example is that here we have cells in every dimension (one cell in each dimension) while $\mathbf{CP}^n$ has only even-dimensional cells.

Let us describe the characteristic mappings. Consider B^q and identify on ∂B^q the diametrically opposing points. This gives a mapping $B^q \to \mathbf{RP}^q = \bar{\sigma}^q \subset \mathbf{RP}^n$.

Consequently $\mathscr{C}_i(\mathbf{RP}^n) = \mathbf{Z}$, $0 \leq i \leq n$.

Now ∂ is not trivial anymore. Let us compute the incidence numbers. For this, we have to compute the degrees of the following mappings: $\partial B^q = S^{q-1} \to \mathbf{RP}^{q-1} \to \to \mathbf{RP}^{q-1}/\mathbf{RP}^{q-2} = S^{q-1}$. Here the last equality holds because there is a single cell in each dimension. The result is a composite $S^{q-1} \to S^{q-1}$. Let us compute the homomorphisms between the homology groups. We have $S^{q-1} \to \mathbf{RP}^{q-1}$ where $\mathbf{RP}^{q-1}$



may be represented as the upper hemisphere with the diametrically opposing points of the equator being pairwise identified with one another. That is, the equator is $\mathbf{RP}^{q-1}$. Further factorization contracts the equator into a single point, so that the upper hemisphere becomes S^{q-1}.

The same mapping can also be described as follows. At first the equator is contracted into a single point, then in the union obtained the two spheres are sewn together in such a way that each point is identified with the diametrically opposing one. Central reflection of the sphere preserves the orientation if the dimension of the sphere is odd and changes it if the dimension is even. Thus $S^{q-1} \to S^{q-1}$ is of degree 0 if q is odd and 2 if q is even.

We conclude that the incidence coefficient $[\sigma^q : \sigma^{q-1}]$ is zero if $q = 2k+1$ and 2 if $q = 2k$.

Thus $\partial\sigma^{2k} = 2\sigma^{2k-1}$ and $\partial\sigma^{2k-1} = 0$, hence

$$\begin{aligned} H_0(\mathbf{RP}^n) &= \mathbf{Z} \\ H_1(\mathbf{RP}^n) &= \mathbf{Z}_2 \\ H_2(\mathbf{RP}^n) &= 0 \\ H_3(\mathbf{RP}^n) &= \mathbf{Z}_2 \\ &\cdots\cdots \end{aligned}$$

for $n = 2k$

$$H_{2k-1}(\mathbf{RP}^n) = \mathbf{Z}_2 \qquad H_{2k}(\mathbf{RP}^n) = 0$$

for $n = 2k+1$

$$H_{2k}(\mathbf{RP}^n) = 0 \qquad H_{2k+1}(\mathbf{RP}^n) = \mathbf{Z}$$

Exercise. Calculate the homology groups of (a) the torus, (b) the Klein bottle, (c) the space $\mathbf{RP}^2 \times \mathbf{RP}^2$.

Homology with coefficients in an arbitrary group

We can build up the singular homology theory by defining singular chains as linear combinations of singular simplexes with coefficients in an arbitrary Abelian group G, in full analogy with the case of integral coefficients. We obtain homology groups which we denote by $H_q(X; G)$ and $H_q(X, Y; G)$. We can extend the basic results of the last two sections for this general case. In particular, if K is a CW complex, then $H_q(K; G) = \operatorname{Ker} \partial_q / \partial_{q+1}(\mathscr{C}_{q+1}(K; G))$, where $\mathscr{C}_i(K; G)$ is the group of finite linear combinations of the form $\Sigma g_k \sigma_k$ (where $g_k \in G$ and σ_k^i are i-dimensional cells) and ∂_i is defined by

$$\partial_i(\Sigma_k g_k \sigma_k^i) = \Sigma_{k,l}\, g_k [\sigma_k^i : \sigma_l^{i-1}] \sigma_l^{i-1}$$

(where σ_k^i and σ_l^{i-1} run through the set with i-dimensional and $(i-1)$-dimensional cells respectively).

For instance,

$$H_i(\mathbf{RP}^n; \mathbf{Z}_2) = \begin{cases} \mathbf{Z}_2, & 0 \le i \le n, \\ 0, & i > n. \end{cases}$$

Indeed, $\mathscr{C}_i(\mathbf{RP}^n; \mathbf{Z}_2) = \mathbf{Z}_2$ for $i \le n$, while ∂_i: $\mathscr{C}_i(\mathbf{RP}^n; \mathbf{Z}_2) \to \mathscr{C}_{i-1}(\mathbf{RP}^n; \mathbf{Z}_2)$ is trivial (as $2 \equiv 0 \bmod 2$).

Further on we shall study homology theories with various coefficients in more detail.

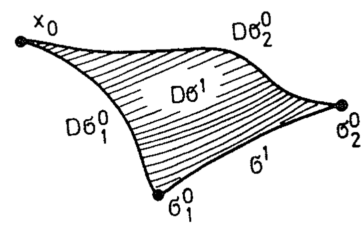
The notation $H_i(X)$ will be kept for $H_i(X; \mathbf{Z})$.

§13. HOMOLOGY AND HOMOTOPY

Theorem. If a space X has trivial homotopy groups (and, in particular, is path-connected), its homology groups are trivial, too, i. e. $H_0(X)=\mathbf{Z}$, $H_q(X)=0$ if $q>0$.

Remark. For CW complexes the statement is trivial: $\pi_i(X)\equiv 0$ for all i implies that X is contractible, thus $H_i(X)\equiv H_i$ (point).

Proof. A 0-dimensional singular simplex of X is in fact a point of X. Let a point $x_0\in X$ be fixed and for each 0-dimensional singular simplex σ^0 let a path $D\sigma^0$ be chosen



that connects σ^0 with x_0. The path $D\sigma^0$ may be regarded as a 1-dimensional singular simplex or a 1-dimensional singular chain. Moreover $\partial(D\sigma^0)=x_0-\sigma^0$. For any 0-dimensional chain $c=\Sigma k_i\sigma_i^0$ we put $Dc=\Sigma k_iD\sigma_i^0$.

If σ^1 is a 1-dimensional singular simplex of X and $\partial\sigma^1=\sigma_1^0-\sigma_2^0$, then the 1-dimensional simplexes $D\sigma_1^0,\sigma^1$ and $D\sigma_2^0$ together constitute a mapping of the boundary of the 2-dimensional simplex Δ^2 into X. Because X is simply connected, the mapping may be extended to a mapping $\Delta^2\to X$, which is a 2-dimensional singular simplex of X. Now for each 1-dimensional singular simplex σ^1 we fix such a 2-dimensional simplex $D\sigma^1$. Clearly $\partial D\sigma^1 = D\partial\sigma^1+\sigma^1$. Continuing this procedure in the subsequent dimensions, by making use of the triviality of the groups $\pi_q(X)$ we succeed in constructing for any singular simplex σ of X another simplex $D\sigma$ whose dimension is larger by one such that $\partial D\sigma=D\partial\sigma+\sigma$ if $\dim\sigma>0$. The mapping D: $\oplus_q C_q(X)\to\oplus_q C_q(X)$ is a chain homotopy connecting the identity mapping $\bar\varphi$: $\oplus_q C_q(X)\to\oplus_q C_q(X)$ with φ: $\oplus_q C_q(X)\to\oplus_q C_q(X)$ which is given by

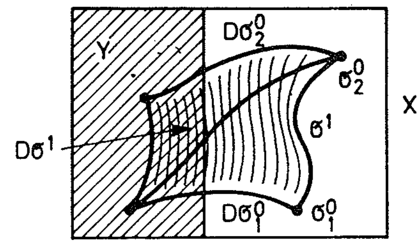
$$\varphi(c)=\begin{cases}(\operatorname{Ind} c)x_0 & \text{for } c\in C_0(X),\\ 0 & \text{for } c\in C_q(X),\ q>0.\end{cases}$$

Then φ clearly induces the trivial mapping of homology groups while it is homotopic to the identity mapping. This is only possible if the homology groups are trivial, i. e. $H_0(X)=\mathbf{Z}$ and $H_q(X)=0$ for $q>0$.

The relative analogue of the theorem is proved in literally the same way:

Theorem. If (X, Y) is a pair of topological spaces with X path-connected and $\pi_q(X, Y)\equiv 0$ for every q, then $H_q(X, Y)=0$ for every q.

The homotopy D: $\oplus_q C_q(X, Y)\to\oplus_q C_q(X, Y)$ to connect the identity with the null mapping is constructed as shown on the picture:



Corollary. If a mapping $f\colon X\to Y$ with Y path-connected induces isomorphisms between the respective homotopy groups, then it induces isomorphisms between the homology groups too.

Proof. Let us apply the relative variant of the theorem to the pair (Zf, X) where Zf is the cylinder of the mapping f (it is path-connected if so is Y). In view of the exact homotopy sequence

$$\begin{array}{ccccccccc} \pi_i(X) & \to & \pi_i(Zf) & \to & \pi_i(Zf, X) & \to & \pi_{i-1}(X) & \to & \pi_{i-1}(Zf) \\ & {\scriptstyle f_*}\searrow & \| & & & & & {\scriptstyle f_*}\searrow & \| \\ & & \pi_i(Y) & & & & & & \pi_{i-1}(Y) \end{array}$$

we have $\pi_i(Zf, X)=0$ for every i (regarded as set if $i=1$). Hence $H_i(Zf, X)=0$ for every i. Now it follows from the exact homology sequence of the pair (Zf, X) that the inclusion $X\to Zf$ as well as $f\colon X\to Y$ induce isomorphisms of the homology groups in every dimension.

Another formulation of the corollary. If $f\colon X\to Y$ is a weak homotopy equivalence, then $f_*\colon H_*(X)\to H_*(Y)$ is an isomorphism.

The Hurewicz theorem

Theorem. Assume that X is a path-connected space and $\pi_\alpha(X)=0$ for $\alpha<q$ and $\pi_q(X)\neq 0$ $(q>1)$. Then $H_\alpha(X)=0$ for $\alpha<q$ and $H_q(X)=\pi_q(X)$.

Proof. The topological space X may be assumed to be a CW complex without loss (by the cellular approximation theorem in §10). It may even be assumed to have no cells at all of dimension less than q, as indicated by a theorem in §3 (a corollary of the cellular approximation theorem).

As shown in §10, $\pi_q(X)$ is an Abelian group whose generators and relations correspond to the cells of dimensions q and $q+1$, respectively. More exactly, for any cell σ_j^{q+1} we consider the characteristic mapping $f_j^{q+1}\colon B^{q+1}\to X$ and restrict it to ∂B^{q+1}. We obtain a mapping of S^q into the q-skeleton of X, i. e. $\vee\, S_i^q$, which means an element of the free Abelian group spanned on the generators σ_i^q. By turning it into zero we get the relation.

After these remarks the theorem is already obvious. Indeed, we have $H_\alpha(X)=0$ for $\alpha<q$, as there are no cells of dimension less than q. In dimension q, $\mathscr{C}_q=Z_q$ because

$\partial \equiv 0$. Now $B_q(X)$ is in direct correspondence with the relations of $\pi_q(X)$. Hence we obtain the isomorphism. Q. e. d.

The Hurewicz theorem permits a more exact formulation which will be important in the sequel.

Let $\alpha \in \pi_q(X)$ be represented by $f\colon S^q \to X$. The standard sphere S^q is assumed once and for all to have a fixed orientation. The element $f_*(1) \in H_q(X)$ is clearly determined by α, which gives rise to a mapping $\pi_q(X) \to H_q(X)$ the so-called *Hurewicz homomorphism* denoted by γ_q. (Exercise. Verify that it is indeed a homomorphism!)

Theorem (Hurewicz). If $\pi_0(X) = \ldots = \pi_{q-1}(X) = 0$, $q > 1$, then γ_q is an isomorphism.

The proof is essentially presented above, the further details are left to the reader.

Exercise. Formulate and prove the relative Hurewicz theorem.

The case $q = 1$

Theorem. If X is path-connected, then $H_1(X) = \pi_1(X)/[\pi_1(X), \pi_1(X)]$. Here $[\pi_1(X)], \pi_1(X)]$ is the commutator-group of $\pi_1(X)$, i. e. its subgroup generated by all elements of the form $aba^{-1}b^{-1}$, where $a, b \in \pi_1(X)$.

A more precise formulation: For path-connected X the homomorphism $\gamma_1\colon \pi_1(X) \to$ $\to H_1(X)$ is an epimorphism with the kernel $[\pi_1(X), \pi_1(X)]$.

Proof. The space X may be assumed to be a CW complex with a single vertex. Then $\pi_1(X)$ is a group whose generators and relations correspond to one- and two-dimensional cells, respectively. Now $H_1(X)$ is an *Abelian* group having the same generators and relations as $\pi_1(X)$. In other words, we obtain $H_1(X)$ from $\pi_1(X)$ by adding to the relations those of the pairwise commutativity. Q.e.d. (The proof of the second variant is left to the reader.)

The inverse Hurewicz theorem

Theorem. Assume that X is path-connected and $H_\alpha(X) = 0$ for $\alpha < q$, $\pi_1(X) = 0$. Then $\pi_\alpha(X) = 0$ for $1 < \alpha < q$ and $\pi_q(X) = H_q(X)$.

Proof. Assume that $\pi_r(X)$ is different from zero with some $r < q$. Then the first non-trivial homotopy group is equal to the first non-trivial homology group, in contradiction with the original assumption. Thus $\pi_2(X) = \pi_3(X) = \ldots = \pi_{q-1}(X) = 0$ and by the Hurewicz theorem, $\pi_q(X) = H_q(X)$.

The inverse Hurewicz theorem also has its relative variant.

The Whitehead theorem

Theorem. Assume that X and Y are pathwise and simply connected spaces and f: $X \to Y$ is any mapping such that $f_*: \pi_2(X) \to \pi_2(Y)$ is an epimorphism. Then the following statements are equivalent:

(1) $f_*: \pi_r(X) \to \pi_r(Y)$ is an isomorphism for $r<q$ and an epimorphism for $r=q$.

(2) $f_*: H_r(X) \to H_r(Y)$ is an isomorphism for $r<q$ and an epimorphism for $r=q$.

The theorem immediately follows from the relative Hurewicz theorem. One only has to consider the pair (Zf, X), where Zf is the cylinder of the mapping f.

§14. COHOMOLOGY

Cochains. Let us consider the chains $C_q(X)$ of a space X and let G be a fixed Abelian group. A cochain of the space X with coefficients in G is a homomorphism of $C_q(X)$ into G. The natural addition turns the set Hom $(C_q(X), G)$ of cochains into a group denoted by $C^q(X; G)$. (In general Hom (A, B) denotes the set of all homomorphisms of a given Abelian group A into a given Abelian group B. It is an Abelian group.)

The operator δ. Let F_1, F_2 be a pair of groups and G a third group. Let us be given a homomorphism φ: $F_1 \to F_2$. Then φ induces a homomorphism φ^*: Hom$(F_2, G) \to$ $\to$ Hom(F_1, G) defined by the formula $(\varphi^* f)(a) = f(\varphi a), f \in$ Hom(F_2, G). In particular, let φ be the boundary operator ∂: $C_q(X) \to C_{q-1}(X)$. Then the so-called coboundary operator is $\delta = \varphi^*: C^{q-1}(X) \to C^q(X)$.

For any chain $c \in C_q(X)$, $(\delta\zeta)c = \zeta(\partial c)$. Since $\partial^2 = 0$, we also have $\delta^2 = 0$. Thus the sequence

$$\ldots \xleftarrow{\delta^2} C^2 \xleftarrow{\delta^1} C^1 \xleftarrow{\delta^0} C^0 \leftarrow G \ldots$$

is a complex (called a cochain complex).

Cohomology groups. Similarly to the case of homology groups and boundary operators, we shall consider Ker $\delta^q = Z^q(X; G)$ and Im $\delta^{q-1} = B^q(X; G)$. The quotient group $H^q(X; G) = \text{Ker}\,\delta^q / \text{Im}\,\delta^{q-1}$ is the q-th cohomology group of the space X with coefficients in G. It is denoted by $H^q(X; G)$.

Null-dimensional cohomology. Let X be connected. None of the nonzero elements is a coboundary: $B^0(X; G) = 0$. The cochain $\xi \in C^0(X; G)$ is a cocycle i. e. $\delta^0 \xi = 0$ if and only if ξ is constant, i. e. sends C^0 into a single element of G. Indeed, a null-dimensional cochain is a function on X taking its values in G. Let a, b be a pair of points such that $\zeta(a) \neq \zeta(b)$. Consider a one-dimensional simplex that connects a and b. We have $(\delta\zeta)(\sigma^1) = \zeta(\partial\sigma^1) = \zeta(a) - \zeta(b) \neq 0$, i. e. $\delta\zeta \neq 0$.

We have obtained a natural equality $H^0(X; G) = G$. Similarly, if $X = \cup_i X_i$, where X_i are the connected components of X, we have $H^0(X; G) = \oplus_i G$.

Relative cohomology. Suppose that Y is a closed subset of X. We have a natural inclusion $C_q(Y) \subset C_q(X)$.

The subgroup $C^q(X, Y) \subset \text{Hom}\,(C_q(X); G)$ consists of all cochains ζ whose values on the whole $C_q(Y)$ are zero. Clearly $\delta(C^q(X, Y)) \subset C^{q+1}(X, Y)$. Hence the groups $H^q(X, Y; G)$ are defined in an obvious way.

The exact cohomology sequence. Similarly to the case of homology, we may assign to a pair (X, Y) an exact sequence of cohomology groups. The inclusion $i\colon Y \to X$ gives rise to $i^*\colon H^q(X; G) \to H^q(Y; G)$. Let ζ be a cocycle in $C^q(Y; G)$, i. e. a homomorphism $C_q(Y) \to G$. Let the homomorphism $\bar{\zeta}\colon C_q(X) \to G$ be one of the extensions of ζ on $C_q(X)$. Then $\delta\bar{\zeta}$ is a relative cocycle mod Y, as $\zeta' = \delta\bar{\zeta}$ vanishes on $C_q(Y)$. So we obtain a homomorphism $H^q(Y; G) \to H^{q+1}(X, Y; G)$. The correctness of the definition is easily verified, i. e. the homomorphism does not depend on the particular choice of the elements ζ and their extensions.

As any relative cocycle may as well be regarded as an absolute cocycle, there is a natural homomorphism $H^{q+1}(X, Y; G) \to H^{q+1}(X; G)$.

As a result we have a sequence

$$\ldots \to H^q(X, Y; G) \to H^q(X; G) \to H^q(Y; G) \to H^{q+1}(X, Y; G) \to \ldots$$

Exercise. Prove that the above sequence is exact.

The exact sequence of a triple is a simple generalization of the sequence of a pair. Let $X \supset Y \supset Z$. We only have to replace formally absolute groups by relative ones mod Z. We obtain

$$\ldots \to H^q(X, Y; G) \to H^q(X, Z; G) \to H^q(Y, Z; G) \to H^{q+1}(X, Y; G) \to \ldots$$

which again is an exact sequence.

Cellular cohomology. Similarly to the case of homology, cellular cochains may be defined by $\mathscr{C}^q(X) = H^q(X^q, X^{q-1}; G)$ where X^q denotes the q-dimensional skeleton of X. The coboundary operator

$$\delta\colon H^q(X^q, X^{q-1}; G) \to H^{q+1}(X^{q+1}, X^q; G), \text{ that is } \delta\colon \mathscr{C}^q(X) \to \mathscr{C}^{q+1}(X)$$

arises as the operator at the corresponding place in the sequence of the triple (X^{q+1}, X^q, X^{q-1}).

Let us note an important difference between homology and cohomology. There is no such rule that cochains should vanish except on finitely many elementary cycles of the form $1 \cdot \sigma^q$. Thus cochains in general may only be written as infinite linear combinations $\Sigma g_k \sigma^q_k$.

Scalar product between $H^q(X; \mathbf{Z})$ and $H_q(X; \mathbf{Z})$ is defined in the following way. Let $\alpha \in H^q(X; \mathbf{Z})$, $\beta \in H_q(X; \mathbf{Z})$ and let a and b represent α and β, respectively. Put $(\alpha, \beta) = a(b)$. It is clearly independent of the particular representatives of α and β, since

$$\begin{aligned}(a + \delta a')(b + \partial b') &= a(b) + \delta a'(b) + a(\partial b') + \delta a'(\partial b') = \\ &= a(b) + a'(\partial b) + \delta a(b') + \delta^2 a'(b') = a(b),\end{aligned}$$

for a and b are a cocycle and a cycle respectively.

It may easily happen that an element $\alpha \in H^q(X; \mathbf{Z})$ is not fully determined by its scalar products with all elements of $H_q(X)$. For example let α have a finite order in $H^q(X; \mathbf{Z})$. Then $(\alpha, \beta) = 0$ for every $\beta \in H_q(X)$.

Exercise. Let $H_q(X) = \mathbf{Z}^{m_q} \oplus T_q$ and $H^q(X; \mathbf{Z}) = \mathbf{Z}^{n_q} \oplus T^q$ where $\mathbf{Z}^{m_q} = \mathbf{Z} \oplus \ldots \oplus \mathbf{Z}$ (m_q terms), $\mathbf{Z}^{n_q} = \mathbf{Z} \oplus \ldots \oplus \mathbf{Z}$ (n_q terms), and T_q and T^q are torsion groups. Prove that $m_q = n_q$, and $T_q = T^{q+1}$ for any q.

Cohomology and homology with coefficients in a field

Let us consider $H^q(X; k)$ and $H_q(X; k)$ for a finite CW complex X when k is assumed to be a field. Then the group $\mathscr{C}_q(X; k)$ is a finite dimensional linear space over k and the cycles $Z_q(X; k)$ and boundaries $B_q(X; k)$ form subspaces in it. Their quotient $H_q(X; k)$ is again a linear space, implying that $H_q(X; k) = k \oplus \ldots \oplus k$.

The group $\mathscr{C}^q(X; k)$ of cochains may be regarded as the space of linear functionals over $\mathscr{C}_q(X; k)$ with values in k. In other words, $\mathscr{C}_q(X; k)$ and $\mathscr{C}^q(X; k)$ are adjoint linear spaces, $\partial\colon \mathscr{C}_q(X; k) \to \mathscr{C}_{q-1}(X; k)$ and $\delta\colon \mathscr{C}^{q-1}(X; k) \to \mathscr{C}^q(X; k)$ are adjoint operators. It follows that $\operatorname{Ker} \partial$ and $\operatorname{Coker} \delta = \mathscr{C}^q(X; k)/\operatorname{Im} \delta$, further $\operatorname{Im} \delta$ and $\operatorname{Coim} \delta = \mathscr{C}^{q-1}(X; k)/\operatorname{Ker} \delta$, as well as $H_q(X; k) = \operatorname{Ker} \partial / \operatorname{Im} \partial$ and the kernel of the projection $\operatorname{Coker} \delta \to \operatorname{Coim} \delta$, i. e. $\operatorname{Ker} \delta / \operatorname{Im} \delta = H^q(X; k)$, are pairs of adjoint linear spaces. Thus $H_q(X; k)$ and $H^q(X; k)$ are adjoint linear spaces and have, among others, equal dimensions. The scalar product between $H^q(X; k)$ and $H_q(X; k)$ is obviously nondegenerate, in contrast to the integral case (see above).

§15. CHANGE OF COEFFICIENTS

Theorem. $H_i(X; \mathbf{Q}) = H_i(X) \otimes \mathbf{Q}$, where $\mathbf{Q}$ is the group of rational numbers. In other words, if

$$H_i(X) = \underbrace{\mathbf{Z} \oplus \ldots \oplus \mathbf{Z}}_{p} \oplus \text{(finite group)}$$

then

$$H_i(X; \mathbf{Q}) = \underbrace{\mathbf{Q} \oplus \ldots \oplus \mathbf{Q}}_{p}.$$

Proof. The inclusion $\mathbf{Z} \to \mathbf{Q}$ defines an imbedding $C_i(X) \to C_i(X; \mathbf{Q})$. In view of the commutative diagram

$$\begin{array}{ccc} C_i(X) & \xrightarrow{\partial} & C_{i-1}(X) \\ \cup & & \cup \\ C_i(X; \mathbf{Q}) & \xrightarrow{\partial} & C_{i-1}(X; \mathbf{Q}) \end{array}$$

a chain of $C_i(X)$ is or is not a cycle together in $C_i(X)$ and $C_i(X; \mathbf{Q})$, i. e. $Z_i(X) = Z_i(X; \mathbf{Q}) \cap C_i(X)$. Clearly for any $\alpha \in C_i(X; \mathbf{Q})$ there exists such N that $N\alpha \in C_i(X)$. Therefore, if $\alpha \in Z_i(X; \mathbf{Q})$ is a boundary then, for some M, $M\alpha$ is a boundary in $Z_i(X)$. Thus $H_i(X) \to H_i(X; \mathbf{Q})$ has finite kernel and every element of $H_i(X; \mathbf{Q})$ will get into the image of $H_i(X)$ if multiplied by a suitable integer, which implies that

$$H_i(X) \otimes \mathbf{Q} \to H_i(X; \mathbf{Q}) \otimes \mathbf{Q} = H_i(X; \mathbf{Q})$$

is an isomorphism.

An analogous **Theorem.** For any finite CW complex X we have

$$H^i(X; \mathbf{Q}) = H^i(X; \mathbf{Z}) \otimes \mathbf{Q}.$$

Because $\dim H^i(X; \mathbf{Q}) = \dim H_i(X; \mathbf{Q})$ we have $\operatorname{rank} H^i(X; \mathbf{Z}) = \operatorname{rank} H_i(X)$ for any finite CW complex (cf. the exercise above).

Assume now that $G_1 \subset G$ and $G_2 = G/G_1$.

We write out the exact sequence $0 \longrightarrow G_1 \xrightarrow{i} G \xrightarrow{j} G_2 \longrightarrow 0$ and the corresponding exact sequence of chain complexes

$$0 \to \mathscr{C} \otimes G_1 \to \mathscr{C} \otimes G \to \mathscr{C} \otimes G_2 \to 0$$

where $\mathscr{C} = \mathscr{C}(X; \mathbf{Z})$ is the complex of cellular chains of a complex X (verify the exactness of the last sequence).

As it will be shown there exists an exact homology sequence

$$\ldots \to H_i(X; G_1) \xrightarrow{i_*} H_i(X; G) \xrightarrow{j_*} H_i(X; G_2) \xrightarrow{\beta} H_{i-1}(X; G_1) \to \ldots$$

where i_* and j_* are induced by the homomorphisms i and j while β is the *Bockstein homomorphism* defined as follows. Let $a \in \mathscr{C}_i(X; G_2)$, $\partial a = 0$ represent the homology class $\alpha \in H_i(X; G_2)$. Because $j: G \to G_2$ is an epimorphism, we can construct a chain $a' \in \mathscr{C}_i(X; G)$ corresponding to the chain a. Now a' is not necessarily a cycle, nevertheless it has the property $j_*(\partial a') = 0$.

Hence all coefficients of the chain a' belong to the same coset mod G_1, namely G_1 itself. Thus $\partial a' \in \mathscr{C}_{i-1}(X; G_1)$. The homomorphism β is then defined as assigning to α the homology class of a'. We notice that $a' \notin \mathscr{C}_i(X; G)$ if $a' \to a \neq 0$ by $G \to G_2$. Therefore if a' is no cycle in $\mathscr{C}_i(X; G)$ then $\partial a' = 0$ is not homological to zero in $\mathscr{C}_{i-1}(X; G_1)$, i. e. $\beta(a) \neq 0$. If $a' \in Z_i(X; G)$, then $\partial a' = 0$ and $\beta(a) = 0$. This implies $\operatorname{Im} j_* = \operatorname{Ker} \beta$. Exactness in the terms $H_i(X; G)$ and $H_i(X; G_1)$ is clear by the definitions of i_* and j_*.

Exercise. Verify the correctness of the definition of the Bockstein homomorphism.

We have an analogous exact sequence in cohomology:

$$\ldots \to H^i(X; G_1) \to H^i(X; G) \to H^i(X; G_2) \xrightarrow{\beta} H^{i+1}(X; G_1) \to \ldots$$

where β is the analogous cohomology Bockstein homomorphism.

The formula of universal coefficients

In view of the formula $H_i(X; G_1 \oplus G_2) = H_i(X; G_1) \oplus H_i(X; G_2)$, where G_1 and G_2 are arbitrary Abelian groups, we are able to compute the homology groups of X with arbitrary coefficients, once $H_i(X; \mathbf{Z})$ and all $H_i(X; \mathbf{Z}_k)$ are known. Let us therefore express $H_i(X; \mathbf{Z}_k)$ through $H_i(X)$. Consider the exact sequence

$$0 \longrightarrow \mathbf{Z} \xrightarrow{i} \mathbf{Z} \xrightarrow{j} \mathbf{Z}_k \longrightarrow 0$$

where i is multiplication by the number k. We have an exact sequence

$$\ldots \to H_i(X; \mathbf{Z}) \xrightarrow{i_*} H_i(X; \mathbf{Z}) \xrightarrow{j_*} H_i(X; \mathbf{Z}_k) \xrightarrow{\beta} H_{i-1}(X; \mathbf{Z}) \to \ldots$$

where i_* is again multiplication by k. The group $H_i(X; \mathbf{Z}_k)$ is to be determined while $H_i(X; \mathbf{Z})$ and $H_{i-1}(X; \mathbf{Z})$ are supposed to be known.

Let the above segment of the sequence be substituted by

$$0 \to H_i(X; \mathbf{Z})/k \cdot H_i(X; \mathbf{Z}) \to H_i(X; \mathbf{Z}_k) \to \{\text{elements of order } k \text{ in } H_{i-1}(X; \mathbf{Z})\} \to 0,$$

where $k \cdot H_i(X; \mathbf{Z})$ denotes the subgroup of $H_i(X; \mathbf{Z})$ of the elements of form $b' = kb$, $b \in H_i(X; \mathbf{Z})$. In short, in $H_i(X; \mathbf{Z})$ we switched to congruence mod k.

For any Abelian group G we shall use the notation $\mathrm{Tor}(\mathbf{Z}_k; G)$ for the subgroup of all elements b such that $kb = 0$. The the exactness of the short sequence above is equivalent to

$$H_i(X; \mathbf{Z}_k) = (H_i(X; \mathbf{Z}) \otimes \mathbf{Z}_k) \oplus \mathrm{Tor}(\mathbf{Z}_k;\ H_{i-1}(X; \mathbf{Z}))$$

which is called the *universal coefficients formula.*

We mention that $\mathrm{Tor}(A,B)$ is defined in algebra for any pair A, B of Abelian groups and is called the *torsion product* of A and B. For finitely-generated A and B one has simply $\mathrm{Tor}(A, B) = \mathrm{Tors}\, A \otimes \mathrm{Tors}\, B$ where $\mathrm{Tors}\, G$ denotes the subgroup of G consisting of the elements of finite order. It turns out that

$$H_i(X; G) = (H_i(X; \mathbf{Z}) \otimes G) \oplus \mathrm{Tor}(G; H_{i-1}(X; \mathbf{Z}))$$

for any finitely-generated G. (Prove this formula for finitely-generated G!)

Example. Let us compute $H_i(\mathbf{RP}^n; \mathbf{Z}_2)$. The homology groups with coefficients in $\mathbf{Z}$ are already known:

$$\mathbf{Z}, \mathbf{Z}_2, 0, \mathbf{Z}_2, 0, \mathbf{Z}_2, 0, \mathbf{Z}_2, 0, \ldots.$$

We may apply the universal coefficients formula. By writing out in succession the groups $H_i(\mathbf{RP}^n; \mathbf{Z}) \otimes \mathbf{Z}_2$ we obtain $\mathbf{Z}_2, \mathbf{Z}_2, 0, \mathbf{Z}_2, 0, \mathbf{Z}_2, \ldots.$

For the second term of the formula we have

$$0, 0, \mathbf{Z}_2, 0, \mathbf{Z}_2, 0, \mathbf{Z}_2, 0, \ldots$$

Finally

$$H_*(\mathbf{RP}^n;\ \mathbf{Z}_2)=\{\mathbf{Z}_2,\ \mathbf{Z}_2,\ \mathbf{Z}_2,\ \ldots,\ \mathbf{Z}_2,\ 0,\ 0,\ \ldots\}.$$

The universal coefficient formula for cohomology is proved analogously. For any finitely-generated group G we have

$$H^i(X;\ G)=H^i(X;\ \mathbf{Z})\otimes G\oplus \mathrm{Tor}\,(G;\ H^{i+1}(X;\ \mathbf{Z})).$$

The Künneth formula

Let K and L be a pair of chain complexes of Abelian groups where all K_n are torsion-free. Then there is an exact sequence

$$0\longrightarrow \bigoplus_{m+n=r}(H_m(K)\otimes H_n(L))\xrightarrow{\ p\ }H_r(K\otimes L)\xrightarrow{\ \beta\ }\bigoplus_{m+n=r-1}\mathrm{Tor}\,(H_m(K),\ H_n(L))\longrightarrow 0$$

where p is the tensor homology multiplication (see §16 below) and β is the natural homomorphism. Moreover this sequence is split, i. e.

$$H_r(K\otimes L)=\bigoplus_{m+n=r}(H_m(K)\otimes H_n(L))\oplus\left(\bigoplus_{m+n=r-1}\mathrm{Tor}\,(H_m(K),H_n(L))\right)$$

although the splitting homomorphism is not natural. This is the formula of Künneth (for complexes of Abelian groups). It has an important particular case, the so-called "tensor formula of Künneth".

Assume that the chains $\mathscr{C}_n(K)$ and the groups $H_n(K)$ are free Abelian groups. Then

$$H_r(K\otimes L)\cong\bigoplus_{m+n=r}H_m(K)\otimes H_n(L).$$

If we choose a field k for the group of coefficients we have

$$H_r(K\otimes L;\ k)\cong\bigoplus_{m+n=r}H_m(K;\ k)\otimes_k H_n(L;\ k).$$

§16. MULTIPLICATION

The *tensor product* of a pair $\mathscr{C}$, $\mathscr{C}'$ of chain complexes is a chain complex $\mathscr{C}''=$ $=\mathscr{C}\otimes\mathscr{C}''$ such that

$$\mathscr{C}''_n=\oplus_{i+j=n}\mathscr{C}_i\otimes\mathscr{C}'_j$$

and

$$\partial''(\mathscr{C}_i\otimes\mathscr{C}'_j)=(\partial\mathscr{C}_i)\otimes\mathscr{C}'_j+(-1)^i\mathscr{C}_i\otimes(\partial'\mathscr{C}'_j)\ \text{ for }\ \mathscr{C}_i\in\mathscr{C}_i,\ \mathscr{C}'_j\in\mathscr{C}'_j.$$

Then $(\partial'')^2 = 0$, as it can easily be seen.

We may translate this definition into the language of geometry. Let K_1 and K_2 be CW complexes. Their direct product is again a CW complex consisting of the direct products of the cells of K_1 and K_2.

false as stated (locally finite K_1 suffices)

Because chains are finite functions on the cells with their values taken in G, provided that G is a ring, they may tensorially be multiplied. Their domain of definition will be the cells of $K_1 \times K_2$.

Let $a \in \mathscr{C}_i(K_1; \mathbf{Z})$, $b \in \mathscr{C}_j(K_2; \mathbf{Z})$, $a = \Sigma_k a_k \sigma_k^i$, $b = \Sigma_l b_l \tau_l^j$. Then $a \otimes b = \Sigma_{k,l} a_k b_l (\sigma_k^i \times \times \tau_l^j)$. By geometric consideration $\partial(a \otimes b) = (\partial a) \otimes b + (-1)^i a \otimes (\partial b)$. Hence $a \in Z_i(K_1, \mathbf{Z})$ and $b \in Z_j(K_2; \mathbf{Z})$ imply $a \otimes b \in Z_{i+j}(K_1 \times K_2; \mathbf{Z})$, so we may speak about the *tensor product* of homology classes of $H_i(K_1; \mathbf{Z})$ and $H_j(K_2; \mathbf{Z})$, and $\alpha \otimes \beta \in H_{i+j}(K_1 \times K_2; \mathbf{Z})$.

The tensor product is *natural*, i. e. if $f\colon K_1 \to L_1$ and $g\colon K_2 \to L_2$ are continuous mappings $f \times g\colon K_1 \times K_2 \to L_1 \times L_2$ is their product, then for arbitrary $a \in H_i(K_1)$, $b \in H_j(K_2)$, $(f_* a) \otimes (g_* b) = (f \times g)_* (a \otimes b)$.

It is *associative*, i. e. for any $a \in H_i(K_1)$, $b \in H_j(K_2)$, $c \in H_k(K_3)$ the elements $(a \otimes b) \otimes c$ and $a \otimes (b \otimes c)$ of $H_{i+j+k}(K_1 \times K_2 \times K_3)$ coincide.

It is *anticommutative*, i. e. if $a \in H_i(K)$, $b \in H_j(K)$, and $f\colon K \times K \to K \times K$ is defined by $f(x, y) = (y, x)$ then $f_*(a \otimes b) = (-1)^{ij}(b \otimes a)$.

All these facts can easily be proved by the reader.

The tensor product can be similarly defined in homology, and also in cohomology, with coefficients in an arbitrary commutative ring.

Let G be a commutative ring. The diagonal mapping $\Delta\colon K \to K \times K$ induces a mapping $\Delta^*\colon H^*(K \times K;\ G) \to H^*(K;\ G)$. For arbitrary a, $b \in H^*(K;\ G)$ we set $a \cdot b = \Delta^*(a \otimes b)$.

The multiplication that we have turns $H^*(K; G)$ into an anticommutative ring. The naturalness (functorial property) of multiplication, as well as associativity, distibutivity and anticommutativity follow from those of the tensor product.

Existence of unity

If the commutative ring G has a unit element, then so has $H^*(K; G)$. Consider the mapping $f\colon K \to *$. As $H^*(*; G) = G$, there is a unit element in this ring. Let it be denoted by 1. Consider a chain of mappings $f^*(1) \mid H^0(K;\ G)$, it will be a unit element of the ring $H^*(K;\ G)$.

Indeed, the chain of mappings

$$K \xrightarrow{\ \Delta\ } K \times K \xrightarrow{\ p\ } K \times (*),$$

where $p(x,y) = x$, sends for any $a \in H^*(K; G)$ an element of the type $a \otimes 1 \in H^*(K \times (*); G)$, $a \in H^*(K; G)$, first into $a \otimes f^*(1) \in H^*(K \times K; G)$ and then into $a \cdot f^*(1) \in H^*(K; G)$. On the other hand, $K = K \times (*) \to K$ is the identity mapping, consequently $a \cdot f^*(1) = f^*(1) \cdot a = a$.

The cohomology rings have no analogy in the case of homology, where natural multiplication exists only in a few particular cases, for example when K is a group.

The Hopf invariant

As the first application of the ring structure of $H^*(K; G)$ we prove the following non-trivial fact: the group $\pi_{4n-1}(S^{2n})$ is infinite. This will be done by constructing a non-trivial homomorphism $\pi_{4n-1}(S^{2n}) \to \mathbf{Z}$.

Let $\alpha \in \pi_{4n-1}(S^{2n})$ be represented by f. We construct a CW complex $X_\alpha = S^{2n} \cup_f e^{4n}$ by attaching to S^{2n} a $4n$-dimensional cell along the mapping f. It consists of three cells of dimensions 0, $2n$ and $4n$. As for the coboundary operator in the cellular cochain complex we have $\delta \equiv 0$, $H^*(X_\alpha; \mathbf{Z}) = \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$.

Let a and b be generators of $H^{2n}(X_\alpha; \mathbf{Z})$ and $H^{4n}(X_\alpha; \mathbf{Z})$, respectively. Since $\dim a = 2n$, we have $a^2 \in H^{4n}(X_\alpha)$, i. e. $a^2 = hb$ with $h \in \mathbf{Z}$. This number h will be assigned to α and will be called the *Hopf invariant*. The definition is correct, i. e. $h(\alpha) = h$ clearly does not depend on the particular choice of f within the homotopy class α.

Theorem. $h(\alpha)$ is additive: $h(\alpha+\beta) = h(\alpha) + h(\beta)$.

Proof. Together with $X_{\alpha+\beta} = S^{2n} \cup_{f+g} e^{4n}$ a similar complex $Y_{\alpha,\beta}$ will be considered, too. It is defined in the following way. Let $f \in \alpha$ and $g \in \beta$. We have a mapping $f \vee g \colon S^{4n-1} \vee S^{4n-1} \to S^{2n}$. We attach $e^{4n} \vee e^{4n}$ to S^{2n} along the mapping to obtain a complex that consists of one null-dimensional, one $2n$-dimensional and two $4n$-dimensional cells. Next S^{4n-1} is mapped onto $S^{4n-1} \vee S^{4n-1}$ by contracting to a single point the equator S^{4n-1}.

$$\begin{array}{ccc} S^{2n} & \xleftarrow{\;f+g\;} & S^{4n-1} \\ \| & & \downarrow \\ S^{2n} & \longleftarrow & S^{4n-1} \vee S^{4n-1} \end{array}$$

The two horizontal mappings here describe the complexes $X_{\alpha+\beta}$ and $Y_{\alpha,\beta}$, the vertical mapping $S^{4n-1} \to S^{4n-1} \vee S^{4n-1}$ gives rise to a mapping $X_{\alpha+\beta} \to Y_{\alpha,\beta}$ which identifies the $2n$-dimensional cells of $X_{\alpha+\beta}$ and $Y_{\alpha,\beta}$ while the single $4n$-dimensional cell of $X_{\alpha+\beta}$ covers both $4n$-dimensional spheres of $Y_{\alpha,\beta}$. We obtain $H^*(Y_{\alpha,\beta}; \mathbf{Z}) \to H^*(X_{\alpha+\beta}; \mathbf{Z})$ where $H^*(Y_{\alpha,\beta}; \mathbf{Z}) = \underset{(0)}{\mathbf{Z}} \oplus \underset{(2n)}{\mathbf{Z}} \oplus \underset{(4n)}{\mathbf{Z}} \oplus \underset{(4n)}{\mathbf{Z}}$ and $H^*(X_{\alpha+\beta}; \mathbf{Z}) = \underset{(0)}{\mathbf{Z}} \oplus \underset{(2n)}{\mathbf{Z}} \oplus \underset{(4n)}{\mathbf{Z}}$. The generators in dimensions $2n$ and $4n$ are a', b'_1, b'_2 and a, b. By the definition of $Y_{\alpha,\beta}$, $b'_1 \mapsto b$, $b'_2 \mapsto b$ and $a' \mapsto a$. Now in $H^*(Y_{\alpha,\beta}; \mathbf{Z})$ we have $(a')^2 = h_1 b'_1 + h_2 b'_2$ where $h_1, h_2 \in \mathbf{Z}$.

By the naturality property of cohomology groups with respect to mapping of complexes this implies $a^2 = (h_1 + h_2)b$. On the other hand $a^2 = h(\alpha+\beta)b$, i. e. $h(\alpha+\beta) = h_1 + h_2$. Now we notice that h depends on $f \in \alpha$ alone, i. e. is independent of $g \in \beta$.

By putting $\beta=0$ we obtain $h_1=h(\alpha)$. Similarly $h_2=h(\beta)$. Thus $h(\alpha+\beta)=h(\alpha)+h(\beta)$ as stated. Q. e. d.

In view of the theorem, the mapping $\alpha\mapsto h(\alpha)$ is a homomorphism $\pi_{4n-1}(S^{2n})\to\mathbf{Z}$. We have to prove its nontriviality. Let us consider the product $S^{2n}\times S^{2n}$ and attach the spheres $S^{2n}\times(*)$ and $(*)\times S^{2n}$ to one another by identifying the corresponding points of the two spheres. As a result we have a complex X which consists of one cell in each dimension 0, $2n$ and $4n$.

The attaching mapping of the $4n$-cell σ^{4n} maps S^{4n-1} into the $(4n-1)$-dimensional skeleton of X which consists of a single cell σ^{2n}. Let α: $S^{4n-1}\to S^{2n}$ be this mapping, and α be its homotopy class. In other words, the complex X is obtained by attaching to S^{2n} a ball e^{4n} along a mapping α, i. e. X is of the type X_α.

It turns out that the element of $\pi_{4n-1}(S^{2n})$ described by the mapping α has nonzero $h(\alpha)$. Indeed, consider

$$S^{2n}\times S^{2n}\to X_\alpha = S^{2n}\cup_\alpha e^{4n}.$$

Let a_1 and a_2 be generators of $H^{2n}(S^{2n}\times S^{2n};\ \mathbf{Z})=\mathbf{Z}\oplus\mathbf{Z}$ and a, b, d be generators of $H^{2n}(X_\alpha;\ \mathbf{Z})$, $H^{4n}(X_\alpha;\ \mathbf{Z})$ and $H^{4n}(S^{2n}\times S^{2n};\ \mathbf{Z})$, respectively. In the cohomology we have then the correspondences $a\mapsto a_1+a_2$, $b\mapsto d$, and in consequence $a^2\mapsto(a_1+a_2)^2=$ $=a_1^2+2a_1a_2+a_2^2$, i. e. $a^2=h(\alpha)b$, $h(\alpha)b\to a_1^2+2a_1a_2+a_2^2$.

If K_1 and K_2 are torsion-free complexes over $\mathbf{Z}$ then clearly

$$H^*(K_1\times K_2;\ \mathbf{Z}) = H^*(K_1;\ \mathbf{Z})\otimes H^*(K_2;\ \mathbf{Z}),$$

i. e., in the present case,

$$H^*(S^{2n}\times S^{2n};\ \mathbf{Z}) = H^*(S^{2n};\ \mathbf{Z})\otimes H^*(S^{2n};\ \mathbf{Z}),$$

hence $d=a_1\cdot a_2$. Thus $a_1^2=a_2^2=0$, and finally $h(\alpha)\cdot b\mapsto 2d$, $h(\alpha)\neq 0$ as claimed.

Actually we have $\pi_{4n-1}(S^{2n})=\mathbf{Z}\oplus\{\text{a finite group}\}$, which we are not going to prove here. It is known that

$$\pi_3(S^2)=\mathbf{Z},\quad \pi_7(S^4)=\mathbf{Z}\oplus\mathbf{Z}_{12},$$
$$\pi_{11}(S^6)=\mathbf{Z},\quad \pi_{15}(S^8)=\mathbf{Z}\oplus\mathbf{Z}_{120}.$$

§17. OBSTRUCTION THEORY

Let X be a topological space, K be a CW complex and g: $K^{n-1}\to X$ be a mapping, where K^{n-1} denotes the $(n-1)$-dimensional skeleton of K. We want to extend g to a mapping of K^n into X. (This is the step of induction in the course of defining a mapping $K\to X$ by successively extending it from each skeleton to the next one.) We shall only consider the case when X is $(n-1)$-simple (for example, simply-connected).

Let us be given a cell $e^n \subset K$ with characteristic mapping $\chi: B^n \to K$. Since $\dot{e}^n \subset K^{n-1}$ and $f: K^{n-1} \to X$ is already defined, we have $\dot{B}^n = S^{n-1} \to K \to X$.

We remind the reader that the cell e^n will be fixed during the whole procedure. If we succeed in extending f from the boundary of e^n to a mapping defined over the whole cell, we shall be satisfied that by analogous construction we are able to extend it onto the other cells as well, and obtain a mapping $K^n \to X$ as required.

So the question of extending $f: \dot{e}^n \to X$ to $\tilde{f}: e^n \to X$ is reduced to the question of extending the $S^{n-1} \to X$ to $B^n \to X$.

Now $S^{n-1} \to X$ defines an element of $\pi_{n-1}(X)$. (It will be recalled that in the view of the $(n-1)$-simplicity this element does not depend on the behaviour of the base points.) Consequently $S^{n-1} \to X$ extends to some mapping $B^n \to X$ if and only if the homotopy class of the former is zero in $\pi_{n-1}(X)$.

So we have assigned to each $e^n \subset K$ an element of $\pi_{n-1}(X)$.

Let this correspondence be continued as a homomorphism $\mathscr{C}_n(K) \to \pi_{n-1}(X)$ by performing the above construction for each cell $e^n \subset K^n$ and then defining the mapping on $\mathscr{C}_n(K)$ by linearity. The result is a cochain c_f with coefficients in $\pi_{n-1}(X)$; $c_f \in \mathscr{C}^n(K, \pi_{n-1}(X))$, called the *obstruction* to extending $f: K^{n-1} \to X$ onto K^n.

Clearly $f: K^{n-1} \to X$ may be extended to a mapping of the n-skeleton of K if and only if $c_f = 0$.

So far we have only been reformulating the problem and as yet the relation $c_f = 0$ does not carry any new information. As it will soon turn out, c_f has many interesting properties.

Theorem 1. The cochain $c_f \in \mathscr{C}^n(K, \pi_{n-1}(X))$ is a cocycle, i. e. $\delta c_f = 0$.

Proof. We shall need the relative Hurewicz theorem which has been mentioned in an exercise. Here it is. If $Y \subset X$, $x_0 \in Y$, $\pi_1(X) = \pi_1(Y) = 0$ and $\pi_k(X, Y, x_0) = 0$, $k < n$, then $H_k(X, Y) = 0$ for $k < n$ and $\pi_n(X, Y, x_0) = H_n(X, Y, x_0)$.

In the diagram

$$
\begin{array}{ccccc}
\mathscr{C}_{n+1}(K) = H_{n+1}(K^{n+1}, K^n) & = & \pi_{n+1}(K^{n+1}, K^n) & & \\
 & & \tilde{\partial}\downarrow & & \\
\downarrow\partial & & \pi_n(K^n) & & \\
 & & j_*\downarrow & & \\
\mathscr{C}_n(K) = H_n(K^n, K^{n-1}) & = & \pi'(K^n, K^{n-1}) & & \\
 & \searrow^{i} & \partial'\downarrow & & \\
 & & \pi_{n-1}(K^{n-1}) & & \\
 & & f_*\downarrow & & \\
 & & \pi_{n-1}(X) = \text{the coefficient group,} & &
\end{array}
$$

the horizontal lines represent the Hurewicz isomorphisms. By definition the boundary operator ∂: $\mathscr{C}_{n+1}(K) \to \mathscr{C}_n(K)$ is the corresponding operator in the exact sequence of the triple (K^{n+1}, K^n, K^{n-1}). By the Hurewicz theorem it reduces to an operator in the exact homotopy sequence for the same triple:

$$\pi_{n+1}(K^{n+1}, K^n) \to \pi_n(K^n, K^{n-1}).$$

By construction it decomposes into $\tilde{\partial}$ and j_*:

$$\pi_{n+1}(K^{n+1}, K^n) \xrightarrow{\tilde{\partial}} \pi_n(K^n) \xrightarrow{j_*} \pi_n(K^n, K^{n-1}),$$

the ordinary boundary operator and the transition from absolute to relative spheroids. We notice that the three-term sequence obtained is not contained in the sequence of the triple. The mapping ∂' is again ordinary boundary operator and f_* is the homomorphism induced by f: $K^{n-1} \to X$.

The square consisting of $\mathscr{C}_{n+1}$, $\mathscr{C}_n$, π_{n+1} and π_n is clearly commutative. Consider the homomorphism c_f^n: $\mathscr{C}_n(K) \to \pi_{n-1}(X)$. It is a composite $f_* \circ \partial'$; $\delta c_f^n \in \mathscr{C}^{n+1}(K)$.

The cochain δc_f^n defines a homomorphism $\mathscr{C}_{n+1}(K) \to \pi_{n-1}(X)$ such that $\delta c_f^n = c_f^n \circ \partial = f_* \circ \partial' \circ j_* \circ \tilde{\partial} = 0$, as $\partial' \circ j_* = 0$ (∂' and j_* being two successive homomorphisms in the exact sequence of the pair (K^n, K^{n-1})). Q. e. d.

The cohomology class of the cocycle c_f^n will be denoted by C_f^n. So we have $C_f^n \in H^n(K; \pi_{n-1}(X))$.

Theorem 2. $C_f^n = 0$ if and only if f: $K^{n-1} \to X$ after having been appropriately changed on K^{n-1} without being altered on K^{n-2} may be extended to a mapping $\tilde{f}$: $K^n \to X$.

Before the proof we introduce a notion which will prove useful in the sequel.

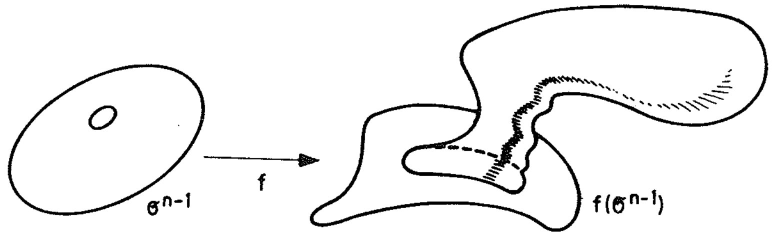
Let f and g: $K^{n-1} \to X$ be such that $f(x) = g(x)$ for $x \in K^{n-2}$. Let $\sigma^{n-1} \subset K^{n-1}$ be an arbitrary cell with characteristic mapping χ: $B^{n-1} \to K$. Because $\chi(\dot{B}^{n-1}) \subset K^{n-2}$ the mappings $f \circ \chi$ and $g \circ \chi$ coincide on $\dot{B}^{n-1}$. Now f and g are different on the cell σ^{n-1}, their images are nevertheless attached to each other along the image of $\dot{B}^{n-1}$, i. e. we have a spheroid in X, defining an element of $\pi_{n-1}(X)$. Actually we obtained a cochain which assigns an element of $\pi_{n-1}(X)$ to every cell σ^{n-1}. It shall be denoted by $d_{f,g}^{n-1}$ and called the *difference cochain* of f and g. Thus $d_{f,g}^{n-1} \in \mathscr{C}^{n-1}(K, \pi_{n-1}(X))$.

Obviously $d_{f,g}^{n-1} = 0$ if and only if there exists a homotopy which connects f and g and is constant on K^{n-2} (where $f = g$).

Lemma 1. For every mapping f: $K^{n-1} \to X$ and cochain $d \in \mathscr{C}^{n-1}(K; \pi_{n-1}(X))$ there may be found a mapping g: $K^{n-1} \to X$ such that $d = d_{f,g}^{n-1}$ and $g|_{K^{n-2}} = f|_{K^{n-2}}$.

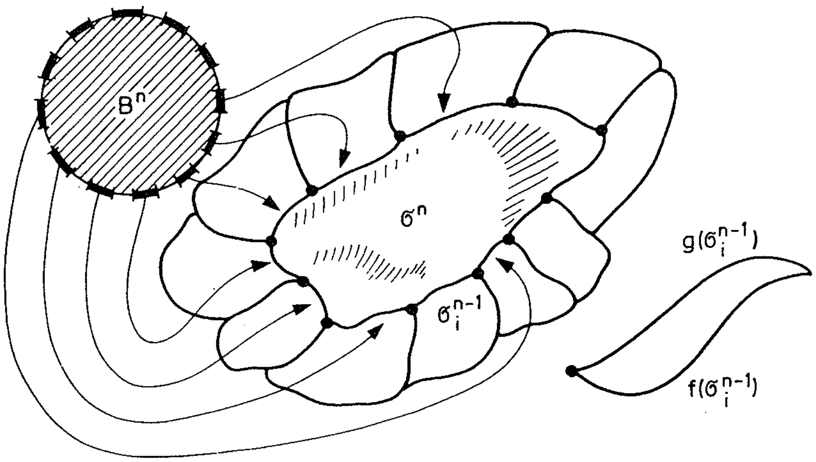
Indeed, let us consider an arbitrary cell σ^{n-1} and its image $f(\sigma^{n-1}) \subset X$. We take a small ball in the centre of the cell and cut off its image from $f(\sigma^{n-1})$. Next we attach to its place a spheroid representing the value in $\pi_{n-1}(X)$ of the cochain d at σ^{n-1}.

Now g is defined as coinciding with f everywhere except on the ball where it is blown up into the spheroid. By similarly altering f on every cell we finally get a mapping g for which $d = d_{f,g}^{n-1}$ as stated by the lemma.



Lemma 2. $\delta d^{n-1}_{f,g} = c^n_f - c^n_g$ (i. e. $d^{n-1}_{f,g}$ is not a cocycle anymore).

Proof. We have $(\delta d^{n-1}_{f,g})(\sigma^n) = d^{n-1}_{f,g}(\partial\sigma^n) = \Sigma_i[\sigma^n : \sigma^{n-1}_i] d^{n-1}_{f,g}(\sigma^{n-1}_i)$. Let us recall the definition of $[\sigma^n : \sigma^{n-1}_i]$. There exists a characteristic mapping $B^n \to K$ for the cell σ^n such that $\dot{B}^n = S^{n-1} \to K^{n-1} \to K^{n-1}/K^{n-2} = \vee\, S^{n-1}_i$. Then $[\sigma^n : \sigma^{n-1}_i]$ is the degree of the mapping $S^{n-1} \to S^{n-1}_i$. As it had been proved earlier, $S^{n-1} \to K^{n-1}$ is homotopic to a mapping which maps the whole sphere, except for a finite number of balls B^{n-1}_k, into K^{n-2} while the balls are mapped onto cells σ^{n-1}_i with degrees ± 1. The number of balls mapped onto σ^{n-1}_i is actually equal to the incidence number.



On the picture the heavy segments are meant to denote the small balls selected on S^{n-1}. Let us now examine the value taken by the cochain $c^n_f - c^n_g$ on the cell σ^n, i. e. where do the little balls go when f resp. g is applied. Because $f(x) = g(x)$ for $x \in K^{n-2}$, the value of $c^n_f - c^n_g$ on each cell σ^{n-1}_i is the spheroid which is the value on σ^{n-1}_i of the difference cochain $d^{n-1}_{f,g}$, taken as many times as a little ball is mapped onto σ^{n-1}_i, i. e. taken with the incidence number. To the whole cell σ^n the sum $\Sigma_i[\sigma^n : \sigma^{n-1}_i] d^{n-1}_{f,g}(\sigma^{n-1}_i)$ is then assigned. As it clearly coincides with $(\delta d^{n-1}_{f,g})(\sigma^n)$, the lemma is proved.

Proof of theorem 2. Assume that there exists g: $K^{n-1} \to X$ which extends to $K^n \to X$ such that $g|_{K^{n-2}} = f|_{K^{n-2}}$. Then $c^n_g = 0$, $\delta d^{n-1}_{f,g} = c^n_f$ (by lemma 2) and $C^n_f = 0$. Conversely, if $C^n_f = 0$, then $c^n_f = \delta d$, $d \in \mathscr{C}^{n-1}(K, \pi_{n-1}(X))$. According to lemma 1 there exists g: $K^{n-1} \to X$ coinciding with f on K^{n-2} such that $d^{n-1}_{f,g} = d$. We have $c^n_g = c^n_f - \delta d^{n-1}_{f,g} =$ (by lemma 2) $c^n_f - \delta d = 0$, i. e. g extends to a mapping $K^n \to X$. Q. e. d.

We mention some obvious properties of the obstructions and the difference cochains.

1. If f and g: $K^{n-1}\to X$ are homotopic, then $c_f^n = c_g^n$.

2. Let K_1 and K_2 be two complexes and let φ: $K_1\to K_2$ be a cellular mapping. Consider the mapping g: $K_1^{n-1}\to X$ defined by $g(x) = f(\varphi(x))$ where f: $K_2^{n-1}\to X$. If $\hat{\varphi}$: $\mathscr{C}^n(K_2)\to\mathscr{C}^n(K_1)$ denotes the homomorphism induced by φ, then $c_g^n = \hat{\varphi}(c_f^n)$, i. e. the obstruction is natural with respect to mappings of complexes.

3. If f and g are mappings of K^{n-1} to X and $f=g=h$ on K^{n-2}, then

$$d_{f,g}^{n-1} + d_{g,h}^{n-1} = d_{f,h}^{n-1}.$$

4. Let f, g: $K_2^{n-1}\to X$; $f=g$ on K_2^{n-2} and let φ: $K_1^{n-1}\to K_2^{n-1}$. Then

$$d_{f\circ\varphi,\, g\circ\varphi}^{n-1} = \hat{\varphi}\, d_{f,g}^{n-1}.$$

5. $$d_{f,g}^{n-1} = -d_{g,f}^{n-1}.$$

The relative case

If $L\subset K$ is a subcomplex and f is defined on $K^{n-1}\cup L$, then the obstruction to extending f to a mapping $K^n\cup L\to X$ is found in $\mathscr{C}^n(K, L; \pi_{n-1}(X))$. It is a (relative) cocycle whose cohomology class is in $H^n(K, L; \pi_{n-1}(X))$. The theory of "relative" obstructions is parallel to its absolute variant and we are not going to go into the details. We only mention the interesting connection between relative obstructions and difference cochains; further on it will play important role at several instances. Assume that f, g: $K\to X$ coincide (or are homotopic) on the n-skeleton of K. We then have a mapping F: $(K^n\times I)\cup(K\times\partial I)\to X$. denote $L=K\times I$, $M=K\times\partial I$. Clearly $K^n\times\times I\cup K\times\partial I = L^{n+1}\cup M$. The obstruction to extending F on $L^{n+2}\cup M$ is in $\mathscr{C}^{n+2}(L, M; \pi_{n+1}(X))=\mathscr{C}^{n+2}(\Sigma K, \pi_{n+1}(X))=\mathscr{C}^{n+1}(K, \pi_{n+1}(X))$ and it can easily be seen that it equals to $d_{f,g}^{n+1}$. Now $\delta d_{f,g}^{n+1} = c_f^{n+2}-c_g^{n+2} = 0$ as f and g are defined on the whole K. Thus if f, g: $K\to X$ coincide on K^n and they are defined *on the whole* K, their difference cochain may be represented as an obstruction.

By applying theorem 2 to this case we get the following statement.

Theorem 3. Let f, g: $K\to X$ coincide on K^n. Then $d_{f,g}^{n+1}\sim 0$ if and only if $f|_{K^{n+1}}\sim\sim g|_{K^{n+1}}$ relatively to K^{n-1} (i. e. they may be connected by a homotopy which is constant on K^{n-1}).

As an application of this theorem we indicate the connection between cohomology and the mappings into $K(\pi, n)$ mentioned in §2:

Corollary. $H^n(X, \pi) = \Pi(X, K(\pi, n))$.

Consider the composition $\mathscr{C}_n(K(\pi, n))\to H_n(K(\pi, n)) = \pi_n(K(\pi, n)) = \pi$, where the first homomorphism arises in consequence of $\mathscr{C}_{n-1}(K(\pi, n)) = 0$ (the cell structure of $K(\pi, n)$ is given as in §10) and $\mathscr{C}_n$ coincides with the group of cycles, i. e. H_n is a quotient group of $\mathscr{C}_n$. The result is a cochain in $\mathscr{C}^n(K(\pi, n); \pi)$. An alternative description: each n-

dimensional cell of $K(\pi, n)$ corresponds to an element of the group π, which extends by linearity to a homomorphism $\mathscr{C}_n(K(\pi, n)) \to \pi$.

The cochain $e \in \mathscr{C}^n(K(\pi, n); \pi)$ arising may also be described as the difference cochain $d^n_{f,g}$ for the inclusion $f\colon K^n(\pi, n) \to K(\pi, n)$ and the constant mapping $g\colon K^n(\pi, n) \to K(\pi, n)$. Both of them extend to $K(\pi, n) \to L(\pi, n)$, thus $c^n_f = c^n_g = 0$ and $\delta e = \delta d^n_{f,g} = 0$, i. e. e is a cocycle.

The cohomology class of e is called the *fundamental cohomology class* of $K(\pi, n)$ and will be denoted by $e \in H^n(K(\pi, n); \pi)$, too. We remark that the fundamental class $e_X \in H^n(X; \pi_n(X))$ of any space X for which $\pi_0(X) = \ldots = \pi_{n-1}(X) = 0, (n > 1)$ can be defined similarly. Later on we shall return to this notion.

Theorem. For any CW complex X the mapping assigning to $f\colon X \to K(\pi, n)$ the class $f^*(e) \in H^n(X; \pi)$ gives rise to a one-to-one correspondence between $H^n(X; \pi)$ and the set $\Pi(X; K(\pi, n))$ of homotopy classes of mappings of X into $K(\pi, n)$.

This theorem was already announced in §2.

Proof. Let $\alpha \in H^n(X, \pi)$. We prove that there exists some f such that $f^*(e) = \alpha$.

Let a represent α. We construct $f\colon X \to K(\pi, n)$. Let X^{n-1} be mapped onto the base point of $K(\pi, n)$. Next we define f on X^n. Let e^n be an n-cell of X. Then $a(e^n) \in \pi$ Because the boundary of the cell e^n is mapped onto a single point, the mapping must be a spheroid in $K(\pi, n)$. We choose for it an arbitrary spheroid that represents $a(e^n)$. Clearly $a \in \mathscr{C}^n(X; \pi)$ is a difference cochain between $f\colon X^n \to K(\pi, n)$ and the constant mapping $g\colon X^n \to K(\pi, n)$. We have $0 = \delta a = c^n_f - c^n_g$ ($c^n_g = 0$, as g extends to the whole X), i. e. $c^n_f = 0$, and f may be continued on X^{n-1}. The obstructions to extending it onto $X^{n+2}, X^{n+3}, \ldots$, etc. are in trivial groups, i. e. $\pi_{n+1}(K(\pi, n)) = \pi_{n+2}(K(\pi, n)) = \ldots = 0$. Thus there exists a mapping $f\colon X \to K(\pi, n)$. The composite $\mathscr{C}_n(X) \xrightarrow{f_*} \mathscr{C}_n(K(\pi, n)) \xrightarrow{e} \pi$ obviously coincides with a, so $f^*(e) = \alpha$.

It is left to show that for any pair $f, g\colon X \to K(\pi, n)$, $f^*(e) = g^*(e)$ implies $f \sim g$.

It suffices to consider cellular f and g. Because $K^{n-1}(\pi, n) = *$, $f|_{X^{n-1}} = g|_{X^{n-1}}$ is immediate. Thus $f|_{X^n}$ and $g|_{X^n}$ correspond to certain cocycles $a, b \in \mathscr{C}^n(X; \pi)$ in the same way as above in the first part of the proof. These cocycles are in fact the images of $e \in \mathscr{C}^n(K(\pi, n))$ under the cochain homomorphisms defined by f and g. Further $a = d^n_{f,h}$ and $b = d^n_{g,h}$ where $h\colon X^n \to K(\pi, n)$ is a mapping onto a single point. We have $d^n_{f,h} = -d^n_{g,h} = a - b \sim 0$.

By theorem 3, $f|_{X^n} \sim g|_{X^n}$ i. e. g is homotopic to a mapping $\tilde{g}\colon X \to K(\pi, n)$ such that $f|_{X^n} = \tilde{g}|_{X^n}$ (by the Borsuk theorem). Now f and $\tilde{g}$ are clearly homotopic as the differenc cochains are taken with coefficients in trivial groups. Q.e.d.

Obstruction to extending a section of a fibration

Let (E, B, F, p) be a fibration (whether a locally trivial or Serre one, it does not matter now). We assume that the fibre is simple and the base is simply connected. The latter is unnecessary; actually it would suffice to have "simple" fibrations in the following sense. If $s_1, s_2: I \to B$ are paths which connect the points $x, y \in B$ and φ_t, ψ_t: $p^{-1}(x) \to E$ are homotopies such that $\varphi_0 \equiv \psi_0$, $p^{-1}(x) \subset E$, $\varphi_t(p^{-1}(x)) \subset \subset p^{-1}(s_1(t))$, $\psi_t(p^{-1}(x)) \subset p^{-1}(s_2(t))$ (they exist as guaranteed by the covering homotopy theorem). Then the mappings $\varphi_1, \psi_1: p^{-1}(x) \to E$ are homotopic. This assumption may also be satisfied in the case of a base which is not a simply connected as the example of tangent unit vectors to an *oriented* manifold shows.

Suppose that B is a CW complex and that we are given a section over the $(n-1)$-skeleton of the base. Let e^n be an n-dimensional cell of the base and f: $B^n \to B$ the characteristic mapping. The fibration induced by f over B^n is trivial. The section over the $(n-1)$-skeleton induces a section of the induced fibration over $\dot{B}^n = S^{n-1}$, i. e. a mapping $S^{n-1} \to B^n \times F$, i. e. an element of $\pi_{n-1}(B^n \times F) = \pi_{n-1}(F)$. (Here we assumed the fibration to be locally trivial. The situation is nevertheless the same in Serre's case, too. Indeed, there exists a *canonical* isomorphism between the $(n-1)$-dimensional homotopy group of an arbitrary fibre and the standard copy of the fibre, as it follows from the simply-connectedness of the base.) The function assigning to each cell σ^n an element of $\pi_{n-1}(F)$ gives rise to a cochain $\bar{c}^n$ of $\mathscr{C}^n(B; \pi_{n-1}(F))$. Its properties are proved analogously to those of the obstructions to extending continuous mappings. We list them here without giving the proofs.

1. A section extends to a section over the n-skeleton of the base if and only if $c^n = 0$.
2. $\delta c^n = 0$.
3. The cohomology class of the cocycle c^n is zero if and only if the section may be altered on the $(n-1)$-skeleton of the base, without being changed over the $(n-2)$-skeleton, so that it would extend onto the whole n-skeleton.

The difference cochain is also defined. The notion of obstruction to extending a mapping is a special case of the more general notion of obstruction to extending a section. Indeed, any mapping $K \to X$ is equivalent to its graph $K \to K \times X$ which is a section of trivial fibration. Obstructions to extending a mapping or its graph are the same.

On the other hand, the notion of obstruction to a section does not reduce to the special case of mappings, as it follows from the following remark.

Let $\pi_0(F) = \dots = \pi_{n-1}(F) = 0$. The obstruction to extending sections to the k-skeleton of the base is zero for $k < n$, as its values are taken in trivial groups. Obstruction to extending a section to the n-skeleton is not necessarily trivial anymore. Its cohomology class is in $H^n(B; \pi_{n-1}(F))$ and it is independent of what the section is on the previous skeleton. This fact is almost obvious and it is proved by using difference cochains and the analogue of lemma 2 (i. e. the formula $\delta d^{n-1}_{f,g} = c^n_f - c^n_g$). So the class is defined by the fibration alone and it is called the *characteristic class* of the fibration.

Exercise (the basic property of characteristic classes). If $\xi \in H^n(B;\, \pi_{n-1}(F))$ is the characteristic class of a fibration (E, B, F, p) and $f\colon B' \to B$ is a continuous mapping then the characteristic class of the fibration induced by f is $f^*(\xi)$.

Exercise. The characteristic class of a fibration is independent of the particular cell structure given on the base.

Let us still add the following interesting observation. The characteristic class of the Serre fibration $EX \xrightarrow{\Omega X} X$ regarded as an element of $H^n(X; \pi_{n-1}(\Omega X)) = H^n(X; \pi_n(X))$, where $\pi_0(X) = \ldots = \pi_{n-1}(X) = 0$, is nothing else than the fundamental class of X. (Prove it!) Hence the invariance of the notion.

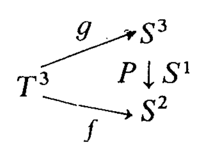
APPENDIX 1
TWO REMARKABLE EXAMPLES OF CONTINUOUS MAPPINGS

Any sufficiently good topological space, say CW complex, whose homotopy groups vanish was proved to be contractible. If in addition the space is simply connected this follows from triviality of the homology groups, as well.

Suppose now that a mapping induces the null homomorphism between the homotopy groups, or homology groups. Is it necessarily homotopic to constant?

The answer is negative even for mappings that are trivial both on homotopy and homology groups, as shown by the following counterexample.

Let $p\colon S^3 \to S^2$ be the Hopf fibration. The three-dimensional torus $T^3 = S^1 \times S^1 \times \times S^1$ is mapped onto S^3 by its 2-dimensional skeleton being contracted to a single point. The composite $f\colon T^3 \to S^2$



is not null homotopic, otherwise by the covering homotopy theorem, the homotopy in point could be covered in S^3, which would imply that $g\colon T^3 \to S^3$ is homotopic to the mapping which sends T^3 into S^1 and so, as S^1 is contractible in S^3, g would be null homotopic, too.

Let us now examine what kind of mappings are induced by $f\colon T^3 \to S^2$ in the homology and homotopy groups.

By consideration of the dimensions one immediately gets that the homomorphisms $H^i(S^2) \to H^i(T^3)$ are trivial. As for the homotopy groups, we have $\pi_1(T^3) = \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$; $\pi_k(T^3) = 0$, $k \geq 2$; $\pi_1(S^2) = 0$; thus the mappings $\pi_i(T^3) \to \pi_i(S^2)$ are trivial, too.

In a mind cultivated by topology the suspicion would naturally arise that the existence of such an extraordinary mapping must be somehow connected with the fact that T^3 is not simply connected, similarly to many other phenomena.

Nevertheless it has no significance in the present case, as it will be shown on the next construction which only involves simply connected spaces. Let $X = S^{2n-2} \times S^3$. Then X is a complex consisting of one cell in each dimension 0, 3, $2n-2$, and $2n+1$. By contracting the $2n$-skeleton to a point we obtain $S^{2n-2} \times S^3 \to S^{2n+1}$. Let S^{2n+1} be fibred over $\mathbf{CP}^n$ *with fibre* S^1. The composite mapping $S^{2n-2} \times S^3 \to S^{2n+1} \to \mathbf{CP}^n$ is not null homotopic as it may be proved analogously to the above while it induces trivial homology and homotopy homomorphisms.

Indeed, the mapping between homology groups is trivial as it goes through S^{2n+1} while $\dim \mathbf{CP}^n = 2n$.

Still easier is the case with homotopy groups. Actually $S^{2n-2} \times S^3 \to S^{2n+1}$ already induces null homomorphism, as

$$S^{2n-2} \vee S^3 \xrightarrow{\text{imbedding}} S^{2n-2} \times S^3 \longrightarrow S^{2n+1}$$

where $S^{2n-2} \vee S^3$ is mapped onto a single point. Let us be given an element of $\pi_k(S^{2n-2} \times \times S^3)$, i. e. a mapping $f: S^k \to S^{2n-2} \times S^3$, i. e. a pair (f_1, f_2) such that $f_1 : S^k \to S^{2n-2}$ and f_2: $S^k \to S^3$. Because the union is mapped into a single point, f is homotopic to constant.

APPENDIX 2
THE EXACT SEQUENCE OF PUPPE

Let X' and X be arbitrary CW complexes and f: $X' \to X$ an arbitrary continuous mapping. Let us consider a further space Y and the pointed sets $\Pi(X', Y)$, $\Pi(X, Y)$, $\Pi(Y, X')$, $\Pi(Y, X)$. For any continuous mapping α we shall denote by $[\alpha]$ the corresponding element of $\Pi(\ldots; \ldots)$.

A three-termed sequence $X' \xrightarrow{f} X \xrightarrow{g} X''$ is said to be *exact* if for any space Y the sequence

$$\Pi(Y, X') \xrightarrow{f_*} \Pi(Y, X) \xrightarrow{g_*} \Pi(Y, X'')$$

is exact. (Exactness means that the pre-image of the base point coincides with the image of the previous set.)

Dually, a sequence is *coexact* if the following sequence

$$\Pi(X'', Y) \xrightarrow{g^*} \Pi(X, Y) \xrightarrow{f^*} \Pi(X', Y)$$

is exact for any Y.

A sequence $\ldots \to X_{n+1} \xrightarrow{f_n} X_n \xrightarrow{f_{n-1}} X_{n-1} \to \ldots$ is said to be exact (resp. coexact) if its three-terms subsequences $X_{i+1} \to X_i \to X_{i-1}$ are exact (coexact).

Let us denote the cone of the mapping f by C_f, i. e. $C_f \equiv X \cup_f CX'$. We construct the following sequence of CW complexes

$$X' \xrightarrow{f} X \xrightarrow{i} C_p \xrightarrow{j} \Sigma X' \xrightarrow{\Sigma f} \Sigma X \xrightarrow{\Sigma i} C_{\Sigma f} \xrightarrow{\Sigma j} \dots$$

where i is imbedding, j the natural projection: j: $C_f \to C_f/X = \Sigma X'$; Σf denotes the suspension mapping over f, etc.

Theorem 1. For any continuous mapping f: $X' \to X$ the sequence

$$X' \xrightarrow{f} X \xrightarrow{i} C_f \longrightarrow \dots \xrightarrow{\Sigma^n f} \Sigma^n X \xrightarrow{\Sigma^n i} \Sigma^n C_f \xrightarrow{\Sigma^n j} \Sigma^{n+1} X' \longrightarrow \dots$$

$$\Sigma^n C_f = C_{\Sigma^n f}$$

is coexact.

Proof. At first we prove the coexactness of $X' \xrightarrow{f} X \xrightarrow{i} C_f$, i. e. for any Y we prove the exactness of

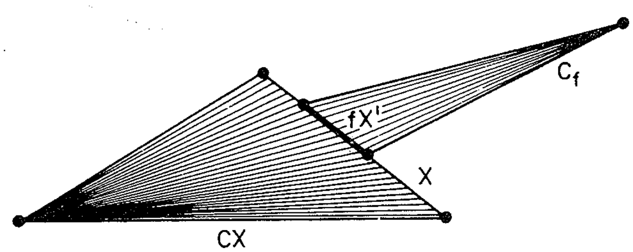
$$\Pi(C_f,\ Y) \xrightarrow{i^*} \Pi(X,\ Y) \xrightarrow{f^*} \Pi(X',\ Y).$$

Let $[\alpha] \in \Pi(X,\ Y)$, $[\beta] \in \Pi(C_f,\ Y)$ and $\alpha = i^*(\beta)$, i. e. let us have a commutative diagram

$$\begin{array}{ccccc} X' & \xrightarrow{f} & X & \xrightarrow{i} & C_f \\ & \searrow^{f^*\alpha} & \downarrow \alpha & \swarrow_{\beta} & \\ & & Y & & \end{array}$$

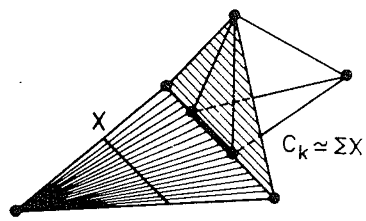
By $f^*\alpha = \alpha \circ f = \beta \circ i \circ f$ the mapping $f^*\alpha$ extends to a mapping of the cone C_f, i. e. $f^*[\alpha] = 0$. Assume now that $f^*[\alpha] = 0$; then $\alpha = \beta \circ i$ by the diagram, where β has been given as a homotopy that connects f with the constant mapping. Coexactness is proved.

Next we notice that $C_i \approx C_i/CX = C_f/X = \Sigma X'$, where CX is the cone over X.



As it has already been proved, the sequence $X' \xrightarrow{f} X \xrightarrow{i} C_f \xrightarrow{K} C_i \xrightarrow{t} C_k$ is coexact. The diagram

$$\begin{array}{ccccc} C_f & \xrightarrow{K} & C_i & \xrightarrow{t} & C_k \\ & \searrow & p_1 \downarrow \approx & & p_2 \downarrow \approx \\ & & \Sigma X' & \xrightarrow[\Sigma f]{} & \Sigma X \end{array}$$



is clearly commutative. Here p_1 and p_2 are homotopy equivalences, j is the natural projection $C_f \to C_f/X$ and Σf is the suspension over f. This proves the coexactness of the five-term sequence $X' \xrightarrow{f} X \xrightarrow{i} C_f \xrightarrow{j} \Sigma X' \xrightarrow{\Sigma f} \Sigma X$.

In order to extend it to the right without loosing its coexactness we notice the following simple fact: if $X' \xrightarrow{f} X \xrightarrow{g} X''$ is coexact, then so is the sequence $\Sigma X' \xrightarrow{\Sigma f} \Sigma X \xrightarrow{\Sigma g} \Sigma X''$. Indeed, in view of $\Pi(\Sigma X, Y) = \Pi(X, \Omega Y)$ the diagram

$$\begin{array}{ccccc} \Pi(\Sigma X'', Y) & \xrightarrow{(\Sigma g)^*} & \Pi(\Sigma X, Y) & \xrightarrow{(\Sigma f)^*} & \Pi(\Sigma X', Y) \\ \updownarrow = & & \updownarrow = & & \updownarrow = \\ \Pi(X'', \Omega Y) & \xrightarrow{g^*} & \Pi(X, \Omega Y) & \xrightarrow{f^*} & \Pi(X', \Omega Y) \end{array}$$

is commutative. The second row is known to be exact, thus the first row is exact as well.

We obtain that

$$\Sigma^n X' \xrightarrow{\Sigma^n f} \Sigma^n X \xrightarrow{\Sigma^n i} C_{\Sigma^n f} \xrightarrow{\Sigma^n j} \Sigma^{n+1} X' \xrightarrow{\Sigma^{n+1} f} \Sigma^{n+1} X$$

is coexact. Q.e.d.

Let us now consider the exact sequence

$$\Pi(X', Y) \xleftarrow{f^*} \Pi(X, Y) \xleftarrow{i^*} \Pi(C_f, Y) \xleftarrow{j^*} \Pi(\Sigma X', Y) \leftarrow \dots$$

$$\dots \leftarrow \Pi(\Sigma^n X', Y) \xleftarrow{\Sigma^n f^*} \Pi(\Sigma^n X, Y) \xleftarrow{\Sigma^n i^*} \Pi(C_{\Sigma^n f}, Y) \xleftarrow{\Sigma^n j^*} \Pi(\Sigma^{n+1} X', Y) \leftarrow \dots$$

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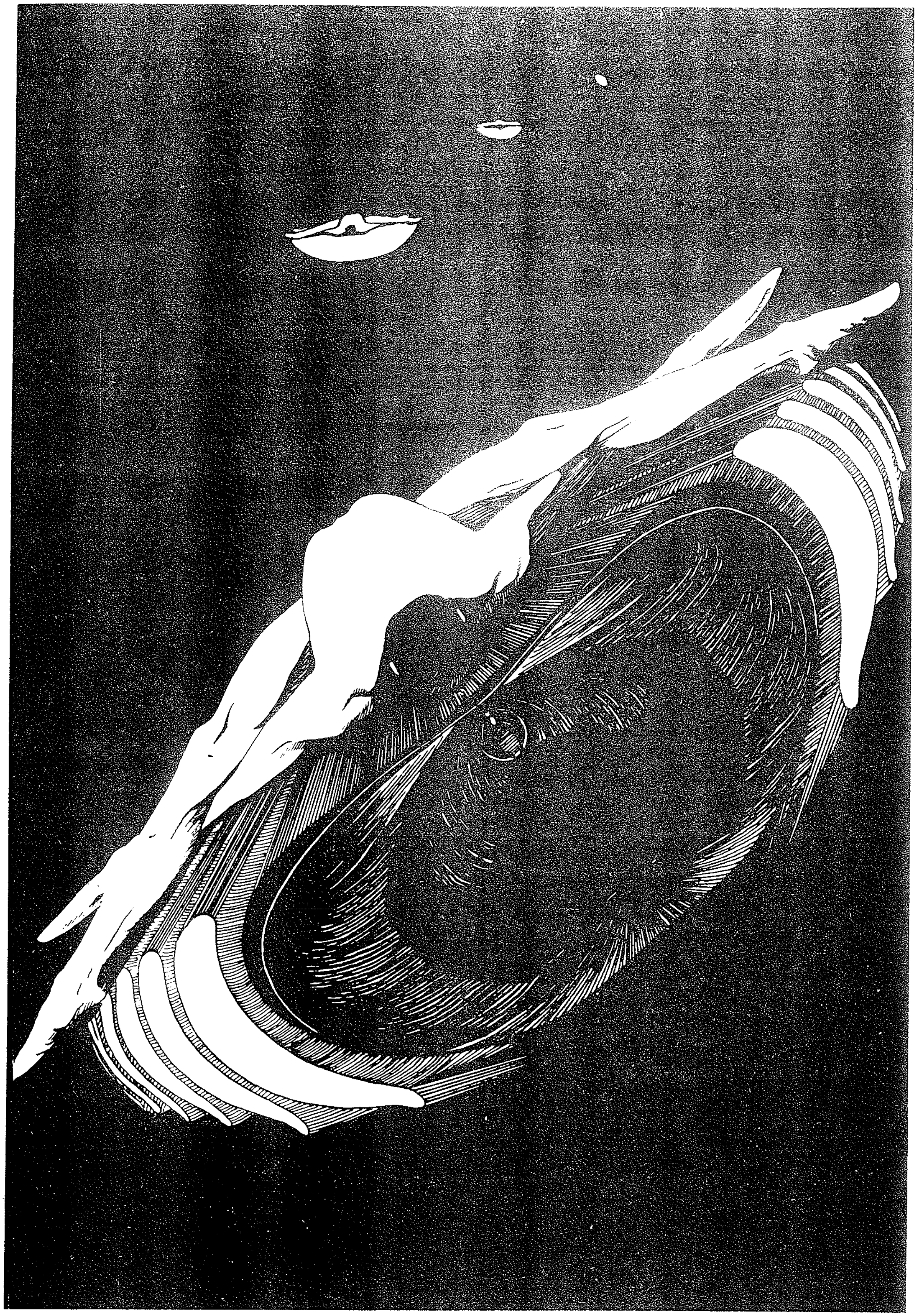
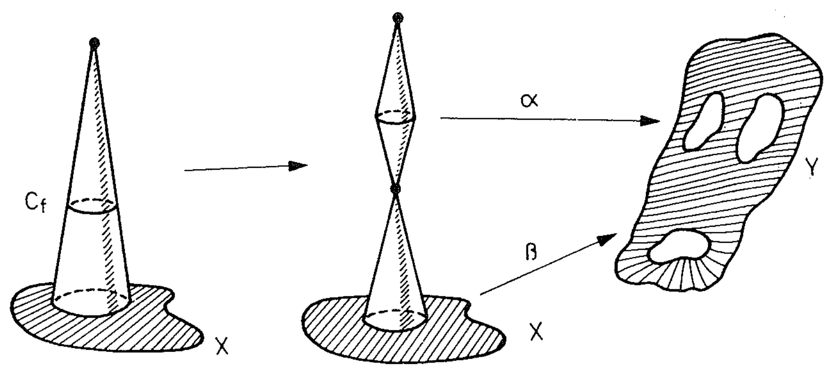


All terms of the sequence, except the first three ones are groups, and all but the first three groups are Abelian. The mappings between the groups are homomorphisms. The first three terms are simply sets with base points and f^*, i^* and j^* are mappings of sets compatible with base points.

As it turns out there is a left action of the group $\Pi(\Sigma X', Y)$ on the set $\Pi(C_f, Y)$, dividing it into a set of orbits. Then this orbit space is injectively mapped by i^* into the set $\Pi(X, Y)$. The action of $\Pi(\Sigma X', Y)$ on $\Pi(C_f, Y)$ is given in the following way. For $[\alpha] \in \Pi(\Sigma X', Y)$ and $[\beta] \in \Pi(C_f, Y)$ we denote by $\alpha \oplus \beta$: $C_f \to Y$ the mapping

$$\begin{cases} (\alpha \oplus \beta)(x', t) = \begin{cases} \alpha(x', 2t), & 0 \le t \le \frac{1}{2},\ x' \in X', \\ \beta(x', 2t-1), & \frac{1}{2} \le t \le 1,\ x' \in X'; \end{cases} \\ (\alpha \oplus \beta)(x) = \beta(x),\ x \in X. \end{cases}$$



Remark. Obviously $(\alpha \oplus \beta)|_X = \beta|_X$.

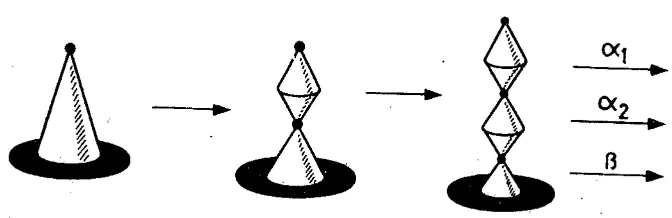
The reader will easily prove the following statements.

(i) If $\alpha_1 \sim \alpha_2$ and $\beta_1 \sim \beta_2$ then $\alpha_1 \oplus \beta_1 \sim \alpha_2 \oplus \beta_2$. Similarly if $\alpha_1 \sim \alpha_2$; $\beta_1 \sim \beta_2$ (rel X), then $\alpha_1 \oplus \beta_1 \sim \alpha_2 \oplus \beta_2$ (rel X). (here (rel X) means the existence of a homotopy stable on X.)

(ii) If $(*)$ is a constant mapping, then $(*) \oplus \beta \sim \beta$ (rel X).

(iii) $(\alpha_1 \oplus \alpha_2) \oplus \beta \sim (\alpha_1 \oplus (\alpha_2 \oplus \beta))$ (rel X), where $\oplus$ is the group operation given in the group $\Pi(\Sigma X', Y)$.

(iv) $\alpha_1 \oplus (\alpha_2 \circ j) \sim ((\alpha_1 + \alpha_2) \circ j)$ (rel X), where j: $C_f \to \Sigma X'$ is the natural projection.

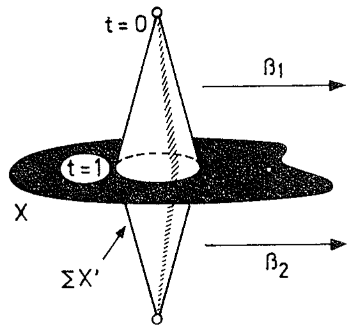


(iv) $\alpha_1 \oplus (_2 \circ j) \sim ((\alpha_1 + \alpha_2) \circ j) \pmod X$, where $j: C_f \to \sum X'$ is the natural projection.

We introduce a further operation $d(\beta_1, \beta_2)$ for classes $[\beta_1], [\beta_2] \in \Pi(C_f, Y)$ which can be represented by β_1 and β_2 for which $\beta_1|_X = \beta_2|_X$.

The mapping $d(\beta_1, \beta_2)$: $\Sigma X' \to Y$ is defined by the formula

$$d(\beta_1, \beta_2)(x', t) = \begin{cases} \beta_1(x', 2t), & \text{for} \quad 0 \le t \le \frac{1}{2}, x' \in X' \\ \beta_2(x', 2-2t), & \text{for} \quad \frac{1}{2} \le t \le 1, x' \in X' \end{cases}$$



$\beta_1(x', 1) = \beta_2(x', 1)$

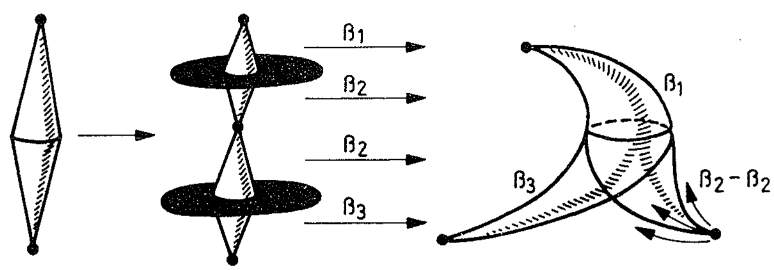
$(x', 1) \in X$

The reader is encouraged to check that

(v) $\beta_1 \sim \beta_1'$ (rel X) and $\beta_2 \sim \beta_2'$ (rel X) imply

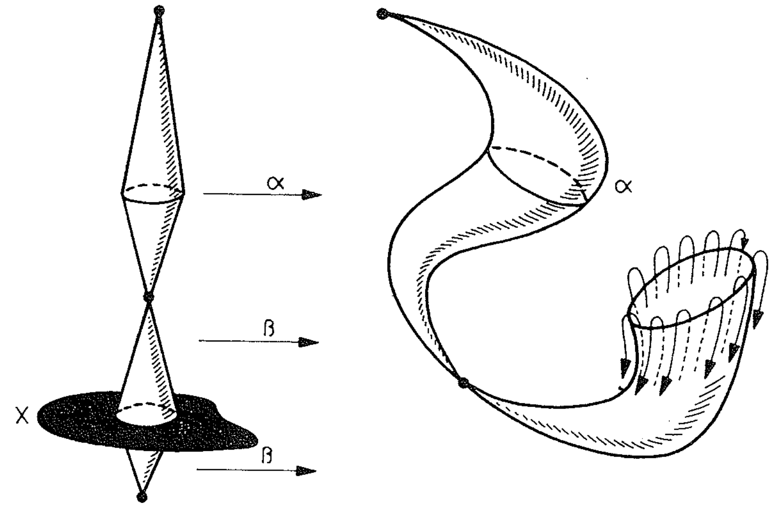
$$d(\beta_1, \beta_2) \sim d(\beta_1', \beta_2');$$

(vi) $d(\beta_1, \beta_2) + d(\beta_2, \beta_3) \sim d(\beta_1, \beta_3)$.

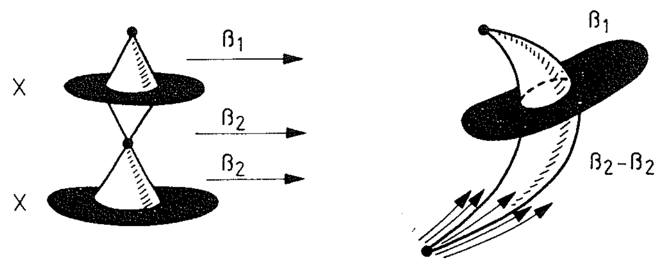


(The cone regarded as having been passed doubly, conditionally denoted by $\beta_2 - \beta_2$ on the picture, is contractible on $\beta_2(f(X'))$.)

(vii) $d(\alpha \oplus \beta, \beta) \sim \alpha$.



(viii) $\beta_1 \sim [d(\beta_1, \beta_2) \textcircled{\scriptsize{I}} \beta_2]$ (rel X).



Suppose now that $d(\beta_1\ \beta_2) \sim *$. By (viii) we have $\beta_1 \sim [d(\beta_1, \beta_2) \textcircled{\scriptsize{I}} \beta_2]$ (rel X) $\sim$ $\sim [(*) \textcircled{\scriptsize{I}} \beta_2]$ (rel X) $\sim \beta_2$ (rel X), i. e. $\beta_1 \sim \beta_2$ (rel X). Conversely, if $\beta_1 \sim$ $\sim \beta_2$ (rel X) then $d(\beta_1, \beta_2) \sim d(\beta_2, \beta_2) \sim d((*) \textcircled{\scriptsize{I}} \beta_2, \beta_2) \sim (*)$, as follows from (vii), i. e. $d(\beta_1\ \beta_2) \sim (*)$. Hence $\beta_1 \sim \beta_2$ (rel X) if and only if the mapping $d(\beta_1\ \beta_2)$ is null homotopic. By (i—iii) there exists an action of $\Pi(\Sigma X'\ Y)$ on the set $\Pi(C_f, Y)$ from the left, given by

$$[\alpha] \textcircled{\scriptsize{I}} [\beta] = [\alpha \textcircled{\scriptsize{I}} \beta],$$

where $[\alpha] \in \Pi(\Sigma X', Y)$, $[\beta] \in \Pi(C_f, Y)$. So we are ready now to formulate the theorem "on the action".

Theorem 2. Let $[\beta_1], [\beta_2] \in \Pi(C_f, Y)$. Then $i^*[\beta_2]$ if and only if there exists an $[\alpha] \in \Pi(\Sigma X', Y)$ such that $[\beta_1] = [\alpha] \textcircled{\scriptsize{I}} [\beta_2]$.

Proof. For $[\beta_1] = [\alpha] \textcircled{\scriptsize{I}} [\beta_2]$ we have $i^*(\beta_1) = i^*(\alpha \textcircled{\scriptsize{I}} \beta_2) = (\alpha \textcircled{\scriptsize{I}} \beta_2)|_X = \beta_2|_X = i^*(\beta_2)$. Conversely, let $i^*[\beta_1] = i^*[\beta_2]$. Then there exist β_1 and β_2 representing $[\beta_1]$ and $[\beta_2]$, respectively, such that $\beta_1|_X = \beta_2|_X$. Taking (viii) into account we get $[\beta_1] = [d(\beta_1, \beta_2) \textcircled{\scriptsize{I}} \beta_2] = [d(\beta_1, \beta_2)] \oplus [\beta_2]$. Q.e.d.

Let us examine the group $\Pi(\Sigma X', Y)$. Since $\Pi(C_f, Y)$ is not a group the equality $j^*[\alpha_1] = j^*[\alpha_2]$ does not imply a relation $[\alpha_2] = [\alpha_1] + (\Sigma f^*)[\gamma]$. Nevertheless the relation is valid, i. e. exactness in the term $\Pi(\Sigma X', Y)$ has a meaning analogous to the case when the left side is a group.

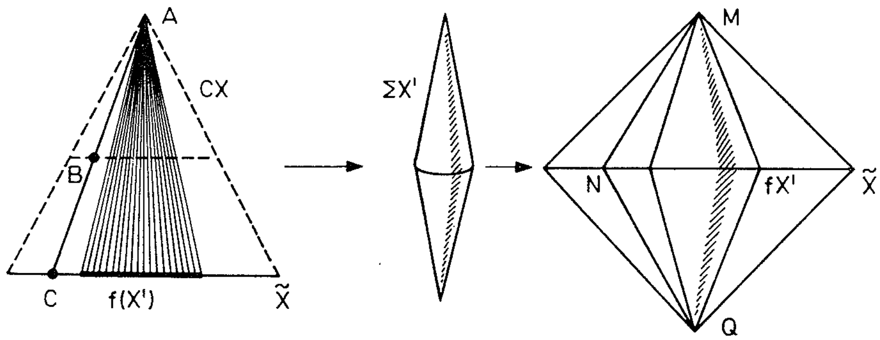
Theorem 3. Let $[\alpha_1], [\alpha_2] \in \Pi(\Sigma X', Y)$. Then $j^*[\alpha_1] = j^*[\alpha_2]$ if and only if there exists a coset $\gamma \in \Pi(\Sigma X, Y)$ such that $[\alpha_2]-[\alpha_1] = (\Sigma f)^*[\gamma]$.

Proof. Assume $[\alpha_2] = [\alpha_1]+(\Sigma f)^*[\gamma]$. Let α_0 and β_0 denote the constant mappings $\Sigma X' \to * \in Y$ and $C_f \to * \in Y$, respectively. Then $j^*[\alpha_0] = [\beta_0]$ and by (iv),

$$j^*[\alpha_1 + (\Sigma f)^*\gamma] = [\alpha_1] \oplus (j^*(\Sigma f)^*[\gamma]).$$

The mapping $j^*(\Sigma f)^*\gamma = \gamma \circ (\Sigma f) \circ j$ is null homotopic.

Indeed, it is defined on C_f, so we have to show that it extends to a mapping of CX (as the mapping f may be substituted by an *imbedding* $X' \to \tilde{X}$ where $\tilde{X}$ is a space homotopy equivalent with X).

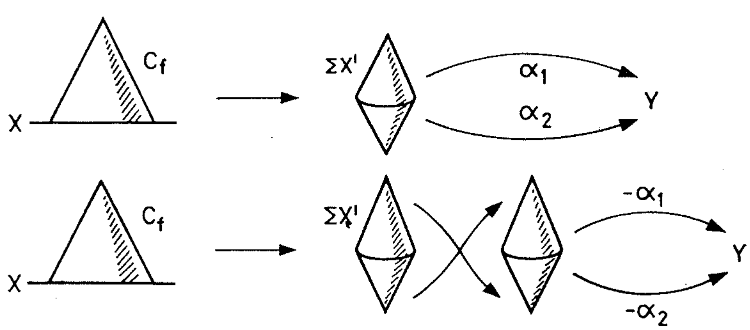


The extension is defined by mapping AB onto MN and BC onto NQ. Thus

$$j^*[\alpha_1 + (\Sigma f)^*\gamma] = [\alpha_1] \oplus [\beta_0] = [\alpha_1] \oplus j^*[\alpha_0] = j^*[\alpha_1 + \alpha_0] = j^*[\alpha_1],$$

i. e. $j^*[\alpha_2] = j^*[\alpha_1]$.

Conversely, assume $j^*[\alpha_1] = j^*[\alpha_2]$ and consider $[\alpha_2 - \alpha_1]$ and $j^*[\alpha_2 - \alpha_1] \in (\alpha_2 - \alpha_1) \circ j$. We have $(\alpha_2 - \alpha_1) \circ j \sim \alpha_1 \oplus ((-\alpha_2) \circ j)$. Now $(-\alpha_2) \circ j \sim (-\alpha_1) \circ j$, as shown on the picture.



Hence

$$\alpha_1 \oplus ((-\alpha_2) \circ j) \sim \alpha_1 \oplus ((-\alpha_1) \circ j) \sim (\alpha_1 - \alpha_1) \circ j \sim \beta_0,$$

i. e. $j^*[\alpha_2 - \alpha_1] = [\beta_0]$, $[\alpha_2 - \alpha_1] = [\alpha_2] - [\alpha_1] = (\Sigma f)^*[\gamma]$ as claimed. Q. e. d.

Remark 1. If we choose $Y = K(\pi, n)$ then the Puppe exact sequence becomes (in part) the exact cohomology sequence of the pair (X', X) (here we suppose that X' is substituted by a homotopy equivalent space such that $f\colon X \to X'$ is an imbedding):

$$H^n(X';\ \pi) \leftarrow H^n(X;\ \pi) \leftarrow H^n(C_f;\ \pi) \leftarrow H^{n-1}(X';\ \pi) \leftarrow H^{n-1}(X;\ \pi) \leftarrow \dots$$

As for large N any mapping $\Sigma^N X \to K(\pi, n)$ is null homotopic, the groups with indexes $n-k$, where $k > n$ vanish and the sequence is continued to the right by zeros. As f is imbedding, $H^n(C_f;\ \pi) = H^n(X'/X;\ \pi) = H^n(X', X;\ \pi)$.

Exercise. The reader is advised to try to dualise the construction used in theorems 1–3. For this, one has to prove the exactness of

$$\dots \to \Omega X \xrightarrow{\Omega f} \Omega X' \longrightarrow W_f \xrightarrow{\tau} X \xrightarrow{f} X'.$$

Here W_f is the space of pairs (x, s) where $x \in X$ and s is a path in X' such that $s(1) = f(x)$, $s(0) = *$ ($*$ stands for the base point of X') and $\tau(x, s) = s(1) = f(x) \in X$, as f may be assumed to be an imbedding. The mapping τ is a fibration with fibre $\Omega X'$.

Remark 2. If $Y = S^n$, the above exact sequence becomes the exact homotopy sequence of the pair (X', X). Indeed

$$\dots \to \Pi(S^n, \Omega X) \to \Pi(S^n, \Omega X') \to \Pi(S^n, W_f) \to \Pi(S^n, X) \to \Pi(S^n, X'),$$

i. e.

$$\dots \to \pi_{n+1}(X) \to \pi_{n+1}(X') \to \pi_n(X', X) \to \pi_n(X) \to \pi_n(X'),$$

because $\pi_n(X', X) = \pi_n(W_f)$ is obvious (simply the absolute spheroids in W_f are the same as the relative spheroids of the pair (X', X)).

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