CHAPTER IV

COHOMOLOGY OPERATIONS

§26. GENERAL THEORY

Let n and q be two numbers and Π and G be two Abelian groups. We say that there is given a *cohomological operation* φ *of type* (n, q, Π, G) if for every CW complex X there is defined a mapping φ_X: $H^n(X; \Pi) \to H^q(X; G)$ natural in the sense that the diagram

$$\begin{array}{ccc} H^n(X;\ \Pi) & \xrightarrow{\varphi_X} & H^q(X;\ G) \\ {\scriptstyle f^*}\uparrow & & {\scriptstyle f^*}\uparrow \\ H^n(Y;\ \Pi) & \xrightarrow{\varphi_Y} & H^n(Y;\ G) \end{array}$$

is commutative for any mapping f: $X \to Y$.

We remark that the mappings φ_X need not be group homomorphisms.

We shall write φ instead of φ_X when this causes no confusion.

As G is an Abelian group, so is the family of all cohomological operations of type (n, q, Π, G). Let it be denoted by $\mathcal{O}(n, q, \Pi, G)$.

Theorem.

$$\mathcal{O}(n, q, \Pi, G) \cong H^q(K(\Pi, n); G).$$

This statement, beautiful and unexpected as it is, is almost obvious, as it will be shown.

Proof. We know that $H^n(X; \Pi) = \Pi(X, K(\Pi, n))$. This equality is established by making use of a remarkable element $e \in H^n(K(\Pi, n); \Pi)$; namely, we assign to every mapping f: $X \to K(\Pi, n)$ (actually, to the class of mappings homotopic to f) the element $f^*(e) \in H^n(X; \Pi)$. This correspondence between $H^n(X; \Pi)$ and $\Pi(X; K(\Pi, n))$ is mono- and epimorphic, as it was shown. Let a cohomological operation $\varphi \in \mathcal{O}(n, q, \Pi, G)$ be given. Then we have, among others, a mapping $\varphi_{K(\Pi,n)}$: $H^n(K(\Pi, n); \Pi) \to H^q(K(\Pi, n); G)$ and an element $\varphi(e) \in H^q(K(\Pi, n); G)$. As it turns out, the value $\varphi(e)$ determines the operation φ in a one-to-one relationship, i. e. once $\varphi(e)$ is known the whole operation can be reconstructed. In particular, $\varphi(e) = 0$ implies $\varphi \equiv 0$. On the other hand, every element $x \in H^q(K(\Pi, n); G)$ may be represented as $\varphi(e)$ with an operation $\varphi \in \mathcal{O}(n, q, \Pi, G)$ (uniquely defined).

In other words, we are going to prove that the homomorphism

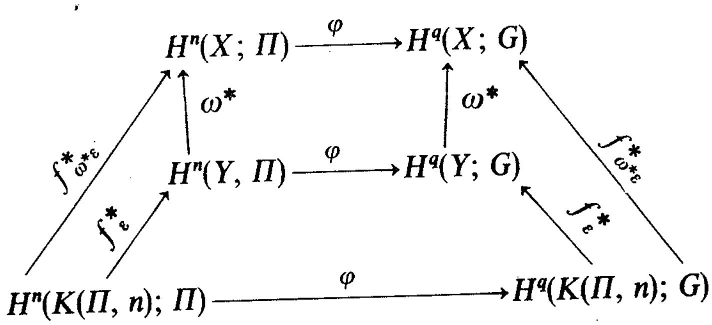
$$\mathcal{O}(n, q, \Pi, G) \to H^q(K(\Pi, n); G),$$

assigning $\varphi(e)$ to φ, is an isomorphism. At first we prove it is a monomorphism.

Let X be an arbitrary CW complex and φ an operation such that $\varphi(e) = 0$, and let $\alpha \in H^n(X; \Pi)$. Then there exists a mapping $f\colon X \to K(\Pi, n)$ such that $f^*(e) = \alpha$. Because φ is natural, $\varphi(\alpha) = \varphi f^*(e) = f^*\varphi(e) = 0$, hence the statement.

Let us examine the epimorphism property. Let x be any element of $H^q(K(\Pi, n); G)$. We set $\varphi(e) = x$ and shall try to extend this correspondence, given on the single element e, to an operation.

Let X be an arbitrary CW complex. Let $\gamma \in H^n(X; \Pi)$; there exists an $f_\gamma\colon X \to K(\Pi, n)$ such that $\gamma = f_\gamma^*(e)$. Define $\varphi(\gamma) = f_\gamma^*(x)$. The mapping is defined; it remained to prove that it is natural. Consider X and Y and $\omega\colon X \to Y$. We have the diagram



We must prove that the square in the centre is commutative i. e. that $\varphi(\omega^*\varepsilon) = \omega^*\varphi(\varepsilon)$. By the definition of φ one has $\varphi(\omega^*\varepsilon) = f^*_{\omega^*\varepsilon}(x)$ and $\varphi(\varepsilon) = f^*_\varepsilon(x)$. Thus we have to prove that $f^*_{\omega^*\varepsilon} = \omega^* f^*_\varepsilon(x)$, i. e. $f^*_{\omega^*\varepsilon}(x) = (f_\varepsilon\omega)^*(x)$.

The mapping $(f_\varepsilon\omega)^*$ sends e into $\omega^*(\varepsilon)$. On the other hand, by the construction of the mapping $f_{\omega^*\varepsilon}, f^*_{\omega^*\varepsilon}(e) = \omega^*(e)$.Then, in view of the theorem about mappings to $K(\Pi, n)$, the mappings $f_\varepsilon\omega\colon X \to K(\Pi, n)$ and $f_{\omega^*\varepsilon}\colon X \to K(\Pi, n)$ are homotopic, i. e. $(f_\varepsilon\omega)^* = f^*_{\omega^*\varepsilon}$; in particular, $f^*_{\omega^*\varepsilon}(x) = \omega^* f^*_\varepsilon(x)$. Q. e. d.

Corollary. A non-trivial cohomology operation will never lower the dimension (i. e. if $0 \neq \varphi \in \mathcal{O}(n, q, \Pi, G)$, then $q \geq n$).

Indeed, $H^q(K(\Pi, n); G) = 0$, for $q < n$, as the complex $K(\Pi, n)$ contains no cells of dimension less than n by construction

Remark. Here is an example of a cohomology operation that is not a homomorphism. Let Π be a commutative ring without elements of degree 2 and n be an even number. Raising to the second power is a mapping $\varphi\colon H^n(X; \Pi) \to H^{2n}(X; \Pi)$ which is obviously no homomorphism. Naturality of φ nevertheless implies that it is a cohomology operation of the type $(n, 2n, \Pi, \Pi)$. It is of course a homomorphism if $\Pi = \mathbf{Z}_2$.

(1) We already know the groups $H^q(K(\Pi, n); \mathbf{Q})$ for all integers n and q and all finitely-generated groups Π. It is possible to interpret these results in terms of cohomology operations. If Π is finite, $H^q(K(\Pi, n); \mathbf{Q}) = 0$ for all $q > 0$. Thus there exist no non-trivial cohomology operations from cohomology with finite coefficients to cohomology with rational coefficients.

If $\Pi = \mathbf{Z}$ and n is odd, then $H^q(K(\Pi, n); \mathbf{Q})$ is only different from zero for $q = n$, when $H^n(K(\Pi, n); \mathbf{Q}) = \mathbf{Q}$. The generator of the group $H^n(K(\Pi, n); \mathbf{Q})$ is the image of the fundamental class $e \in H^n(K(\mathbf{Z}, n); \mathbf{Z})$ under the homomorphism q: $H^n(K(\mathbf{Z}, n); \mathbf{Z}) \to$ $\to H^n(K(\mathbf{Z}, n); \mathbf{Q})$ induced by the natural imbedding $\mathbf{Z} \subset \mathbf{Q}$. Thus every cohomology operation from odd-dimensional integral cohomology to rational preserves dimension, assigning to each element $\alpha \in H^n(X; \mathbf{Z})$ the element $\lambda q(\alpha) \in H^n(X; \mathbf{Q})$ where λ is a rational number fixed for the operation. Finally, if n is even, then $H^*(K(\mathbf{Z}, n); \mathbf{Q}) =$ $= \mathbf{Q}[q(e)]$. Thus every operation from even-dimensional to rational cohomology assigns to each element $\alpha \in H^n(X; \mathbf{Z})$ the element $\lambda\alpha^k \in H^{nk}(X; \mathbf{Q})$ where k is an integer number and $\lambda \in \mathbf{Q}$ ~~an integer~~ respectively fixed for the operation.

Exercise. Prove that for any field F of characteristic 0 any cohomological operation from cohomology with coefficients in F to cohomology with coefficients in F assigns to each element $\alpha \in H^n(X; F)$ the element $\lambda\alpha^{k\ P}\ H^{nk}(X; F)$ where k is an integer and $\lambda \in F$, both of them fixed for the operation.

(2) Let us now interpret the results concerning cohomology modulo p of the spaces $K(\mathbf{Z}_p, n)$ in terms of cohomology operations. First of all, $H^n(K(\mathbf{Z}_p, n); \mathbf{Z}_p) = \mathbf{Z}_p$ and every element of this group is of the form ke where $k \in \mathbf{Z}_p$. Therefore any operation from cohomology with coefficients mod p to cohomology with coefficients mod p, preserving the dimension, is multiplying by a scalar from $\mathbf{Z}_p$.

Further, we have $H^{n+1}(K(\mathbf{Z}_p, n); \mathbf{Z}_p) = \mathbf{Z}_p$. It follows then that for every n there exists a unique (up to a multiplier from $\mathbf{Z}_p$) cohomology operation from the n-dimensional cohomology mod p to the $(n+1)$-dimensional ones. On the other hand it is very easy to construct an example of such operation: the Bockstein homomorphism β. Recall that for $\alpha \in H^n(X; \mathbf{Z}_p)$, $\beta(\alpha) \in H^{n+1}(X; \mathbf{Z}_p)$ is defined in the following way. We take a cocycle $a \in C^n(X; \mathbf{Z}_p)$ representing the element α. It takes the values $0, 1, 2, \ldots,$ $p-1 \in \mathbf{Z}_p$. By considering them as integers we get a cochain $\tilde{a} \in C^n(X; \mathbf{Z})$ whose coboundary $\delta\tilde{a}$ is zero modulo p, i. e. has only values divisible by p. Consider the cochain $\frac{1}{p}\delta\tilde{a}$ and reduce it mod p. We have then a cocycle that represents the element $\beta(\alpha) \in H^{n+1}(X; \mathbf{Z}_p)$. The Bockstein homomorphism obviously defines an operation of $\mathcal{O}(n, n+1, \mathbf{Z}_p, \mathbf{Z}_p)$ which is non-trivial for $n > 0$. (For instance, if X is a complex consisting of two cells σ^n and σ^{n+1} such that $[\sigma^{n+1} : \sigma^n] = p$ then β: $H^n(X; \mathbf{Z}_p) \to$ $\to H^{n+1}(X; \mathbf{Z}_p)$ is an isomorphism.)

We conclude that every operation from cohomology mod p to cohomology mod p increasing the dimension by one has the form $k\beta$ where β is the Bockstein homomorphism and $k \in \mathbf{Z}_p$.

We remark that, by the construction, for any $\alpha \in H^n(X; \mathbf{Z}_p)$, $\beta(\alpha)$ is an integral element of $H^{n+1}(X; \mathbf{Z}_p)$, i. e. it belongs to the image of the reduction homomorphism

$H^{n+1}(X; \mathbf{Z}) \to H^{n+1}(X; \mathbf{Z}_p)$. Actually we have already used this in proving the integrity of the element $d \in H^3(K(\mathbf{Z}_p, 2); \mathbf{Z}_p)$ (in the previous Section).

Because $H^q(K(\mathbf{Z}_p, n); \mathbf{Z}_p) = 0$ for $n+1 < q < n+2p-2$, no operations increasing the dimension by 2, 3, 4, ..., $2p-3$ exist. There is a unique (up to a multiplier) operation increasing the dimension by $2p-1$. (Indeed,

$$\mathcal{O}(n, n+2p-2, \mathbf{Z}_p, \mathbf{Z}_p) = H^{n+2p-2}(K(\mathbf{Z}_p, n); \mathbf{Z}_p) = \mathbf{Z}_p$$

for $n > p$.) It is called the *reduced Steenrod power* and is denoted by P^1 (or sometimes by St^{2p-2}). We also know that (for $n > 4p-5$) $H^{n+2p-1}(K(\mathbf{Z}_p, n); \mathbf{Z}_p) = \mathbf{Z}_p \oplus \mathbf{Z}_p$, $H^{n+2p}(K(\mathbf{Z}_p, n); \mathbf{Z}_p) = \mathbf{Z}_p$ and $H^{n+q}(K(\mathbf{Z}_p, n); \mathbf{Z}_p) = 0$ for $2p < q < 4p-4$. As it turns out the generators in $\mathcal{O}(n, n+2p-1, \mathbf{Z}_p, \mathbf{Z}_p) = \mathbf{Z}_p \oplus \mathbf{Z}_p$ and $H^{n+2p}(n, n+2p, \mathbf{Z}_p, \mathbf{Z}_p) = \mathbf{Z}_p$ are not quite new operations but superpositions of βP^1, $P^1\beta$ and $\beta P^1 \beta$. There are no operations at all increasing the dimension by $2p+1, 2p+2, \ldots, 4p-5$. There exists, however, an operation increasing the dimension by $4p-4$ (and is denoted by P^2). In §28 we shall give a complete classification of the operations of $\mathcal{O}(n, q, \mathbf{Z}_p, \mathbf{Z}_p)$, also proving it in the case $p = 2$.

§27 STABLE OPERATIONS

A *stable cohomology operation* from cohomology with coefficient in Π to cohomology with coefficients in G, increasing the dimension by q, is a sequence of cohomology operations $\varphi_n \in \mathcal{O}(n, n+q, \Pi, G)$ defined for $n = 1, 2, 3, \ldots$ such that for any complex X and number n the diagram

$$\begin{array}{ccc} \Sigma: H^n(X; \Pi) & \longrightarrow & H^{n+1}(\Sigma X; \Pi) \\ \downarrow \varphi_n & & \downarrow \varphi_{n+1} \\ \Sigma: H^{n+q}(X; G) & \longrightarrow & H^{n+q+1}(\Sigma X; G) \end{array}$$

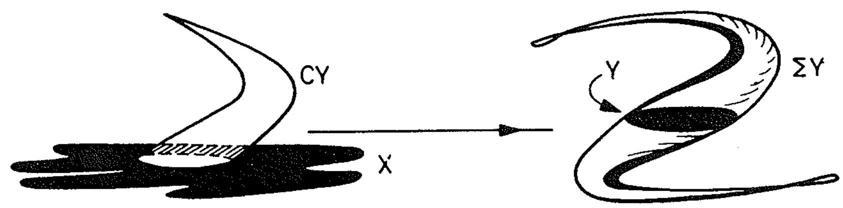
is commutative. (Here Σ denotes the suspension isomorphism.)

Theorem. Any stable cohomology operation commutes with the exact sequence of a CW-pair, i. e. for any (X, Y) the diagram

$$\begin{array}{ccccccccc} \longrightarrow H^n(X, \Pi) & \xrightarrow{i^*} & H^n(Y, \Pi) & \xrightarrow{\delta} & H^{n+1}(X/Y, \Pi) & \xrightarrow{j^*} & H^{n+1}(X, \Pi) \longrightarrow \\ \downarrow \varphi_n & & \downarrow \varphi_n & & \downarrow \varphi_{n+1} & & \downarrow \varphi_{n+1} \\ \longrightarrow H^{n+q}(X, G) & \xrightarrow{i^*} & H^{n+q}(Y, G) & \xrightarrow{\delta} & H^{n+q}(X/Y, G) & \xrightarrow{j^*} & H^{n+q+1}(X, G) \longrightarrow \end{array}$$

is commutative.

Proof. The squares containing i^* and j^* are commutative by the naturality of cohomology operations. It remains to examine the square that contains the coboundary operator δ. Because (X, Y) is a Borsuk pair, we have $X/Y \approx \approx X \cup CY/CY \approx X \cup CY$. Consider the mapping $f\colon X \cup CY \to \Sigma Y$ $(X \cup CY \to \to X \cup CY/X = \Sigma Y)$



and the induced homomorphism $f^*\colon H^{n+1}(\Sigma Y;\ \Pi) \to H^{n+1}(X/Y;\ \Pi)$.

We show that the diagram

$$\begin{array}{ccc} H^n(Y;\ \Pi) & \xrightarrow{\ \delta\ } & H^{n+1}(X, Y;\ \Pi) \\ \downarrow \Sigma & & \downarrow (=) \\ H^{n+1}(\Sigma Y;\ \Pi) & \xrightarrow{\ f^*\ } & H^{n+1}(X \cup CY;\ \Pi) \end{array}$$

is anticommutative, i. e. $f^*\Sigma = -\delta$. We take an element $\zeta \in H^n(Y;\ \Pi)$ and choose a cellular cocycle z representing ζ. Consider the cocycle $\bar{z} \in \mathscr{C}^n(X;\ \Pi)$ which on the cells of Y takes the same values as z and vanishes on the cells of $X \setminus Y$. The cochain $\delta\bar{z}$ is a relative cocycle of X mod Y and defines in $H^{n+1}(X, Y;\ \Pi)$ an element equal to $\delta\zeta$. Let the cochain $\overline{\delta\bar{z}} \in \mathscr{C}^{n+1}(X \cup CY;\ \Pi)$ be defined as the extension of $\delta\bar{z}$ to $X \cup CY$ vanishing on the cells of CY. It is actually a cocycle representing $\delta\zeta$ in $H^{n+1}(X \cup CY;\ \Pi)$. Next we go along the two other sides of the square. Remind that the cells of ΣY are suspensions over the cells of Y.

The cochain $\Sigma\bar{z}$ takes the same value on the cell $\Sigma\sigma$ as $\bar{z}$ on the cell σ. Finally, the cochain $f^*\Sigma z \in \mathscr{C}^{n+1}(X \cup CY;\ \Pi)$ representing the class $f^*\Sigma\zeta$ is zero on all the cells $X \subset X \cup CY$ and is equal to $\bar{z}(\sigma)$ on the cell of CY over $\sigma \subset Y$.

It remains to compare the cochains $f^*\Sigma z$ and $\overline{\delta\bar{z}}$. Consider the cochain $\bar{\bar{z}} \in \mathscr{C}^n(X \cup CY;\ \Pi)$ that coincides with z on $Y \subset X \cup CY$ and vanishes on the cells of $(X \cup CY) \setminus Y$. Clearly $\delta\bar{\bar{z}} = \overline{\delta\bar{z}} + f^*\Sigma z$, hence $\delta\zeta + f^*\Sigma\zeta = 0$.

Now the stable operation φ commutes with Σ and f^*, therefore it commutes with the homomorphism δ as well. Q.e.d.

An important corollary of this theorem is transgressiveness of the stable operations. Namely, let φ be a stable operation, (E, B, F, p) a fibraton with simply-connected base, and suppose $\alpha \in H^r(F;\ G) = E_2^{0r}$ is a transgressive element, i. e. $d_3\alpha = \ldots = d_r\alpha = 0$. Then the element $\varphi(\alpha) \in H^{r+q}(F;\ G) = E_2^{0,r+q}$ is transgressive, too, i. e. $d_3\varphi(\alpha) = \ldots = = d_{r+q}\varphi(\alpha) = 0$. Moreover if $\tau(\alpha) = d_{r+1}\alpha \in E_{r+1}^{r+1,0} = H^{r+1}(B;\ \Pi)/\bigoplus_{s \leq r} \operatorname{Im} d_s$ contains

$\beta \in H^{r+1}(B; \Pi)$ then $\tau(\varphi\alpha)$ contains $\varphi\beta \in H^{r+q+1}(B; G)$ (we may say that the transgression commutes with the operation φ; this is not too exact, but sounds nicely.)

This immediately follows from the representation of the transgression as a composite $H^r(F; \Pi) \xrightarrow{\delta} H^{r+1}(E, F; \Pi) \xrightarrow{(p^*)^{-1}} H^{r+1}(B; \Pi)$ (and the same with G instead of Π) and from the fact that φ commutes with δ and p^*. Indeed, if $\delta\alpha \in \operatorname{Im} p^*$, namely $\delta\alpha = p^*\beta$, then $\delta(\varphi\alpha)$ belongs to Im p^* because $\delta(\varphi\alpha) = \varphi\delta\alpha = \varphi p^*\beta = p^*(\varphi\beta)$.

Let us examine the connection between the groups of the stable operations and the cohomology groups of $K(\pi, n)$.

Let $e \in H^n(K(\pi, n); \pi)$ be the fundamental class. Then $\Sigma e \in H^{n+1}(\Sigma K(\pi, n); \pi)$ gives rise to a mapping $f_n: \Sigma K(\pi, n) \to K(\pi, n+1)$ (we recall that $K(\pi, n) = \Omega K(\pi, n+1)$; the mapping f_n: $\Sigma K(\pi, n) = \Sigma \Omega K(\pi, n+1) \to K(\pi, n+1)$ is defined by $f_n(\varphi, t) = \varphi(t)$ where φ is a loop on $K(\pi, n+1)$, i. e. $\varphi(t) \in K(\pi, n+1)$; $t \in [0, 1]$).

Thus, if $e \in H^n(K(\pi, n); \pi)$ and $e' \in H^{n+1}(K(\pi, n+1); \pi)$ are the fundamental classes, then $f_n^*(e') = \Sigma e$.

Consider a stable operation φ; applying it to the fundamental class e we get $\varphi(e) \in H^{n+q}(K(\pi, n); G)$. As φ is a stable operation, $\varphi(\Sigma e) = \Sigma(\varphi e) \in H^{n+q+1}(\Sigma K(\pi, n); G)$. Now the mapping f_n^* sends e' to Σe, so $\varphi(e') \in H^{n+q+1}(K(\pi, n+1); G)$ is sent to $\varphi(\Sigma e)$. The homomorphism f_n^*: $H^*(K(\pi, n+1); G) \to H^*(\Sigma K(\pi, n); G)$ may be regarded as one decreasing the dimensions by one unit:

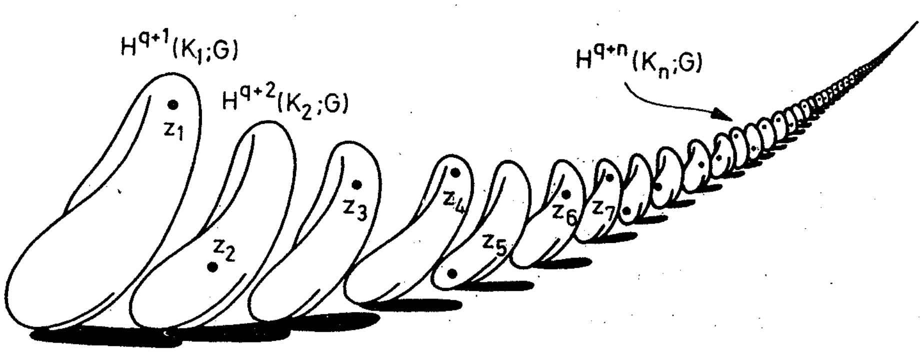
$$H^{n+q+1}(K(\pi, n+1); G) \to H^{n+q}(K(\pi, n); G), \text{ in view of}$$

$$H^{n+q}(K(\pi, n); G) = H^{n+q+1}(\Sigma K(\pi, n); G).$$

We have the sequence

$$\ldots \to H^{n+q}(K(\pi, n); G) \to \ldots \to H^{q+2}(K(\pi, 2); G) \to H^{q+1}(K(\pi, 1); G).$$

Each arrow is an f_n^*. Given a stable operation φ, there is given some element z_n in each $H^{n+q}(K(\pi, n); G)$ (n is arbitrary) such that the homomorphism f_n^* sends z_{s+1} into z_s. Defining a series of cohomology operations is the same as arbitrarily choosing one element in each group $H^{n+q}(K(\pi, n); G)$. But for an arbitrary series of cohomology operations there are no relations between the terms of the sequence $\{z_n\}$.

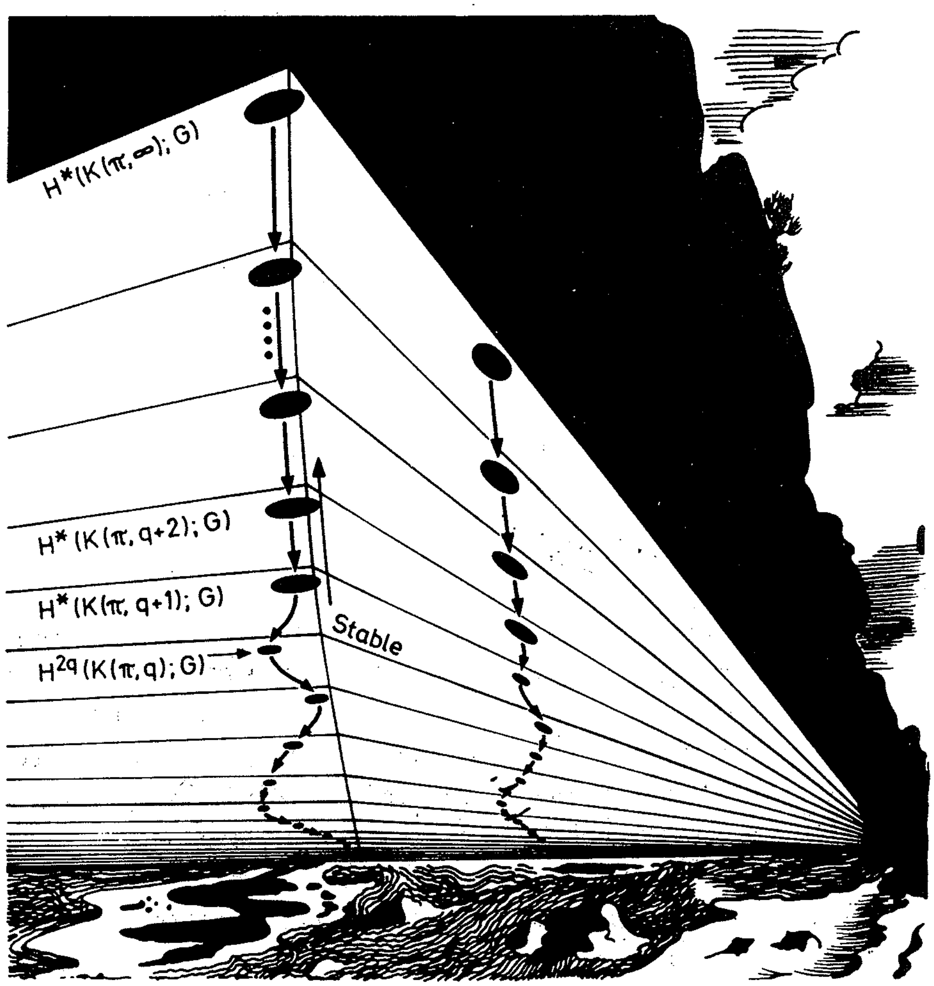


Here we have obtained the condition that distinguishes the stable cohomology operations among the series of operations: the sequence $\{z_n\}$ ($n = 1, 2, 3, \ldots$) must satisfy the condition $z_n = f_n^* z_{n+1}$, i. e. $z_1 = f_1^* z_2$, $z_2 = f_2^* z_3$, ... etc.

Let us formulate the result:

The group $\mathcal{O}^S(q, \Pi, G)$ of all stable cohomology operations which increase the dimensions by q is isomorphic to the inverse limit of the sequence of the groups $H^{q+n}(K(\Pi, n); G)$ and homomorphisms f_n^*.

Let us note that given any $z_s \in H^{q+s}(K(\Pi, s); G)$, all z_k, $k < s$ are automatically defined: $z_k = f_k^* f_{k+1}^* \ldots f_{s-2}^* f_{s-1}^* (z_s)$. So if we are interested only in the action of a



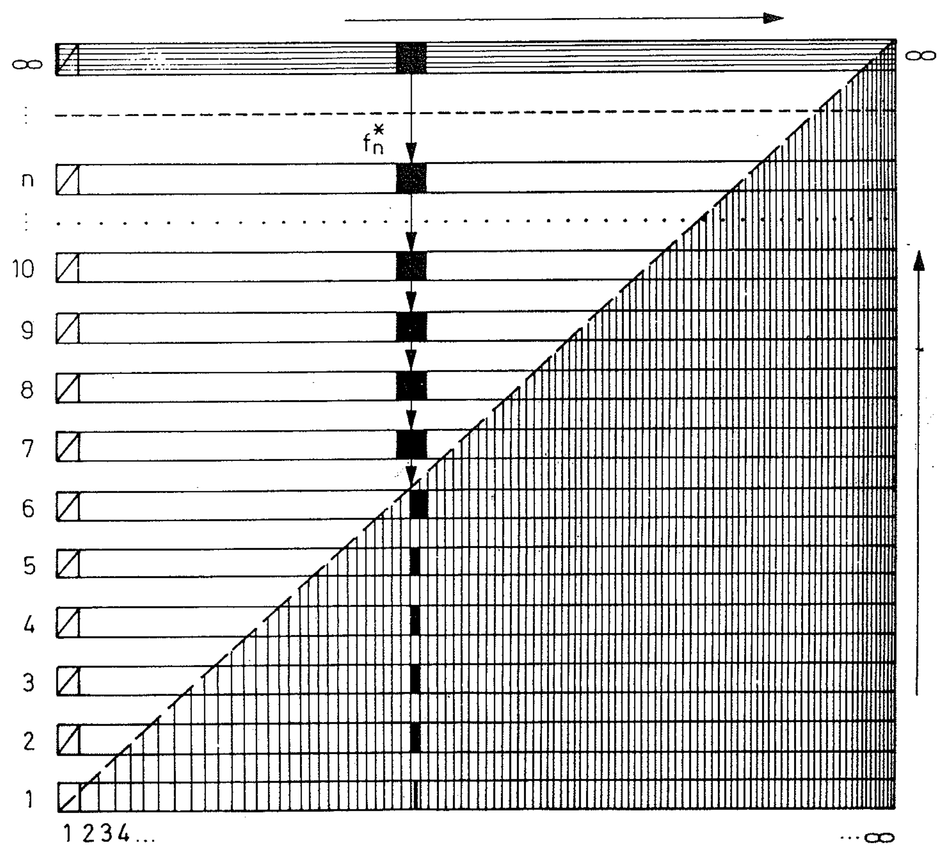
stable operation on the elements whose dimensions are $\leq N$, we may give the operation by giving the single element $z_N \in H^{N+q}(K(\pi, N); G)$.

On the other hand, as $K(\pi, n)$ is $(n-1)$-connected, f^*_{n+1}: $H^{n+q+1}(K(\pi, n+1); G) \to \to H^{n+q}(K(\pi, n); G)$ is an isomorphism for $n > q$. In other words, *with n increasing, each group $H^{q+n}(K(\pi, n); G)$ will stabilize, i. e. cease changing at some N.* Hence

$$\mathcal{O}^S(q, \Pi, G) = H^{N+q}(K(\Pi, N); G)$$

for sufficiently large N (namely for $N > q$).

Here is a diagram for a better explanation of the results:

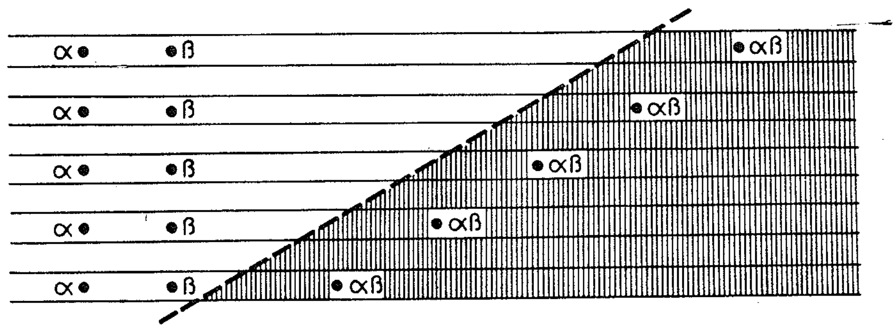


Each row, except that on the top, contains the cohomology of $K(\pi, n)$ with coefficients in G $(n = 1, 2, 3, \ldots)$. On the left end of each row we see the n-th cohomology group (and not the zero-th one). Thus the groups under each other are of different dimensions. On the other hand, the homomorphisms f^*_n are represented by vertical arrows. In the non shaded half of the diagram, all these homomorphisms are isomorphisms, i. e. in each vertical line the groups are identical. The top row consists of these groups which will be reasonably denoted by $H^{\infty+q}(K(\pi, \infty); G)$ with the reservation that they are not the cohomology groups of any space (at any case, not in a natural sense), and are isomorphic to $\mathcal{O}^S(q, \Pi, G)$.

The groups $H^{n+q}(K(\pi, n); G)$ are said to have stable dimension (or simply, to be stable) if $q < n$ and to have unstable dimension (to be unstable) if $q \geq n$. In the diagram the groups of unstable dimensions are shaded.

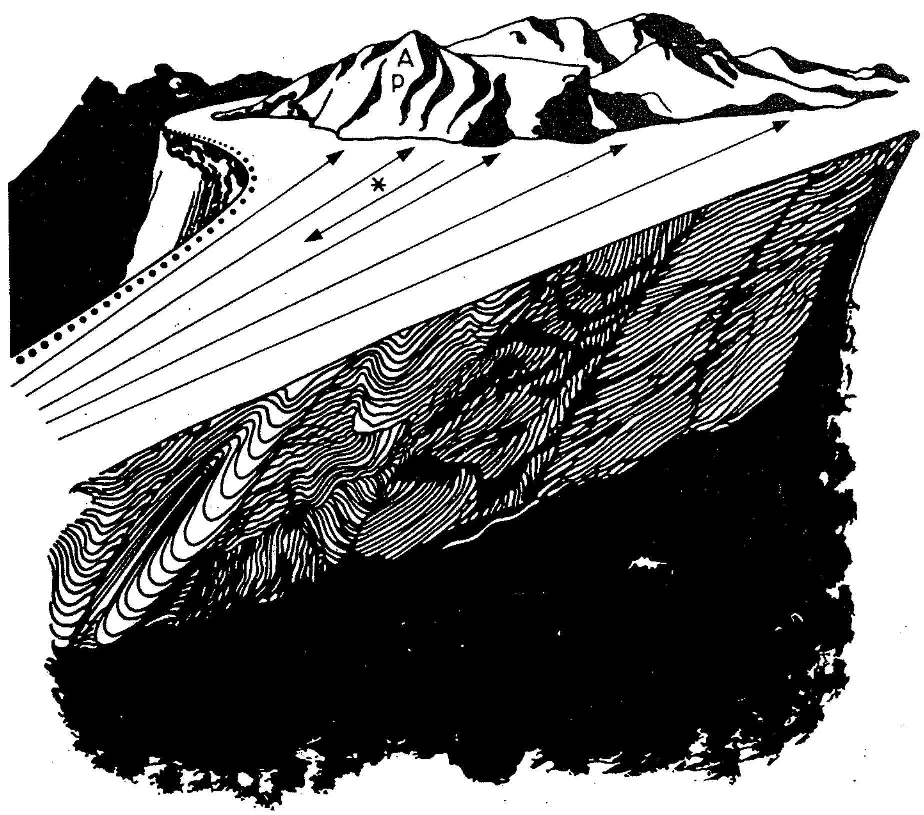
The Steenrod algebra

The multiplicative structure of $H^*(K(\pi, n); G)$ does not induce a similar structure in $\bigoplus_q \mathcal{O}^S(q, \pi, G)$, since the homomorphisms f_n^*: $H^*(K(\pi, n); G) \to H^*(K(\pi, n-1); G)$ are not multiplicative. This is clear by the simple observation that they do not preserve the dimensions, and, in particular, $\dim(f_n^* \alpha f_n^* \beta) = \dim f_n^*(\alpha\beta) - 1$ for any $\alpha, \beta \in H^*(K(\pi, n); G)$;



moreover, as the diagram shows, even if $\alpha, \beta \in H^*(K(\pi, n); G)$ have stable dimensions, $\alpha\beta$ may have unstable dimension.

Nevertheless in the case $\pi = G$ there is another possibility of giving a ring structure to $\bigoplus_q \mathcal{O}^S(q, \pi, G)$. Indeed, for any pair $\varphi' \in \mathcal{O}^S(q', G, G)$, $\varphi'' \in \mathcal{O}^S(q'', G, G)$ we may consider the composite $\varphi' \circ \varphi'' \in \mathcal{O}^S(q' + q'', G, G)$ which is again a stable operation. Multi-



plication defined by the composition turns $\oplus_q \mathcal{O}^S(q, G, G)$ into an associative (non-commutative) ring.

If $G = \pi = \mathbf{Z}_p$, this ring is also a $\mathbf{Z}_p$-algebra. It will be denoted by A_p and called the *Steenrod algebra.* Much in the following §§ will be devoted to a thorough study of it, especially of the case $p=2$.

§28. THE STEENROD SQUARES

Next we construct and examine some particular elements, called Steenrod squares, of the Steenrod algebra A_2.

Steenrod squares are stable cohomology operations (denoted by Sq^i) and at the same time additive homomorphisms

$$Sq^i: H^n(X; \mathbf{Z}_2) \to H^{n+i}(X; \mathbf{Z}_2).$$

(Thus $Sq^i \in H^{n+i}(K(\mathbf{Z}_2, n); \mathbf{Z}_2)$.) They are defined for all $i \geq 0$ and have the following properties:

(1)

$$Sq^i(\alpha) = \begin{cases} 0 & \text{for } i > \deg \alpha, \\ \alpha^2 & \text{for } i = \deg \alpha, \\ \alpha & \text{for } i = 0; \end{cases}$$

(2) (the Cartan formula)

$$Sq^i(\alpha\beta) = \sum_{p+q=i} Sq^p(\alpha) \cdot Sq^q(\beta).$$

Remark. Consider the formal series $Sq = Sq^0 + Sq^1 + \ldots + Sq^i + \ldots$. Then the condition (2) may be written in the following form: $Sq(\alpha\beta) = Sq\alpha\, Sq\beta$, i. e. Sq is a *ring homomorphism* $H^*(X; \mathbf{Z}_2) \to H^*(X; \mathbf{Z}_2)$. The condition (1) may be written in the following form: If α is a homogeneous element of degree k then $Sq(\alpha) = \alpha + Sq^1(\alpha) + \ldots + Sq^{(k-1)}(\alpha) + \alpha^2$.

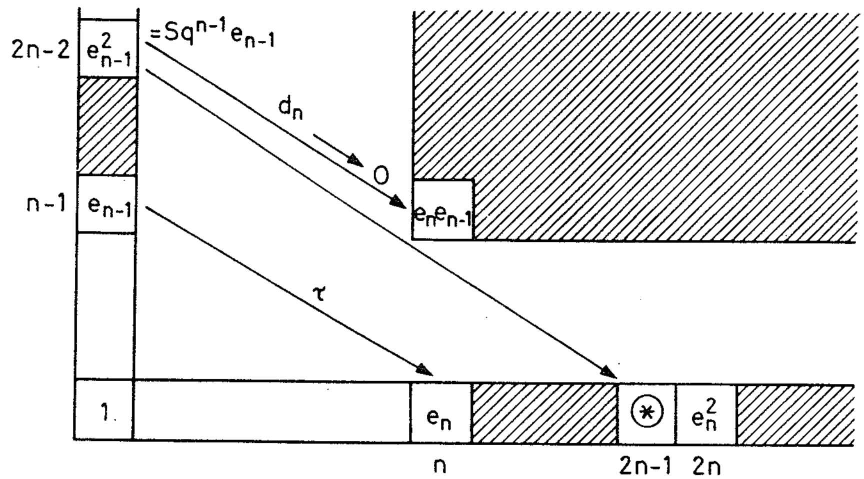
The theorem of existence and unicity of Sq^i

We prove the existence and unicity of the stable cohomology operation satisfying the conditions (1), (2). Unicity actually follows from stability and (1), so the Cartan formula is already their consequence.

Consider the fundamental class $e_n \in H^n(K(\mathbf{Z}_2, n); \mathbf{Z}_2)$. Set $Sq^i(e_n) = 0$ for $i > n$. We want to define $Sq^1(e_n), Sq^2(e_n), \ldots, Sq^{n-1}(e_n), Sq^n(e_n)$. Set $Sq^n(e_n) = e_n^2$.

Why is $e_n^2 \neq 0$? Because there exists at least one CW complex X and an element $0 \neq x \in H^n(X; \mathbf{Z}_2)$ such that $x^2 \neq 0$ (For instance, $X = \mathbf{RP}^\infty$.) (Moreover, all the powers e_n^k, $k \geqslant 1$ are different from zero.)

Let us define $Sq^{n-1}(e_n)$. Consider the fibration $* \sim E \xrightarrow{K(\mathbf{Z}_2, n-1)} K(\mathbf{Z}_2, n)$. We assume that $n>1$, otherwise $K(\mathbf{Z}_2, n-1)$ does not exist. For E_2 we have



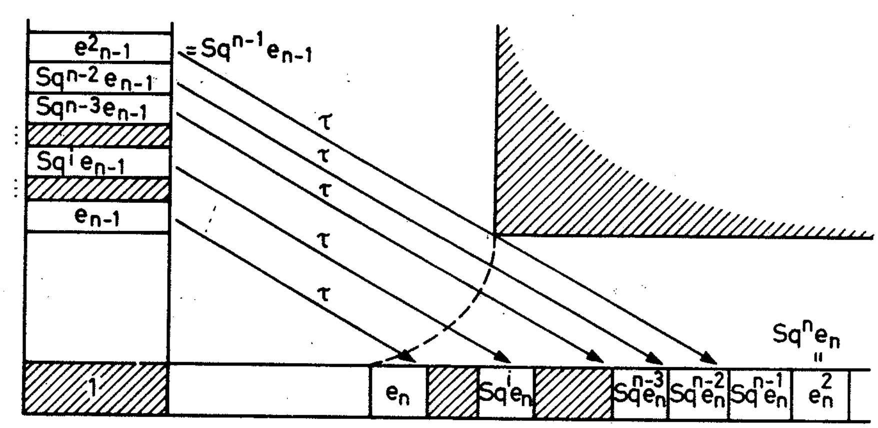
We see that every element under e_{n-1}^2 in ~~row~~ column zero is transgressive. Transgressivity of e_{n-1}^2 does not follow from consideration of the dimensions, as the differential d_n: $E_n^{0,2n-2} \to E_n^{n,n-1}$ is not necessarily trivial.

Nevertheless $d_n(e_{n-1}^2) = 2(d_n e_{n-1})e_{n-1} = 2e_n e_{n-1} = 0$ in $\mathbf{Z}_2$ (the other proof: $E_2^{n,n-1} = E_2^{n,n-1} = \mathbf{Z}_2$ and $e_n e_{n-1}$ is not in the image of $d_n(e_n e_{n-1}) = e_n^2 \neq 0$). Hence $e_{n-1}^2 = Sq^{n-1}e_{n-1}$ is transgressive. It is mapped by the transgression into some element $f \in E_{2n-1}^{2n-1,0}, f \neq 0$. (Actually it *must* be, as this remained the last possibility for it to vanish.) By $E_{2n-1}^{2n-1,0} = E_2^{2n-1,0} = H^{2n-1}(K(\mathbf{Z}_2, n); \mathbf{Z}_2)$ we may write $0 \neq$ $\neq f \in H^{2n-1}(K(\mathbf{Z}_2, n); \mathbf{Z}_2)$. Set $Sq^{n-1}(e_n) = f$. In this way we have defined $Sq^{n-1}(e_n)$ for every $n>1$.

The construction of the remaining $Sq^{n-k}(e_n)$ is very simple.

Let $n>2$; then $Sq^{n-2}(e_{n-1})$ is already defined and belongs to the group $E_2^{0,2n-3}$ of the spectral sequence.

Clearly it is transgressive, and is sent by the transgression to some nonzero element of $E_2^{2n-2,0} = H^{2n-2}(K(\mathbf{Z}_2, n)\, \mathbf{Z}_2)$. We choose this element for the value of the operation Sq^{n-2} on e_n. So we have defined $Sq^{n-2}e_n$ for $n>2$. Let us



make now a step backwards and define $Sq^{n-3}(e_n)$ for every $n>3$ as the image of $Sq^{n-3}(e_{n-1})$ by the transgression; and so on, until $Sq^1(e_n)$ has been reached.

By now we have $Sq^{n-k}e_n \in H^{2n-k}(K(\mathbf{Z}_2, n); \mathbf{Z}_2)$ for every $n>k>0$ (i. e. we defined $Sq^k e_n$ for $k>0$ and $n>0$). In order to make a stable operation the elements $Sq^k(e_n) \in H^{n+k}(K(\mathbf{Z}_2, n); \mathbf{Z}_2)$ must satisfy the equality

$$f_n^*(Sq^k e_n) = Sq^k e_{n-1}$$

for every n. Let it be proved.

(1) For $k \geq n+1$ the equality is obvious, for both sides are zero.

(2) Let $k \leq n-1$. By definition,

$$f_n^*: H^*(K(\mathbf{Z}_2, n); \mathbf{Z}_2) \to H^*(K(\mathbf{Z}_2, n-1); \mathbf{Z}_2)$$

is the composite mapping induced by $\Sigma K(\mathbf{Z}_2, n-1) = \Sigma\Omega K(\mathbf{Z}_2, n) \to K(\mathbf{Z}_2, n)$ and the suspension isomorphism. It was shown in §21 to be the inverse mapping of the transgression τ of the fibration $* \xrightarrow{K(\mathbf{Z}_2, n-1)} K(\mathbf{Z}_2, n)$. Now, by construction, $\tau(Sq^k e_{n-1}) = Sq^k e_n$ for $k \leq n-1$, we have $f_n^*(Sq^k e_n) = Sq^k e_{n-1}$ as needed.

(3) Let $k=n$. Then $Sq^k e_{n-1} = 0$ and we have to show that $f_k^*(Sq^k e_k) = f_k^* e_k^2 = 0$. Again let us recall that f_k^* is a composition $H^*(K(\mathbf{Z}_2, n); \mathbf{Z}_2) \to H^*(\Sigma K(\mathbf{Z}_2, n-1); \mathbf{Z}_2) \to$ $\to H^*(K(\mathbf{Z}_2, n-1), \mathbf{Z}_2)$ in which the first homomorphism is induced by a continuous mapping. It sends e_k into Σe_{k-1} and so e_k^2 into $(\Sigma e_{k-1})^2$. The proof will be completed if we show that $(\Sigma e_{k-1})^2 = 0$. This indeed follows from an elementary observation.

Lemma. In any suspension the cohomology multiplication is trivial. I. e., for any $\alpha \in H^p(\Sigma X; A)$, $\beta \in H^q(\Sigma X, A)$, $p>0$, $q>0$, and any ring A we have $\alpha\beta = 0$.

Clearly the diagonal mapping $\Sigma X \to \Sigma X \times \Sigma X$ is homotopic to the composite $\Sigma X \to$ $\to \Sigma X \vee \Sigma X \subset \Sigma X \times \Sigma X$ (where the first mapping is defined above and the second is the natural imbedding). Constructing the homotopy and deducing the lemma is left to the reader.

It remained to set $Sq^0 e_n = e_n$ for every n to finish the proof of the existence of stable operations Sq^i satisfying (1). Actually we also proved the unicity of such operations, as we *computed* rather than constructed the elements $Sq^k e_n$ by using the equalities $Sq^n e_n =$ $= e_n^2$ and the transgressivity of the operations Sq^k.

The Cartan formula will be proved somewhat later.

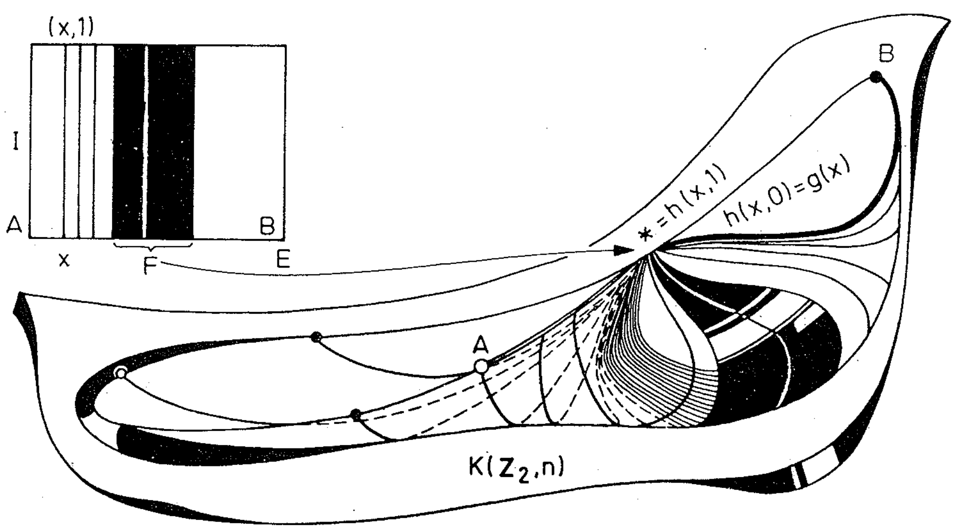
We think it will be worth-while to make the reader acquainted with a direct proof of the transgressivity of Sq^i. If the reader is on the opposite opinion, he may continue reading this book at the proof of the Cartan formula.

Consider an arbitrary Serre fibration $p: E \xrightarrow{F} B$. Choose an element $b \in H^n(B; \mathbf{Z}_2)$. There exists a mapping $f: B \to K(\mathbf{Z}_2, n)$ such that $b = f^*(e_n)$. We want to lift it to a mapping of the fibrations

$$\begin{array}{ccc} E & \xrightarrow{\tilde{f}} & E(K) \\ \Big\downarrow F & \dashrightarrow & \Big\downarrow \; K(\mathbf{Z}_2, n-1) \\ B & \xrightarrow{f} & K(\mathbf{Z}_2, n) \end{array}$$

Clearly that cannot always be done. We only need the case when b represents a coset $\bar{b} \in H^n(B; \mathbf{Z}_2)/\Sigma \operatorname{Im} d_s$ covered by transgression by some element $a \in H^{n-1}(F; \mathbf{Z}_2)$. Because $\bar{b} = \tau(a)$, $p^*(b) = 0$ in $H^n(E; \mathbf{Z}_2)$.

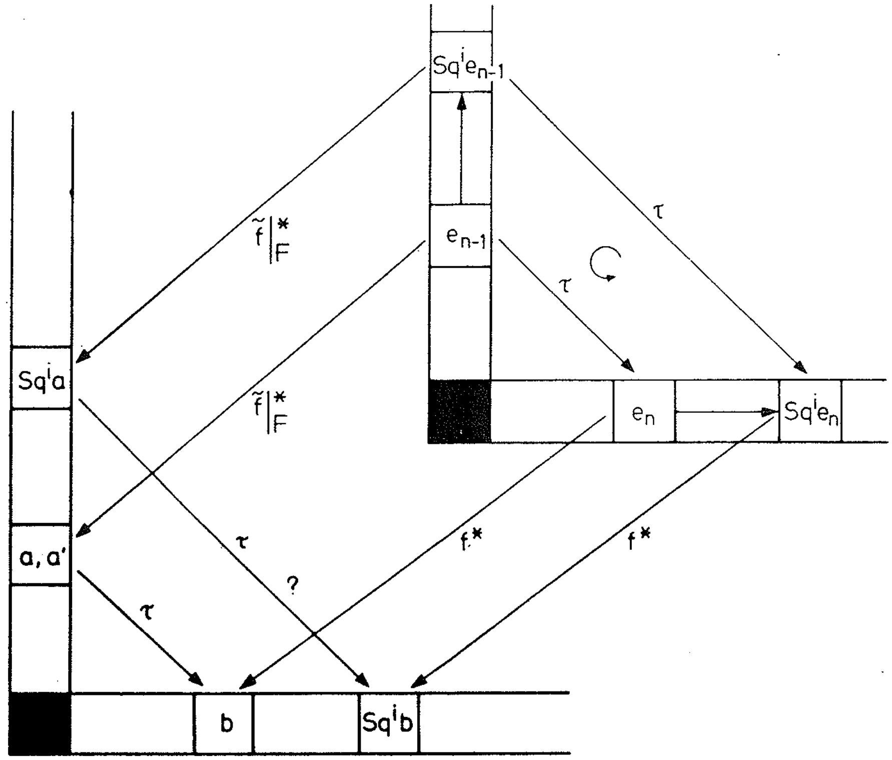
The composite mapping $g = fp$: $E \to K(\mathbf{Z}_2, n)$ is null homotopic, as $g^*(e_n) = (fp)^*(e_n) = p^*(b) = 0$, i. e. there exists a h: $E \times I \to K(\mathbf{Z}_2, n)$ such that $h|_{E\times 0} = g$ and $h(E \times 1) = * \in K(\mathbf{Z}_2, n)$. As $E(K)$ is the space of the paths starting from the point $*$, thus the problem is to assign to each point $x \in E$ a path in $K(\mathbf{Z}_2, n)$ starting from $* = h(E \times 1)$. This is very easy: Let $f(x) = h(x \times I)$; $\tilde{f}$: $E \to E(K)$.



Clearly we have a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\tilde{f}} & E(K) \sim * \\ p \Big\downarrow F & & p' \Big\downarrow K(\mathbf{Z}_2, n-1) \\ B & \xrightarrow{f} & K(\mathbf{Z}_2, n) \end{array}$$

(Indeed, $p' \tilde{f}(x) = h(x, 0) = g(x) = fp(x)$.)
We have then a homomorphism between the two spectral sequences.



Here $f^*(e_n)=b$; $f^*Sq^if^*(e_n)=Sq^i(b)$, $\tau(e_{n-1})=e_n$, $\tau Sq^i(e_{n-1})=Sq^i(e_n)$ because in the universal fibration Sq^i commute with τ *by construction.*

The homomorphism sends e_{n-1} into some $a'\in H^{n-1}(F;\mathbf{Z}_2)$ where $\tau(a')=f^*\tau(e_{n-1})=$ $=f^*(e_n)=b$. The same is true for $a\in H^{n-1}(F;\mathbf{Z}_2)$. If a' and a are equal the proof may be finished easily:

$$\tau(Sq^ia) = f^*(\tau(Sq^ie_{n-1})) = f^*(Sq^ie_n) = Sq^if^*(e_n) = Sq^i(b).$$

However, $a'=a$ is not true in general. We can obtain that, however, by proper use of the freedom left in the construction of $\tilde{f}$.

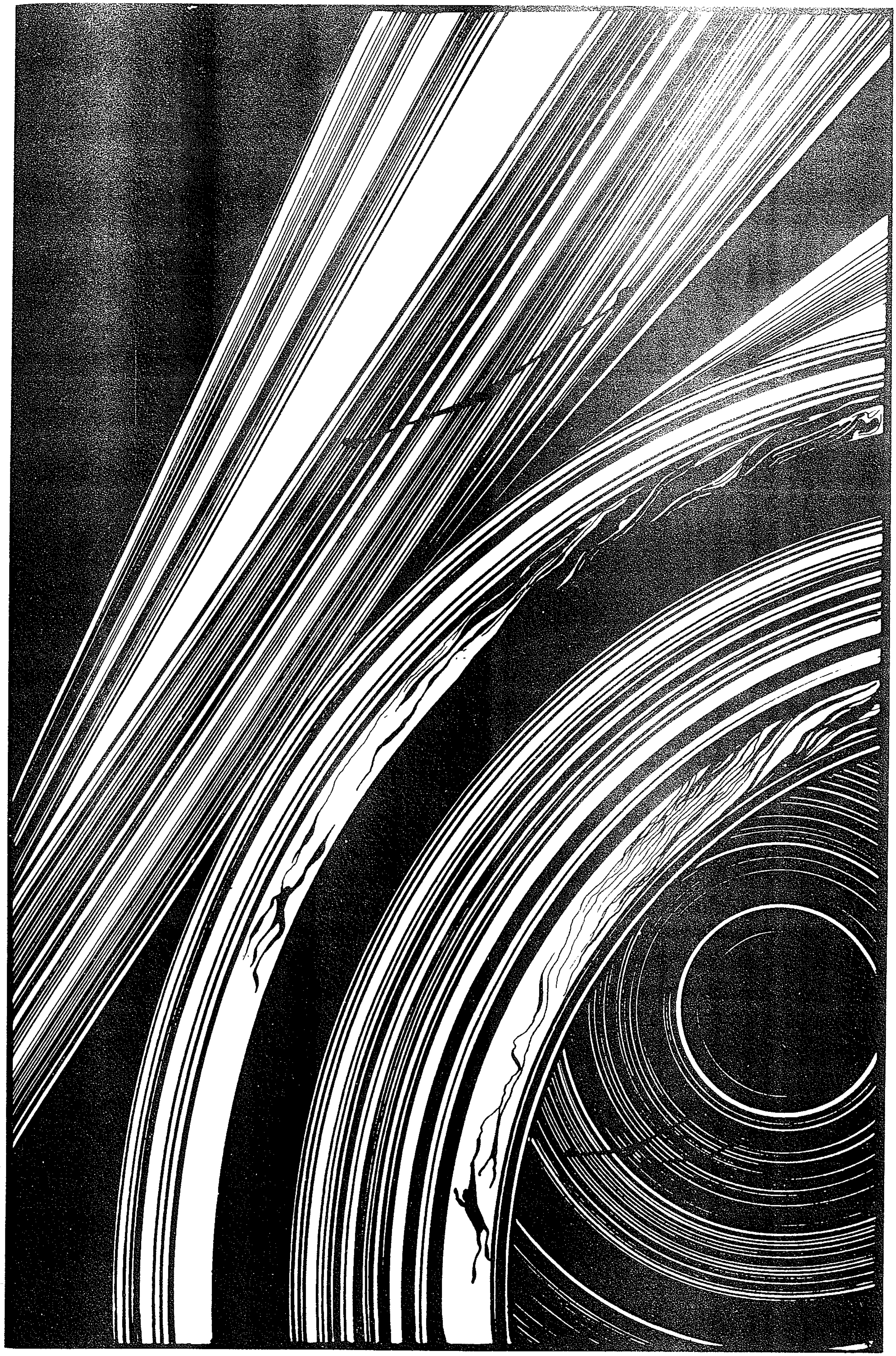
Consider an arbitrary mapping φ: $E\to\Omega K(\mathbf{Z}_2, n) = K(\mathbf{Z}_2, n-1)$. We define a mapping $\tilde{f}'$: $E\to E(K)$ by adding for every $x\in E$ to the path $\tilde{f}(x)$ starting in $*\in K(\mathbf{Z}_2, n)$ and ending in $f(x)\in K(\mathbf{Z}_2, n)$, the loop $\varphi(x)$ with its vertex in $*\in K(\mathbf{Z}_2, n)$. Clearly $\tilde{f}'$ is continuous and by no means worse than $\tilde{f}$ (i. e. it may be substituted for $\tilde{f}$ in the construction). The reader can verify that

$$(\tilde{f}'|_F)^*(e_{n-1}) = (\tilde{f}\,|_F)^*(e_{n-1})+i^*\varphi^*(e_{n-1})$$

where i: $F\to E$ is imbedding of the fibre into the fibred space. Now $(\tilde{f}\,|_F)^*(e_{n-1}) = a'$ while $\varphi^*(e_{n-1})\in H^{n-1}(E)$ by proper choice of the mapping φ: $E\to K(\mathbf{Z}_2, n-1)$, may be any element of $H^{n-1}(E, \mathbf{Z}_2)$. Because $\tau(a-a') = 0$, $a-a'\in H^{n-1}(F;\mathbf{Z}_2)$ belongs to the image of the homomorphism $i^*: H^{n-1}(E;\mathbf{Z}_2)\to H^{n-1}(F;\mathbf{Z}_2)$. So the mapping φ may be chosen such that $i^*\varphi^*(e_{n-1}) = a-a'$ and

$$(\tilde{f}'|_F)^*(e_{n-1}) = a'+(a-a') = a$$

The transgressivity is proved.



Proof of the Cartan formula

We have to prove that for any CW complex X, any number i and any elements x and y of $H^*(X; \mathbf{Z}_2)$ the relation

$$Sq^i(x \cdot y) = \sum_{p+q=i} Sq^p(x) \cdot Sq^q(y) \qquad (\#)$$

holds.

Let X and Y be arbitrary CW complexes and let $x \in H^*(X, \mathbf{Z}_2)$, $y \in H^*(Y, \mathbf{Z}_2)$. As $\mathbf{Z}_2$ is a field, then

$$x \otimes y \in H^*(X \times Y; \mathbf{Z}_2) = H^*(X; \mathbf{Z}_2) \otimes H^*(Y; \mathbf{Z}_2).$$

By the definition of the cohomology multiplication, it is sufficient to prove the following formula:

$$Sq^i(x \otimes y) = \sum_{p+q=i} Sq^p(x) \otimes Sq^q(y)$$

instead of (#).

Clearly it is sufficient to consider homogeneous elements. So let $x \in H^n(X; \mathbf{Z}_2)$ and $y \in H^m(Y; \mathbf{Z}_2)$. There exist mappings $f\colon X \to K(\mathbf{Z}_2, n)$ and $g\colon Y \to K(\mathbf{Z}_2, m)$ such that $f^*(e_n) = x$ and $g^*(e_m) = y$. Now only the following relation is left to be verified:

$$Sq^i(e_m \otimes e_n) = \sum_{p+q=i} Sq^p(e_m) \otimes Sq^q(e_n).$$

Assume at the beginning that $i > m+n$. Then both sides are zero. Now let $i = m+n$. Then

$$Sq^{m+n}(e_m \otimes e_n) = (e_m \otimes e_n)^2 = e_m^2 \otimes e_n^2 = Sq^m(e_m) \otimes Sq^n(e_n).$$

On the right hand side in the whole sum $\sum_{p+q=m+n}$ there is one single term left which is equal to $Sq^m(e_m) \otimes Sq^n(e_n)$. The case $i < m+n$ is left to be examined. Suppose that the formula holds for $i = m+n-(s-1)$; let us prove it for $i = m+n-s$ (s is fixed).

We emphasize that m and n are arbitrary numbers, while the induction is on the difference $m+n-i$.

Let us consider $Sq^{m+n-s}(e_m \otimes e_n)$. As before, we write K_n instead of $K(\mathbf{Z}_2, n)$. Take the tensor product $K_m \otimes K_n$ of the complexes K_m and K_n. Let us recall that for a pair of spaces X, Y the tensor product is defined as $X \times Y / X \vee Y$; for example, $S^p \otimes X = \Sigma^p X$, and in particular, $S^1 \otimes X = \Sigma X$.

By the associativity of the tensor product we have $(S^1 \otimes X) \otimes Y = S^1 \otimes (X \otimes Y)$, i. e. $(\Sigma X) \otimes Y = \Sigma(X \otimes Y)$. Moreover $H^*(X \otimes Y) = H^*(X \times Y)/J$ where J is the subgroup generated by the elements $x \otimes 1 + 1 \otimes y$ where $x \in H^*(X)$ and $y \in H^*(Y)$.

In §27 we constructed a mapping $f_{m-1}\colon \Sigma K_{m-1} \to K_m$. The following pair of mappings

$$K_m \otimes K_n \begin{array}{l} \xleftarrow{f_{m-1} \otimes 1_{K_n}} (\Sigma K_{m-1}) \otimes K_n = \Sigma(K_{m-1} \otimes K_n) \\ \xleftarrow{1_{K_m} \otimes f_{n-1}} K_m \otimes (\Sigma K_{n-1}) = \Sigma(K_m \otimes K_{n-1}) \end{array}$$

gives rise to a pair of homomorphisms

$$H^r(K_m \otimes K_n; \mathbf{Z}_2) \begin{array}{l} \xrightarrow{(f_{m-1} \otimes 1_{K_n})^*} H^r(\Sigma(K_{m-1} \otimes K_n); \mathbf{Z}_2) \\ \xrightarrow{(1_{K_m} \otimes f_{n-1})^*} H^r(\Sigma(K_m \otimes K_{n-1}); \mathbf{Z}_2) \end{array}$$

or

$$H^r(K_m \otimes K_n; \mathbf{Z}_2) \begin{array}{l} \xrightarrow{\Sigma^{-1}(f_{m-1} \otimes 1_{K_n})^*} H^{r-1}(K_{m-1} \otimes K_n; \mathbf{Z}_2) \\ \xrightarrow{\Sigma^{-1}(1_{K_m} \otimes f_{n-1})^*} H^{r-1}(K_m \otimes K_{n-1}; \mathbf{Z}_2) \end{array}$$

where Σ is the suspension isomorphism. The homomorphisms $\Sigma^{-1}(f_{m-1} \otimes 1_{K_n})^*$ and $\Sigma^{-1}(1_{K_m} \otimes f_{n-1})^*$ send $e_m \otimes e_n$ into $e_{m-1} \otimes e_n$ and $e_m \otimes e_{n-1}$, respectively. As shown in §27, $f^*_{m-1}: H^q(K_m) \to H^q(\Sigma K_{m-1}) = H^{q-1}(K_{m-1})$ is the inverse of the transgression in the spectral sequence of the Serre fibration of K_m, and is an isomorphism for $0 < q < 2m$.

Consider the intersection

$$[\text{Ker } \Sigma^{-1}(f_{m-1} \otimes 1)^*] \cap [\text{Ker } \Sigma^{-1}(1_{K_m} \otimes f_{n-1})^*]$$

in $H^r(K_m \otimes K_n; \mathbf{Z}_2)$. Let $\rho = \Sigma_i \alpha_i \otimes \beta_i$ be in this set.

Then $\Sigma_i (f^*_{m-1} \alpha_i) \otimes \beta_i = 0$ and $\Sigma_i \alpha_i \otimes (f^*_{n-1} \beta_i) = 0$, hence $f^*_{m-1} \alpha_i = 0$ and $f^*_{n-1} \beta_i = 0$ for every i. Because ρ is in the cohomology of $K_m \otimes K_n$, we may assume the elements α_i and β_i to be different from 1, and in consequence $\deg \alpha_i \geq 2m$, $\deg \beta_i \geq 2n$, and $\deg \rho \geq 2(m+n)$. We have proved that the intersection of the kernels contains no elements of degree smaller than $2(m+n)$.

Now it is time to return to the Cartan formula. Suppose that

$$\rho = Sq^{m+n-s}(e_m \otimes e_n) - \sum_{p+q=m+n-s} Sq^p(e_m) \otimes Sq^q(e_n)$$

is different from zero in $H^{2(m+n)-s}(K_m \otimes K_n; \mathbf{Z}_2)$.

We have

$$\Sigma^{-1}(f_{m-1} \otimes 1_{K_n})^* \rho = Sq^{m+n-s}(\Sigma^{-1} f^*_{m-1} e_m \otimes e_n) -$$

$$- \sum_{p+q=m+n-s} Sq^p(\Sigma^{-1} f^*_{m-1} e_m) \otimes Sq^q e_n =$$

$$= Sq^{m+n-s}(e_{m-1} \otimes e_n) - \sum_{p+q=m+n-s} Sq^p e_{m-1} \otimes Sq^q e_n = 0$$

by the induction. Similarly $\Sigma^{-1}(1_{K_m} \otimes f_{n-1})^* \rho = 0$.

As dim $\rho = 2(m+n)-s < 2(m+n)$, we have $\rho = 0$, which ends the proof of the Cartan formula.

§29. THE STEENROD ALGEBRA

The Steenrod algebra A is the algebra of all stable cohomology operations over the field $\mathbf{Z}_2$, with multiplication defined as by composition of operations. As it will turn out, for a system of multiplicative generators of A we may choose the operations

$$1, Sq^1, Sq^2, Sq^3, \ldots, Sq^n, \ldots$$

i. e. every stable operation is linear combination of composites of the Steenrod squares.

The set of all Steenrod squares does not make a free generating system of A. We can choose for an additive basis of A the set of all iteratives of the Steenrod squares

$$Sq^I = Sq^{i_1} Sq^{i_2} \ldots Sq^{i_{k-1}} Sq^{i_k}$$

such that the numbers of $I = (i_1, i_2, \ldots, i_{k-1}, i_k)$ satisfy the condition $i_1 \geq 2i_2$, $i_2 \geq 2i_3$, $\ldots, i_{k-1} \geq 2i_k$. Such sequences I will be called *admissible*. An iterate Sq^I is *admissible* if I is admissible.

The multiplicative structure of A is defined by the *Adem relations*

$$Sq^a Sq^b = \sum_c \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c.$$

We notice that on the right side we have linear combinations of admissible iterates; actually the iterate of any two Steenrod squares is a combination of two-term admissible iterates. It follows then that any iterate Sq^I may be written as a linear combination of admissible ones.

Indeed, consider all the sequences I with the sum $\sum_{j=1}^{k} i_j$ equal to a fixed positive number. We have a finite set which may be equipped with lexicographic ordering, i. e. $(i'_1, i'_2, \ldots, i'_k) \succ (i_1, i_2, \ldots, i_k)$ whenever $i'_1 = i_1, \ldots, i'_{s-1} = i_{s-1}$ and $i'_s > i_s$ for some s. Consider $Sq^I = Sq^{i_1} Sq^{i_2} \ldots Sq^{i_k}$. Either it is admissible or $i_s < 2i_{s+1}$ for some s. Using the Adem formula for $Sq^{i_s} Sq^{i_{s+1}}$, we replace it by a linear combination of $Sq I'_r$ where $I'_r \succ I$ for each r. Next we replace in a similar way other pairs of neighbouring Steenrod squares. Again the Adem formula guarantees that in each term the index has increased in the lexicographic ordering. Thus the process is finite and ends with a combination of admissible iterates.

As particular cases of the Adem relations we have

$$Sq^1 Sq^k = \begin{cases} Sq^{k+1} & \text{for } k \text{ even} \\ 0 & \text{for } k \text{ odd.} \end{cases}$$

For example, $Sq^1 Sq^1 = 0$ as we have already known, as $Sq^1 = \beta$ and $\beta\beta = 0$. For $a = b = 2$ we have $Sq^2 Sq^2 = Sq^3 Sq^1$. By the way we see that the Steenrod algebra is non-commutative: $Sq^1 Sq^2 \neq Sq^2 Sq^1$.

The rest of this Section is devoted to proving all these statements. The main tool will be the Borel theorem.

The Borel theorem

Theorem. Assume that we are given a spectral sequence of some fibration such that
(1) $E_\infty = 0$;
(2) in the fibre we have a skew-symmetric algebra with a system of transgressive generators $a_1, a_2, a_3, \ldots$;
(3) this system is *simple*, i. e. the monomials $a_{i_1} a_{i_2} \ldots a_{i_k}$, $i_1 < i_2 < \ldots < i_k$ make an additive basis of the algebra. (It follows then that either $a_i^2 = 0$ or the elements a_i^2 decompose into sums of monomials, with each term containing each a_j no more than once. For example, if

$$H^*(F; \mathbf{Z}_2) = \Lambda_2(a_1, a_2, \ldots, a_k)$$

then clearly $a_1, a_2, \ldots, a_k$ is a simple generating system. If

$$H^*(F, \mathbf{Z}_2) = \mathbf{Z}_2[a_1, a_2, \ldots, a_k]$$

then again $a_1, \ldots, a_k$; $a_1^2, \ldots, a_k^2$; $a_1^4, \ldots, a_k^4$; $a_1^8, \ldots, a_k^8$; $\ldots$ is a simple generating system.)

Now the Borel theorem claims that if conditions (1)–(3) are satisfied, then in the base we have the algebra of polynomials of the generators $b_i = \tau(a_i)$.

Proof. We assume that the dimension of the generators a_i is non-decreasing with the index i.

We are going to construct some abstract spectral sequence $(\tilde{E}_r^{p,q}; \tilde{d}_r)$ satisfying the conditions and then to prove that it coincides with the original.

Let $\tilde{A}$ denote the algebra in the fibre of the spectral sequence. Consider the tensor product $\tilde{A} \otimes_{\mathbf{Z}_2} \mathbf{Z}_2[b_1, b_2, b_3, \ldots]$ where the generators $b_1, b_2, b_3, \ldots$ are in a one-to-one correspondence with $a_1, a_2, a_3, \ldots$ and $\deg b_n = \deg a_n + 1$ $(n = 1, 2, 3, \ldots)$. Let $\tilde{E}_2 = \tilde{A} \otimes_{\mathbf{Z}_2} \mathbf{Z}_2[b_1, b_2, b_3, \ldots]$ (with the natural bigrading).

Next we define the differentials $\tilde{d}_r$. It is natural to make them equal to zero on elements of the bottom row. On the fibre (i. e. on elements of the left column) we set $\tilde{d}_r(a_i) = 0$ for $r < \deg a_i$ (implying that all generators will be transgressive in the new spectral sequence) and $\tilde{d}_r(a_i) = b_i$ if $r = \deg a_i$ (for brevity we shall sometimes omit the sign of tensor product). Now let us be given an element $\alpha \in \tilde{E}_2$. It may be written as a linear combination of elements $a_{i_1} a_{i_2} \ldots a_{i_k} \otimes b_{j_1}^{s_1} b_{j_2}^{s_2} \ldots b_{j_p}^{s_p}$ where $i_1 < i_2 < \ldots < i_k$; $j_1 < \ldots < j_p$; $s_m > 0$.

Set $\tilde{d}_r(a_{i_1}\ldots a_{i_k}\otimes b_{j_1}^{s_1}\ldots b_{j_p}^{s_p})=$

$$=\begin{cases}0 & \text{for } r<\deg a_{i_1} \text{ if } i_1\le j_1,\\ a_{i_2}\ldots a_{i_k}\otimes b_{i_1}b_{j_1}^{s_1}\ldots b_{j_p}^{s_p} & \text{for } r=\deg a_{i_1} \text{ if } i_1\le j_1,\\ 0 & \text{for all } r \quad \text{if } i>j_1.\end{cases}$$

Clearly we have an (additive) spectral sequence. In other words, $\tilde{d}_2\tilde{d}_2==0$; we set $\tilde{E}_3=\operatorname{Ker}\tilde{d}_2/\operatorname{Im}\tilde{d}_2$; the differential $\tilde{d}_3$ is defined as usual; then $\tilde{d}_3\tilde{d}_3=0$, and we set $\tilde{E}_4=\operatorname{Ker}\tilde{d}_3/\operatorname{Im}\tilde{d}_3$, etc. Here $E_\infty=0$ (every generator annulled by all differentials belongs to the image of some differentials). It remains to verify the existence of multiplication. Leaving the details to the reader we only consider the crucial equality

$$\tilde{d}_r(AA')=\tilde{d}_r(A)A+A\tilde{d}_r(A')$$

where $A=a_{i_1}\ldots a_{i_k}\otimes b_{j_1}^{s_1}\ldots b_{j_p}^{s_p}$, $A'=a_{i'_1}\ldots a_{i'_k}\otimes b_{j'_1}^{s'_1}\ldots b_{j'_p}^{s'_p}$, $i_1=i'_1=r$, $i_1<j_1$, $i_1<j'_1$. We have

$$\tilde{d}_r(A)\cdot A'=(a_{i_2}\ldots a_{i_k}\otimes b_{i_1}b_{j_1}^{s_1}\ldots b_{j_p}^{s_p})(a_{i'_1}a_{i'_2}\ldots a_{i'_k}\otimes b_{j'_1}^{s'_1}\ldots b_{j'_p}^{s'_{p'}})=$$

$$=a_{i_1}(a_{i_2}\ldots a_{i_k})(a_{i'_2}\ldots a_{i'_k})\otimes b_{i_1}(b_{j_1}^{s_1}\ldots b_{j_p}^{s_p})(b_{j'_1}^{s'_1}\ldots b_{j'_p}^{s'_{p'}})$$

and the analoguous formula

$$A\cdot\tilde{d}_r(A')=a_{i_1}(a_{i_2}\ldots a_{i_k})(a_{i'_2}\ldots a_{i'_k})\otimes b_{i_1}(b_{j_1}^{s_1}\ldots b_{j_p}^{s_p})(b_{j'_1}^{s'_1}\ldots b_{j'_p}^{s'_{p'}})$$

i. e. $\tilde{d}_r(A)\cdot A'+A\cdot\tilde{d}_r(A')=0$.

We must prove $\tilde{d}_r(AA')=0$. But $AA'=a_{i_1}^2\ldots\otimes\ldots$ We show that $a_{i_1}^2$ is a sum of generators $a_{l_1}+\ldots+a_{l_m}$.

Suppose it is not so, i. e. $a_{i_1}^2$ is a sum of products of the type $a_{l_1}\ldots a_{l_m}$ with $l_1<i_1$. This contradicts the conditions of the theorem, as on the one hand $d_{l_1}a_{i_1}^2==2d_{l_1}a_{i_1}=0$, on the other hand $d_{l_1}(a_{l_1}\ldots a_{l_m})=a_{l_2}\ldots a_{l_m}\otimes b_{l_1}$ (the differentials in consideration are of the original spectral sequence).

Thus AA' begins with a factor whose index is larger than i_1, so $\tilde{d}_{i_1}(AA')=0$.

We are almost ready. We have two spectral sequences. Both of them are multiplicative, with $\acute{E}_\infty=0$, and have identical left columns in the E_2-term. The bottom row of E_2 is $H^*(B;\mathbf{Z}_2)$ in the case of the first sequence, and something equal to $H^*(B;\mathbf{Z}_2)$ by the Borel theorem, in the second case.

Suppose that $H^*(B;\mathbf{Z}_2)$ is isomorphic to the algebra $\mathbf{Z}_2[b_1,b_2,\ldots]$ (where $b_i==\tau(a_i)\in H^*(B;\mathbf{Z}_2)$) up to the dimension q, i.e. in the dimension q there occurs in $H^*(B;\mathbf{Z}_2)$ either a new generator c or a relation which we do not have in $\mathbf{Z}_2[b_1,b_2,\ldots]$. Then both E_2-terms have the same columns up to the q-th one; the differentials are also identical. The difference in column q results that $E_\infty\neq0$: either the new generator remains in E_∞ or there will be left in E_∞ an element which in the constructed spectral sequence is mapped onto an element which is zero in $H^*(B; \mathbf{Z}_2)$ because of the relation. Q.e.d.

Let us return to the Steenrod algebra A.

Theorem (Serre). $H^*(K(\mathbf{Z}_2, n); \mathbf{Z}_2)$ is the polynomial algebra of the generators $Sq^I e_n$ where $e_n \in H^n(K(\mathbf{Z}_2, n); \mathbf{Z}_2)$ is the fundamental class, $I = (i_1, i_2, \ldots, i_k)$ is admissible, i.e. $i_1 \geq 2i_2$, $i_2 \geq 2i_3, \ldots, i_{k-1} \geq 2i_k$ and $\operatorname{exc} I < n$. Here $\operatorname{exc} I$ means the "excess" of the sequence I defined by $\operatorname{exc} I = (i_1 - 2i_2) + (i_2 - 2i_3) + \ldots + (i_{k-1} - 2i_k) + i_k = 2i_1 - (i_1 + i_2 + + \ldots + i_k)$.

Remark 1. $\operatorname{exc} I \geq 1$ for $I \neq (0)$.

Remark 2. Let us examine the condition $\operatorname{exc} I < n$. Consider an iterate Sq^I on the element e_n: $Sq^{i_1} Sq^{i_2} \ldots Sq^{i_k}(e_n)$. Suppose that all Sq^{i_s} (as operations) are not identically zero on the preceding element $Sq^{i_{s+1}} \ldots Sq^{i_k}(e_n)$. We get the inequalities $i_k \leqslant n$, $i_{k-1} \leqslant \leqslant n + i_k$, $i_{k-2} \leqslant n + i_k + i_{k-1}, \ldots, i_1 \leqslant n + i_k + i_{k-1} + \ldots + i_2$; the last inequality implying that $2i_1 \leq n + |I|$, i. e. $\operatorname{exc} I \leq n$.

Proof (by induction).

For $n = 1$ the statement of the Serre theorem is obvious. Indeed, $K(\mathbf{Z}_2, 1) = \mathbf{RP}^\infty$ while $H^*(\mathbf{RP}^\infty; \mathbf{Z}_2) = \mathbf{Z}_2[e_1]$, $\deg e_1 = 1$ as already shown. There exists no admissible sequence of excess < 1 except (0).

Suppose that the theorem is proved for every $k \leq n - 1$. Consider the spectral sequence of the fibration $* \xrightarrow{K_{n-1}} K_n$. Let $|I|$ denote the sum $i_1 + i_2 + i_3 + \ldots + i_k$; the iteration Sq^I will raise the dimension by $|I|$. By definition, $\operatorname{exc} I = 2i_1 - |I|$.

By the induction hypothesis $H^*(K_{n-1}; \mathbf{Z}_2)$ is the polynomial algebra of the multiplicative generators $\rho_I = Sq^I(e_{n-1})$ where $\operatorname{exc} I < n - 1$ and the iteration Sq^I is admissible.

Now $E_\infty = 0$; e_{n-1} is transgressive. Thus all $\rho_I = Sq^I e_{n-1}$ are also transgressive. The multiplicative system $\{\rho_I\}$ is not simple. There is, however, a simple generating system in the algebra $\mathbf{Z}_2[\{\rho_I\}] = H^*(K_{n-1}; \mathbf{Z}_2)$, consisting of all elements $\rho_I^{2^i}$, $i \geq 0$. Clearly

$$(\rho_I)^2 = (Sq^I e_{n-1})^2 = Sq^{|I| + n - 1} \circ Sq^I(e_{n-1}),$$

$$(\rho_I)^{2^2} = Sq^{2(|I| + n - 1)} \circ Sq^{|I| + n - 1} \circ Sq^I(e_{n-1})$$

etc., so each power of the form $(\rho_I)^{2^i}$ admits such representation. Therefore all of them are transgressive elements. Now the elements $(\sigma_I)^{2^i}$ do not belong to the original system $\{\rho_I\}$ because for J, defined by $(\rho_I)^{2^i} = Sq^J e_{n-1}$, we have $\operatorname{exc} J = n - 1$.

Conversely, every element $Sq^I(e_{n-1})$ such that $\operatorname{def} I = n - 1$, is of the type $(\rho_{I'})^{2^i}$ with $\operatorname{exc} I' < n - 1$. Indeed, let $I = (i_1, i_2, \ldots, i_k)$, $Sq^I e_{n-1} = Sq^{i_1} \ldots Sq^{i_k} e_{n-1}$. Then $\operatorname{exc} I = = i_1 - (i_2 + i_3 + \ldots + i_k) = n - 1$. Hence $i_1 = (i_2 + i_3 + \ldots + i_k) + n - 1 = \deg(Sq^{i_2} \ldots Sq^{i_k} e_{n-1})$ and $Sq^I e_{n-1} = (Sq^{i_2} \ldots Sq^{i_k} e_{n-1})^2$. Further, either $I' = (i_2, i_3, \ldots, i_k)$ has excess $< n - 1$ and $Sq^I e_{n-1} = (\rho_{I'})^2$ or $\operatorname{exc} I' = n - 1$, $i_2 = \deg(Sq^{i_3} \ldots Sq^{i_k} e_{n-1})$ and $Sq^I e_{n-1} = (Sq^{i_3} \ldots Sq^{i_k} e_{n-1})^4$. Going on this way we get the statement.

It remained to apply the Borel theorem to the spectral sequence, making use of the transgressivity of the operations Sq^I. Q.e.d.

As it has been shown the Steenrod algebra A can be obtained as the limit of $H^*(K(\mathbf{Z}_2, n); \mathbf{Z}_2)$. Thus we have the following result.

Theorem. The operations Sq^I where I is admissible (without restriction on the excess), make an additive basis of the algebra A.

Here the operation Sq^I is trivial on all elements of dimension $<\operatorname{exc} I$. This statement actually generalizes the equality $Sq^n(x) = 0$ for $\dim x < n$.

Let us remind the reader that the multiplication in A has no relation to the multiplicative structure of $H^*(K(\mathbf{Z}_2, n); \mathbf{Z}_2)$, as it is defined by composition of operations.

The above theorem implies that all stable cohomology operations ranging and taking values in the cohomology mod 2 are linear combinations of iterated Steenrod squares. Moreover, it shows that there are many relations between the iterates because every operation may be represented as linear combination of admissible iterates which make only a small part of the set of all iterates. In what follows, we shall study these relations.

Consider the product $X = \Pi_{i=1}^N \mathbf{RP}^\infty$ of N copies of the infinite-dimensional real projective space $\mathbf{RP}^\infty$. Clearly $H^*(X; \mathbf{Z}_2) = \mathbf{Z}_2[x_1, \ldots, x_N]$, the algebra of polynomials of N one-dimensional generators. Let u denote the product $x_1 x_2 \ldots x_N \in H^N(X; \mathbf{Z}_2)$.

There is a natural grading in A: $A = A_0 \oplus A_1 \oplus \ldots$, where $A_q = \mathcal{O}^S(q, \mathbf{Z}_2, \mathbf{Z}_2)$ is the group of the operations increasing the dimension by q. Let q be fixed and take $N \gg q$.

Let $\varphi \in A_q$ and consider the mapping j: $A_q \to H^*(X; \mathbf{Z}_2)$ by setting $j(\varphi) = \varphi(x_1 x_2 \ldots x_N) = \varphi(u)$.

We have the following remarkable fact:

The mapping j is a monomorphism, i. e. if $\varphi(u) = 0$ in $H^*(X; \mathbf{Z}_2)$ then necessarily $\varphi \equiv 0$, i. e. for any complex Y and element $\alpha \in H^p(Y; \mathbf{Z}_2)$ (of any dimension p) the equality $\varphi(\alpha) = 0$ holds.

Let us prove this statement. Consider the subgroup $B_q = j(A_q) \subset H^*(X; \mathbf{Z}_2)$. We shall try to describe the elements of B_q. Again we shall see that naturality is a very strong property implying a lot of the most interesting consequences. Let ψ be an arbitrary cohomology operation (not necessarily stable); then $\psi(x_1 x_2 \ldots x_N)$ is a *symmetric* polynomial of $x_1, x_2, \ldots, x_N$.

To prove this it suffices to consider the mapping f_{ij}: $X \to X$ permuting the i-th and j-th factors.

A further consequence of the naturalness: $\psi(u)$ is divisible by u, i. e. $\psi(u) = uP(x_1, x_2, \ldots, x_N)$ where $\deg P = q$. To prove this consider the imbedding $\underbrace{\mathbf{RP}^\infty \times \ldots \times \mathbf{RP}^\infty}_{N-1} \to X$ where in the left-hand product the i-th factor is omitted. The mapping induced in the cohomology sends $\psi(u)$ to zero, since u is sent to zero. Hence $\psi(u)$ is divisible by x_i, and this is true for every i.

As an arbitrary stable operation is a linear combination of the operations Sq^I, further study of their behaviour can only be carried out by considering iterates of Steenrod squares. We show that in the polynomial $j(\varphi)\in \mathbf{Z}_2[x_1, x_2, \ldots, x_N]$ each x_i has a degree which is a power of 2. Consider $\mathbf{RP}^\infty$. The Cartan formula immediately gives that

$$Sq^i(x^{2^k}) = \begin{cases} x^{2^k} & \text{for } i=0 \\ x^{2^{k+1}} & \text{for } i=2^k \\ 0 & \text{for other } i \end{cases}$$

$(Sq^i(x^j)=\binom{j}{i}x^{i+j})$, so in the polynomial $Sq^I(u)$ the degree of every variable is a power of 2. Thus $j(\varphi)=uP(x_1, x_2, \ldots, x_N)$ where $\deg P=q$, P is symmetric, and every x_i is on the degree a power 2^k-1.

Conversely, each polynomial of that type is the image of some φ by the mapping j. Indeed, consider the polynomials Symm $(x_1^{2^k}\ldots x_{n_1+1}^{2^{k-1}}\ldots x_{n_2}^{2^{k-1}}\ldots x_{n_{k-1}+1}^2 \ldots x_{n_k}^2 x_{n_k+1}\ldots x_N)$, where Symm denotes symmetrisation and $1\le n_1\le n_2\le \ldots \le n_k\le N$ are arbitrary numbers satisfying $n_1(2^k-1)+(n_2-n_1)(2^{k-1}-1)+\ldots+(n_k-n_{k-1})=q$. Such polynomials form an additive basis in the space of all polynomials. The polynomial considered is the highest term in

$$Sq^{2^{k-1}n_1}\ldots Sq^{2n_{k-1}}Sq^{n_k}(u)$$

if decomposed by the basis ordered lexicographically. Hence the statement is immediate.

Thus we have a complete description of B_q. It remained yet to calculate the dimensions of B_q and A_q (as vector spaces over $\mathbf{Z}_2$). To get the dimension of B_q it suffices to count the representations of a given number as sums of integers of the form 2^k-1.

The dimension of A_q is equal to the number of admissible sequences I with $|I|=q$. Let I be any admissible sequence $I=(i_1, i_2, \ldots, i_k)$; $i_1\ge 2i_2, i_2\ge 2i_3, \ldots, i_{k-1}\ge 2i_k$, and let $i_1+i_2+\ldots+i_k=q$. Consider the sequence $\alpha_1=i_1-2i_2$, $\alpha_2=i_2-2i_3, \ldots, \alpha_{k-1}=$ $=i_{k-1}-2i_k$; $\alpha_k=i_k$. Clearly $q=\sum_{p=1}^{k}\alpha_p(2^p-1)$. Any such partition of q defines an admissible sequence.

Again the number of the admissible sequences I with $|I|=q$ is equal to the number of partitions $q=\sum_i(2^{k_i}-1)$. So $\dim B_q=\dim A_q$, i. e. j is a monomorphism, i. e. $A_q\cong B_q$ as stated.

Example. We give a new proof to the relation $Sq^1Sq^1=0$. Indeed,

$$Sq^1Sq^1(x_1x_2\ldots x_N)=$$
$$=Sq^1(x_1^2x_2\ldots x_N+x_1x_2^2x_3\ldots x_N+\ldots+x_1x_2\ldots x_N^2)=$$
$$=\ldots+x_1\ldots x_i^2\ldots x_j^2\ldots x_N+\ldots+x_1\ldots x_i^2\ldots x_j^2\ldots x_N+\ldots=0$$

since each summand has the coefficient 2.

Exercise. Prove that $Sq^2Sq^2=Sq^3Sq^1$.

The Adem relations

As mentioned before, the Adem relations

$$Sq^a Sq^b = \sum_{c=\max(a-b+1,0)}^{[\frac{a}{2}]} \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c \qquad (a<2b)$$

form a complete system of relations in the algebra A. To prove them it suffices to verify that

$$Sq^a Sq^b(u) = \sum_c \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c(u)$$

where $u = x_1 x_2 \dots x_N \in H^N(X; \mathbf{Z})$. By applying the Cartan formula we bring the two sides of the equality to the form

$$\sum_s \binom{b+a-3s}{b-s} \operatorname{Symm}(x_1^4 \dots x_s^4 x_{s+1}^2 \dots x_{a+b-s}^2 x_{a+b-s+1} \dots x_N)$$

(for the left-hand side) and

$$\sum_c \sum_s \binom{b+a-3s}{c-s} \binom{b-c-1}{a-2c} \operatorname{Symm}(x_1^4 \dots x_s^4 x_{s+1}^2 \dots x_{a+b-s}^2 x_{a+b-s+1} \dots x_N)$$

(for the right-hand side).

So we need to verify the congruence

$$\binom{b+a-3s}{b-s} \equiv \sum_c \binom{b+a-3s}{c-s} \binom{b-c-1}{a-2c} \qquad (\text{mod } 2)$$

or, by substituting $d = a-2s$, $e = b-s$, $f = c-s$,

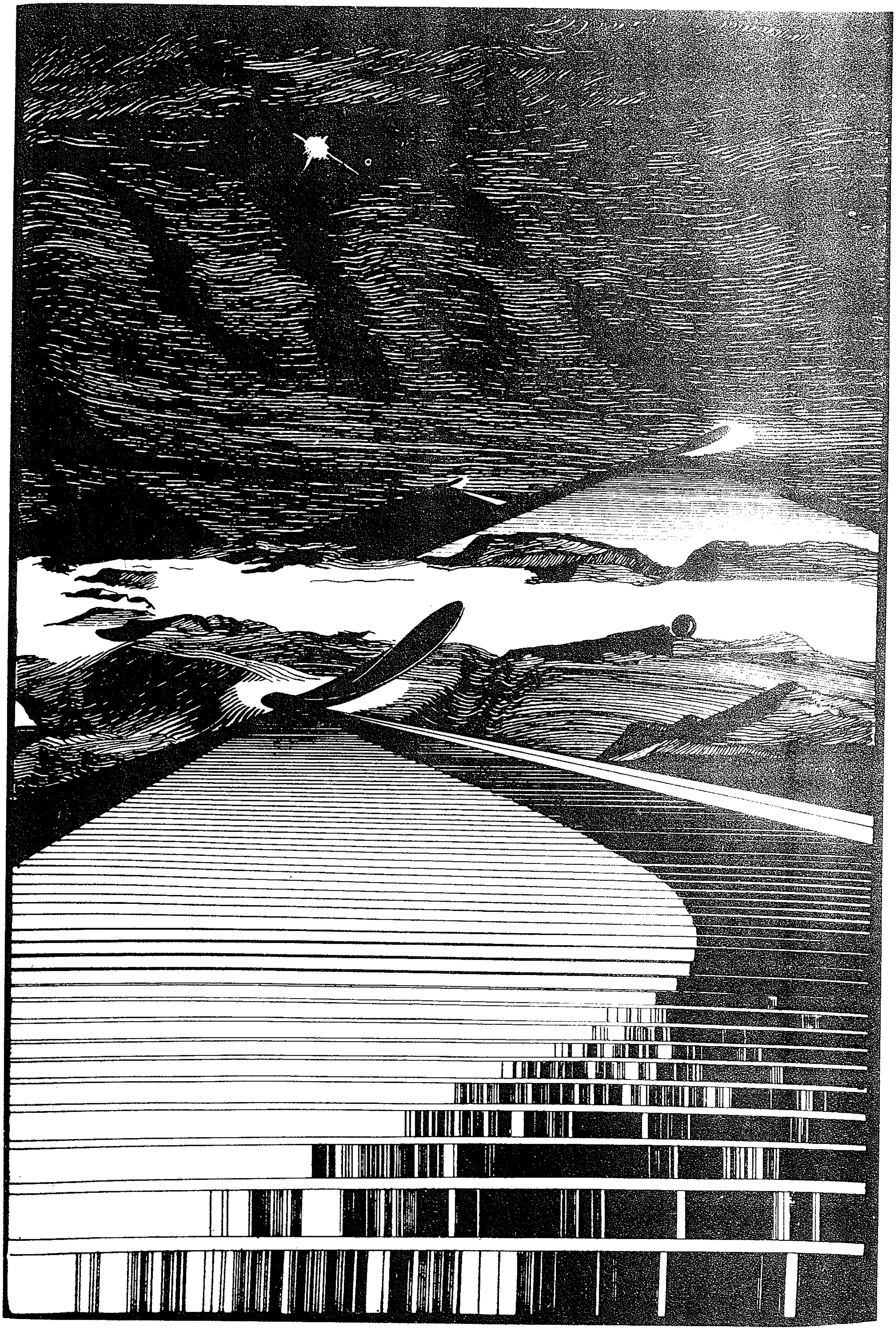
$$\binom{d+e}{e} \equiv \sum_{f=\max(0,d-e+1)}^{[\frac{d}{2}]} \binom{d+e}{f} \binom{e-f-1}{d-2f} \qquad (\text{mod } 2).$$

This can be done by elementary means.

Completeness of the system of Adem relations follows from an earlier remark that any iterate of Steenrod squares can be reduced to a linear combination of admissible ones by using the Adem relations. Because the admissible iterates are proved to be linearly independent, any relation between the iterates is a consequence of the Adem relations. (Let us take a relation $F=0$ and, by using the Adem relations, bring it to the form $F_0=0$ where F_0 is a linear combination of admissible iterates. Then $F_0 \equiv 0$, i. e. F reduces to zero by the Adem relations, i. e. the relation $F=0$ follows from Adem relations.)

Corollary. The system $1, Sq^1, Sq^2, Sq^4, Sq^8, \dots$ is a minimal multiplicative basis of the algebra A.

The proof is left to the reader.



Computing $\oplus_q \mathcal{O}^S(q, \mathbf{Z}, \mathbf{Z}_2)$

The modulo 2 cohomology groups of the spaces $K(\mathbf{Z}, n)$ were determined by Serre at the same time as of $K(\mathbf{Z}_2, n)$, a fact after all not surprising as they need completely analogous computations (induction on n, and application of the Borel theorem).

Theorem. For $n \geq 2$, $H^*(K(\mathbf{Z}, n); \mathbf{Z}_2)$ is the ring of polynomials of the generators $Sq^I \bar{e}_n$ where $\bar{e}_n \in H^n(K(\mathbf{Z}, n); \mathbf{Z}_2)$ is the generator of the cohomology group and $I = (i_1, i_2, \ldots, i_k)$ is any admissible sequence such that $\operatorname{exc} I < n$ and $i_k > 1$. (The last inequality is the only difference between the cases of $K(\mathbf{Z}, n)$ and $K(\mathbf{Z}_2, n)$.)

Passing to the limit we obtain:

Theorem. The group $\oplus_q \mathcal{O}^S(q, \mathbf{Z}, \mathbf{Z}_2)$ considered as a vector space over $\mathbf{Z}_2$ has a basis consisting of all operations Sq^I such that $I = (i_1, i_2, \ldots, i_k)$ is admissible and $i_k > 1$.

The proofs are left to the reader.

Remark 1. We are considering Sq^i as an element of $\mathcal{O}^S(i, \mathbf{Z}, \mathbf{Z}_2)$, i. e. as an operation $H^q(X; \mathbf{Z}) \to H^{q+i}(X; \mathbf{Z}_2)$. Actually Sq^i must not be directly applied to integral elements. We mean that first of all this element is reduced mod 2, i. e. Sq^i and Sq^I are understood to stand for $Sq^i \circ \rho_2$ and $Sq^I \circ \rho_2$ where ρ_2 is the reducing of the integral cohomology mod 2.

Remark 2. One should not believe that Sq^1 acts trivially on $H^*(K(\mathbf{Z}, n); \mathbf{Z}_2)$. There is only the equality $Sq^1 \bar{e}_n = 0$, while, for example, $Sq^1 Sq^2 \bar{e}_n = Sq^3 \bar{e}_n \neq 0$ for $n \geq 3$.

The Steenrod algebra mod p

A theory analogous to the case of $p = 2$ may be developed for the operations $\mathcal{O}^S(q, \mathbf{Z}_p, \mathbf{Z}_p)$ for any prime number $p > 2$.

We may recall one example. Consider the Bockstein homomorphism $\tilde{\beta}_p$ related with the sequence $0 \to \mathbf{Z}_p \to \mathbf{Z}_{p^2} \to \mathbf{Z}_p \to 0$ of the coefficients. Clearly $\Sigma \circ \tilde{\beta}_p = -\tilde{\beta}_p \circ \Sigma$ (this sign was ignored in the case $p = 2$) so the operation β_p given by $\beta_p(x) = (-1)^{\dim x} \tilde{\beta}_p(x)$ is stable. This operation is going to play the role of Sq^1 when $p > 2$.

There also exist operations similar to the other Steenrod squares. Namely there exists a unique stable cohomology operation P_p^i (called a Steenrod power) in $\mathcal{O}^S(2i(p-1); \mathbf{Z}_p, \mathbf{Z}_p)$, $i \geq 0$ such that $P_p^i(x) = x^p$ for $x \in H^{2i}(X; \mathbf{Z}_p)$.

Similarly to the case $p = 2$ the operation P_p^0 is the identity mapping and $P_p^i(x) = 0$ for $\deg x < 2i$. We also agree that $P_2^i = Sq^{2i}$ for $p = 2$.

Let us denote by $A_{q(p)}$ the group $\mathcal{O}^S(q, \mathbf{Z}_p, \mathbf{Z}_p)$ of all stable cohomology operations increasing the dimensions by q. The direct sum $A_{(p)} = A_{0(p)} \oplus A_{1(p)} \oplus A_{2(p)} \oplus \ldots$ will be considered as a vector space over $\mathbf{Z}_p$. Moreover, the composition of operations, as multiplication, provides it with a graded algebra structure. It will be called the Steenrod algebra modulo p. Up to now we have been studying the Steenrod algebra modulo 2: $A = A_{(2)}$. Clearly $A_{0(p)} = \mathbf{Z}_p$, for any operation preserving the dimension is the multiplication by a scalar.

The question of the bases of $A_{(p)}$ arises as it did in the case $p=2$. Let us define the following operations St^k where $k \equiv 0, 1 \pmod{2p-2}$:

$$St^k = \begin{cases} P_p^i & \text{for} \quad k = 2i(p-1) \\ \beta_p \circ P_p^i & \text{for} \quad k = 2i(p-1)+1. \end{cases}$$

Thus far we have been using iterates of Sq^i; we shall now have to deal with iterates of St^k. (For $p=2$, $St^k = Sq^k$.)

Let us be given a sequence $I = (i_1, i_2, \ldots, i_k)$ such that $i_m \equiv 0, 1 \pmod{2p-2}$. We assign to it the operation $St^I = St^{i_1} St^{i_2} \ldots St^{i_k}$.

A sequence I is *admissible* if $i_1 \geq p i_2$, $i_2 \geq p i_3$, $i_3 \geq p i_4$, ...

Theorem. The admissible iterates $\{St^I\}$ form an additive basis of the $\mathbf{Z}_p$-module $A_{(p)}$.

The relations between the operations St^I are generated by the *Adem relations*

$$P_p^a P_p^b = \sum_{c=0}^{[\frac{a}{p}]} (-1)^{c+a} \binom{(p-1)(b-c)-1}{a-pc} P_p^{a+b-c} P_p^c,$$

$$P_p^a \beta_p P_p^b = \sum_{c=0}^{[\frac{a}{p}]} (-1)^{c+a} \binom{(p-1)(b-c)}{a-pc} \beta_p P_p^{a+b-c} P_p^c +$$

$$+ \sum_{c=0}^{[\frac{a-1}{p}]} (-1)^{c+a+1} \binom{(p-1)(b-c)-1}{a-pc-1} P_p^{a+b-c} \beta_p P_p^c, \; a<pb.$$

For a system of multiplicative generators of $A_{(p)}$ (as in the case $p=2$) we may take 1, β_p, P_p^1, P_p^p, $P_p^{p^2}$, $P_p^{p^3}$, ...

Let us recall that we have already obtained some partial information about the algebra $A_{(p)}$ in §25.

The first proof of the theorem was given by H. Cartan. If the reader is willing to get acquainted with it we shall recommend the paper of M. M. Postnikov (Russian Math. Surveys, 1966, Vol. 21. No. 4.).

Let us introduce some notations and definitions. If k is a field, by $\Lambda(m, k)$ we mean the k-algebra with the k-basis $(1, x)$ where $\dim x = m$ and $x^2 = 0$. It will be called the exterior algebra of the generator x. Now let $P(m, k)$ denote the k-algebra with the basis $(1, x^{(1)}, x^{(2)}, \ldots, x^{(k)}, \ldots)$ where $\deg x^{(k)} = km$, and the multiplication formula is $x^{(k)} \cdot x^{(r)} = \binom{k+r}{k} x^{(k+r)}$. It will be called the *algebra of divided polynomials* of the generator $x^{(1)} = x$. Obviously $x = x^{(1)}$ is a generating element of the algebra $P(m, k)$. By the term tensor product we shall always mean the left tensor product, whenever used in the context of the above algebras (i. e. $a \otimes b \cdot c \otimes d = (-1)^{\dim b \cdot \dim c} ac \otimes bd$) with the word "left" omitted, for no other tensor products will be considered.

Definition. Let p be a prime number. A sequence $I = (i_1, i_2, \ldots, i_k)$ is said to satisfy condition (C_p) with respect to the group $\pi = \mathbf{Z}$ or $\mathbf{Z}_{p^s}$, if

(1) $i_1 \geq p i_2$, $i_2 \geq p i_3$, ..., $i_{k-2} \geq p i_{k-1}$, $i_{k-1} \geq 2p-2$;

(2) $i_k = 0$ for $\pi = \mathbf{Z}$;

(3) $i_k = 0$ or 1 for $\pi = \mathbf{Z}_{p^s}$;

(4) $i_t \equiv 0$ or 1 (mod $2p-2$) for $1 \leq t \leq k$.

We shall use the standard notation $H^*(\pi, n; \mathbf{Z}_p) = H^*(K(\pi, n); \mathbf{Z}_p)$.

Theorem. (H. Cartan). For any $n \geq 1$ and any prime $p > 2$ the cohomology algebra $H^*(\pi, n; \mathbf{Z}_p)$, where $\pi = \mathbf{Z}$ or $\mathbf{Z}_{p^s}$, is isomorphic to the tensor product of the exterior algebras $\Lambda(m, \mathbf{Z}_p)$ (with generators of odd degrees) and of ordinary polynomial algebras (with generators of even degrees). For $n \geq 2$, $p = 2$, and $\pi = \mathbf{Z}$ or $\mathbf{Z}_{2^s}$, the algebra $H^*(\pi, n; \mathbf{Z}_2)$ is isomorphic to a tensor product of ordinary polynomial algebras. In each case the number of the generators of degree $n+q$ is equal to the number of sequences I satisfying (C_p), for which $|I| = q$ and $pi_1 < (p-1)(n+q)$.

Remark. The previous results can all be regarded as special cases of this theoem. If $\pi = \mathbf{Z}$, (C_p) implies that $i_k = 0$ and $i_{k-1} \geq 2p-2$. For $p = 2$ the last inequality means that $i_{k-1} \geq 2$, i. e. Sq^1 is not contained in the iterate Sq^I; further, $pi_1 < (p-1)(n+q)$ is equivalent to the well-known condition exc $I < n$.

It turns out that the *homology algebra* of $K(\pi, n)$ also permits full description. (Multiplication is induced by the H-space structure of $K(\pi, n) = \Omega K(\pi, n+1)$.)

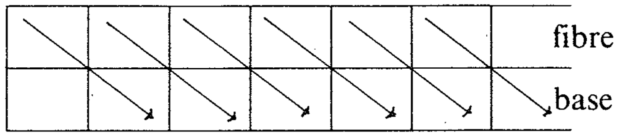
Theorem (H. Cartan). If $n \geq 1$ and $p > 2$ is a prime, the homology algebra $H_*(\pi, n; \mathbf{Z}_p)$, where $\pi = \mathbf{Z}$ or $\mathbf{Z}_{p^s}$, is isomorphic to the tensor product of the exterior algebras $\Lambda(m, \mathbf{Z}_p)$ (with generators of odd degrees) and the *divided polynomials* algebras (with generators of even degrees). If $n \geq 2$, $p = 2$ the homōlogy algebra $H_*(\pi, n; \mathbf{Z}_p)$, where $\pi = \mathbf{Z}$ or $\mathbf{Z}_{2^s}$, is isomorphic to a tensor product of divided polynomials algebras. The number of generators of stable dimension q is equal to the number of sequences I with $|I| = q$, satisfying the condition (C_p).

Theorem (on the choice of a basis; H. Cartan). Let $\pi = \mathbf{Z}$ and $\tilde{e}_n \in H^n(\mathbf{Z}, n; \mathbf{Z}_p)$ be the fundamental class. Then for the generators of exterior and ordinary polynomial algebras composing $H^*(\mathbf{Z}, n; \mathbf{Z}_p)$ we may choose the elements $St_p^I(\tilde{e}_n)$ such that I satisfies (C_p) and $pi_1 < (p-1)$ $(n + |I|)$.

The proofs and further exhaustive information on the integral cohomology of $K(\pi, n)$ can be found in a paper of H. Cartan (Algèbres d'Eilenberg–MacLane et homotopie. Sém. H. Cartan, ENS, 7e année, 1954/1955.).

First applications

Let us return again to the homotopy groups of spheres, more exactly to their 2-components. Consider the first killing space $S^n|_n \xrightarrow{K(\mathbf{Z}, n-1)} S^n$. We shall study the small dimensions and so the effect of the "angle" in the spectral sequence will not concern us; we simply dismiss it. Because we find the ordinary picture of a spectral sequence, containing in the present case only zeros except in a single row and column, not really efficient, we shall use a simplified version more convenient for the calculations but containing no new idea.



full dimension: $n-1$ n $n+1$ $n+2$ $n+3$ ...

Let $\alpha \in H^n(S^n; \mathbf{Z}_2)$ be a generator. Clearly $\tau: e_{n-1} \to a$. As $\pi = \mathbf{Z}$, we have $Sq^1(e_{n-1}) = 0$ and the scheme takes the form

	$n-1$	n	$n+1$	$n+2$	$n+3$	$n+4$	$n+5$
fibre	e_{n-1} ↘	0	$Sq^2 e_{n-1}$	$Sq^3 e_{n-1}$	$Sq^4 e_{n-1}$	$Sq^5 e_{n-1}$	$Sq^4 Sq^2 e_{n-1}$ $Sq^6 e_{n-1}$
base		a	0	0	0	0	0

All elements of the upper row go into E_∞ without being altered or annulled on the way. So it remained to examine the action of the operations in the small stable dimensions in $H^*(S^n|_n; \mathbf{Z}_2)$.

Let us denote by h_1 the image of $Sq^2 e_{n-1}$ in E_∞. Then $Sq^3 e_{n-1} = Sq^1 Sq^2 e_{n-1} = Sq^1 h_1 \cdot Sq^4 e_{n-1}$. Now $Sq^4 e_{n-1}$ has no such representation since the system $\{Sq^{2^i}\}$ is a *minimal* basis, so we must take the image of $Sq^4 e_{n-1}$ in E_∞ as a new multiplicative generator h_2 of degree $n+3$. Thus $Sq^5 e_{n-1} = Sq^1 h_2 = Sq^2 Sq^1 h_1$; $Sq^2 h_1 = Sq^3 h_1 = 0$.

Here we end. For the first four stable dimensions we have

$n+1$	$n+2$	$n+3$	$n+4$	
h_1	$Sq^1 h_1$	h_2	$Sq^1 h_2 = Sq^2 Sq^1 h_1$	...

Thus we have calculated a part of $H^*(S^n|_n; \mathbf{Z}_2)$. Let us determine $\pi_{n+1}(S^n)$. We are interested in the group $H_{n+1}(S^n|_n; \mathbf{Z})$ which is isomorphic to $\pi_{n+1}(S^n)$. In §23 we have shown that $\pi_{n+1}(S^n)$ and $\pi_{n+2}(S^n)$ are finite and have no p-components except the 2-component. We know that $H_{n+1}(S^n|_n; \mathbf{Z})$ is the sum of the free part of $H^{n+1}(S^n|_n; \mathbf{Z})$ and the torsion subgroup of $H^{n+2}(S^n|_n; \mathbf{Z})$. So the first summand is zero and $\pi_{n+1}(S^n) = \text{Tors}\, H^{n+2}(S^n|_n; \mathbf{Z})$.

Now $\text{Tors}\, H^{n+2}(S^n|_n; \mathbf{Z}) = H^{n+2}(S^n|_n; \mathbf{Z}) = \mathbf{Z}_2$. Indeed, $Sq^1 h_1 = \rho_2(\alpha)$ where $\alpha \in H^{n+2}(S^n|_n; \mathbf{Z})$ and α has degree 2. (The latter is true for any element of the form $Sq^1 \xi \in H^*(X; \mathbf{Z}_2)$ where X is any space, as follows from $Sq^1 = \beta$ and the definition of β: the last step in constructing $\beta\xi$ is reducing modulo 2 the element $\frac{1}{2}\delta\xi$ which, by construction, is of degree 2.) We obtain that $H^{n+2}(S^n|_n; \mathbf{Z})$ contains an element of degree 2 which is not divisible by 2 (otherwise $\rho_2 \alpha = 0$).

It remained to apply the universal coefficient formula

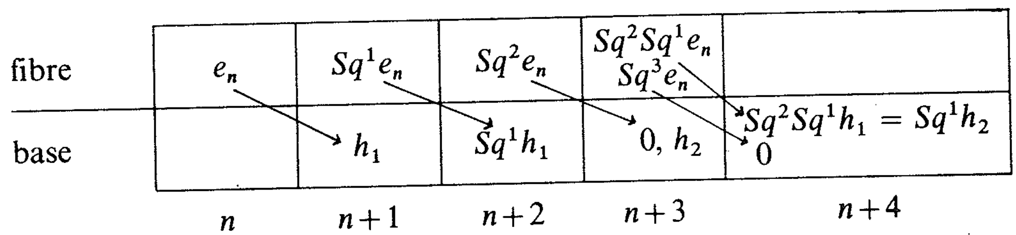
$$H^{n+2}(S^n|_n, \mathbf{Z}_2) = H^{n+2}(S^n|_n; \mathbf{Z}) \oplus \operatorname{Tor}(H^{n+3}(S^n|_n; \mathbf{Z}); \mathbf{Z}_2)$$

and the equality $H^{n+2}(S^n|_n; \mathbf{Z}_2) = \mathbf{Z}_2$.

We have shown that $\pi_{n+1}(S^n) = \mathbf{Z}_2$ for any $n \geq 3$. (As already known, $\pi_3(S^2) = \mathbf{Z}$.)

Let us now calculate $\pi_{n+2}(S^n)$ by using the next killing space:

$$S^n|_{n+1} \xrightarrow{K(\mathbf{Z}_2, n)} S^n|_n$$



We obtain E_∞ by the standard considerations:

0	ρ_1	ρ_2, $Sq^1 \rho_1$	...
$n+1$	$n+2$	$n+3$	

We see that in the dimension $n+3$ two generators occur: ρ_2 and $Sq^1 \rho_1$. The situation is similar to the one above.

Again we have to find Tors $H_{n+2}(Y; \mathbf{Z}) = \pi_{n+2}(S^n)$. Again we have $H^{n+2}(Y; \mathbf{Z}) = 0$ and $H^{n+2}(Y; \mathbf{Z}_2) = \mathbf{Z}_2$. Now $H^{n+3}(Y; \mathbf{Z}_2) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ with ρ_2 and $Sq^1 \rho_1$ as generators.

By repeating the previous reasoning word for word we obtain that Tors $H^{n+3}(Y; \mathbf{Z}) = \mathbf{Z}_2$. Thus $\pi_{n+2}(S^n) = \text{Tors } H_{n+2}(Y; \mathbf{Z}) = \text{Tors } H^{n+3}(Y; \mathbf{Z}) = \mathbf{Z}_2$ for $n \geq 3$.

The reader may attempt to move forward to find $\pi_{n+3}(S^n)$, however this will not be quite trivial.