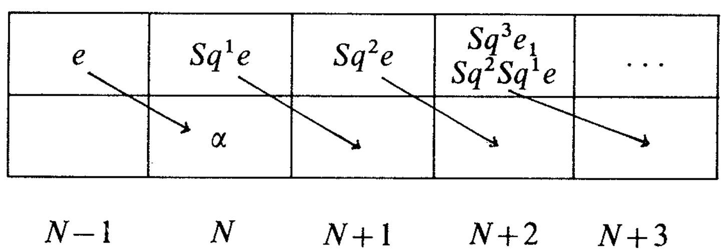
CHAPTER V

THE ADAMS SPECTRAL SEQUENCE

§ 30. GENERAL IDEAS

As it was shown at the end of the preceding Section, the information collected so far is sufficient to find the stable homotopy groups. Once the modulo p cohomology of a space X is known we easily find the "stable part" of the cohomology groups of the first, second, third, etc. killing spaces. In each case the Hurewicz theorem gives the corresponding homology groups. This procedure (Serre's method) however, will not enable us to compute the homotopy groups, at least not without overcoming further difficulties.

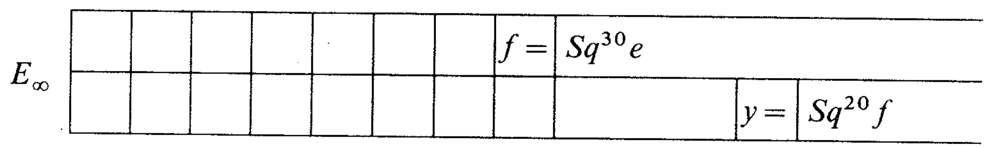
Suppose that, for example we need the stable homotopy groups of some space X while we know the cohomology of X with arbitrary coefficients together with the action of every cohomology operation. Assume, for example, the first non-trivial homotopy group to be $\mathbf{Z}_2$; let it be in the dimension N. Consider the mod 2 cohomology spectral sequence of the fibration $X|_N \xrightarrow{K(\mathbf{Z}_2,\, N-1)} X$.



Here $e \in H^{N-1}(K(\mathbf{Z}_2, N-1);\ \mathbf{Z}_2)$ is the fundamental class and α is the generator of $H^N(X;\ \mathbf{Z}_2)$.

In the upper row we have the cohomology of $K(\mathbf{Z}_2, N-1)$ mod 2, coinciding with the Steenrod algebra $A_{(2)}$ in the stable dimensions. The differential maps e onto α, further for each operation φ the element $\varphi(e)$ is sent onto $\varphi(\alpha)$. The elements that remain in the lower row are those elements of $H^*(X;\ \mathbf{Z}_2)$ which do not belong to the images of α under any operations; the elements that remain in the upper row are elements of form $\varphi(e)$, where $\varphi(\alpha) = 0$. This means that all cohomology groups of $X|_N$ are known. But our knowledge about the action of the operations is not complete. Imagine, for example, that there is a relation $Sq^{20}Sq^{30} = 0$ in the Steenrod algebra (probably there is no such relation but that is not the point) and that $Sq^{30}\alpha = 0$. Then

$Sq^{30}e$ remains in the upper row. In the cohomology of $X|_N$ it cannot be the value of the operation Sq^{30} at any element. Let the operation Sq^{20} be applied to it. As $Sq^{20}Sq^{30} = 0$, the result may not be any element of the upper row. It may be, however, a non-trivial element of the lower row, i. e. $Sq^{20}(f) = y$ lies in $H^*(X|_N;\ \mathbf{Z}_2)$, where f comes from $Sq^{30}e \in H^*(K(\mathbf{Z}_2, N-1);\ \mathbf{Z}_2)$ and y comes from some element of $H^*(X;\ \mathbf{Z}_2)$.



Analogous consequences may follow from a more complicated relation of the form $\sum_i Sq^{b_i}Sq^{c_i} = 0$ as well.

We conclude that our information relating to the action of the cohomology operations is not complete and so the cohomology of the next killing space remains uncertain.

Actually present-day topology has no means to overcome this difficulty: this far not even the homotopy groups of all spheres are computed. There is some means, nevertheless to at least expose the difficulty clearly enough, namely, to collect all calculations related to determining the homotopy groups in a single spectral sequence whose initial term is algorithmically computed, while computing the differentials will contain all the basic difficulties. It is the Adams spectral sequence.

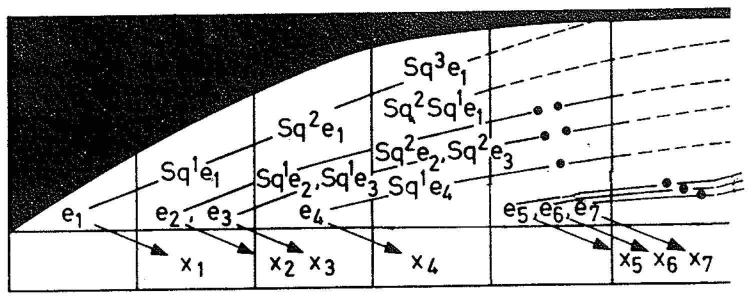
Its usefulness is of course more than merely exposing the difficulties. In fact it may also enable us to overcome a part of them. Namely, the sequence is equipped with a series of further useful structures (for example, with multiplication) which are not at our disposal if we remain at the language of usual killing spaces, and provide us with ample information about the action of the differentials.

Before formulating the results and giving the proofs we are going to expound some basic ideas.

Serre's method means "killing" the cohomology groups in the subsequent dimensions, first the nth, after the $(n+1)$th, etc. The method of Adams is also killing the groups, but not in the same order. Let us be given a space X. Assume it to be $(N-1)$-connected. The aim is to detemine the p-component of the homotopy groups from the dimension N to $N+n$, where $n \ll N$. At the first step we shall kill *every* mod p cohomology in these dimensions. For example, we do the following. Each additive generator of $H^{N+q}(X; \mathbf{Z}_p)$ defines a mapping $X \to K(\mathbf{Z}_p, N+q)$. Together they constitute a mapping of X into a product space $\Pi_i K(\mathbf{Z}_p, N+q_i)$ of a number of $K(\mathbf{Z}_p, N+q)$-spaces. Consider the induced fibration

$$X(1) \xrightarrow{\Pi_i K(\mathbf{Z}_p,\, N+q_i-1)} X$$

Let us consider the spectral sequence. In the upper row we have a so-called free $A_{(p)}$-module (i. e. the operations act freely in this row: there are no relations except those implied by the relations of the algebra $A_{(p)}$).



The differential defines an epimorphism of the upper row to the bottom row and what remains in the former is the kernel of this mapping while in the latter we have zeros. We have all the informations about the cohomology of the space $X(1)$, with the action of the operations included, because E_∞ consists of this single row; so we may repeat the same construction this time for $X(1)$, etc. The result is a sequence of Adams killing spaces: $X(1)$, $X(2)$, $X(3)$, ...

We notice that the same goal, i. e. killing of all cohomology groups of X, could have been reached in a more efficient way. In fact it is not necessary to kill each additive generator of $H^*(X; \mathbf{Z}_p)$ independently. If, for example, we kill an element ξ for which $P^i_p\xi \neq 0$, it will not be necessary to kill $P^i_p\xi$, too, because it will disappear as well without our help.

In other words, we only have to consider the generators of the $A_{(p)}$-module $H^*(X; \mathbf{Z}_p)$, rather than all generators in the additive sense. That is, we consider all the additive generators of $H^N(X; \mathbf{Z}_p)$; then in $H^{N+1}(X; \mathbf{Z}_p)$ we consider the genuinely new generators only, neglecting those elements which are obtained by operations from the previous system of generators. Further we continue the procedure with $H^{N+2}(X; \mathbf{Z}_p)$, etc.

Speaking the language of algebra, we are doing the following. We are given the $A_{(p)}$-module $H^*(X; \mathbf{Z}_p)$ onto which we map a free $A_{(p)}$-module F_1 (the upper row of the spectral sequence of the fibration $X(1) \to X$).

Onto the kernel of this epimorphism (i. e. $H^*(X(1); \mathbf{Z}_p)$) we again map a free $A_{(p)}$-module, and so on. The result is an exact sequence of $A_{(p)}$-modules.

$$\ldots \to F_2 \to F_1 \to H^*(X; \mathbf{Z}_p) \to 0$$

such that all terms except $H^*(X; \mathbf{Z}_p)$ are free. We say we have a *free resolution*, an object with many remarkable properties which we shall discuss later on.

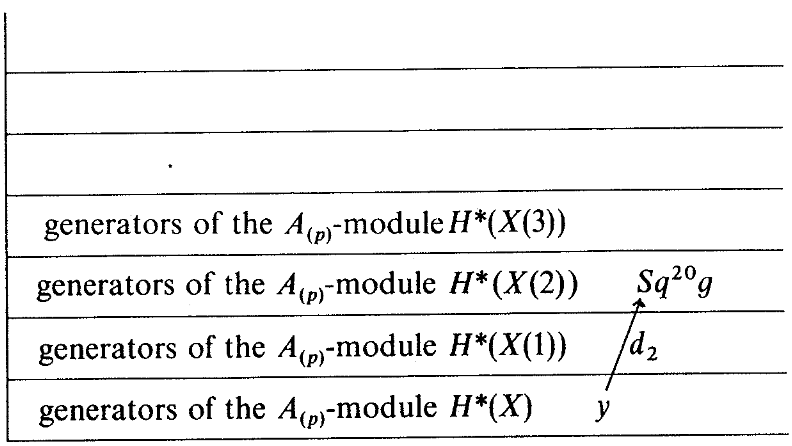
Let us now return to geometry.

We have a process which is convergent in a certain sense, as the subsequent spaces $X(k)$ have smaller and smaller cohomology groups and have none at the limit. In this sense by using the method of Adams we do a more thorough work than with Serre's method: we kill every cohomology group of every killing space.

However, by applying the Serre procedure we make direct use of the homotopy groups of the space *via* the Hurewicz theorem. We always kill cohomology classes in the lower dimensions directly related to certain elements of the homotopy groups. As a rule it is not the case with the method of Adams.

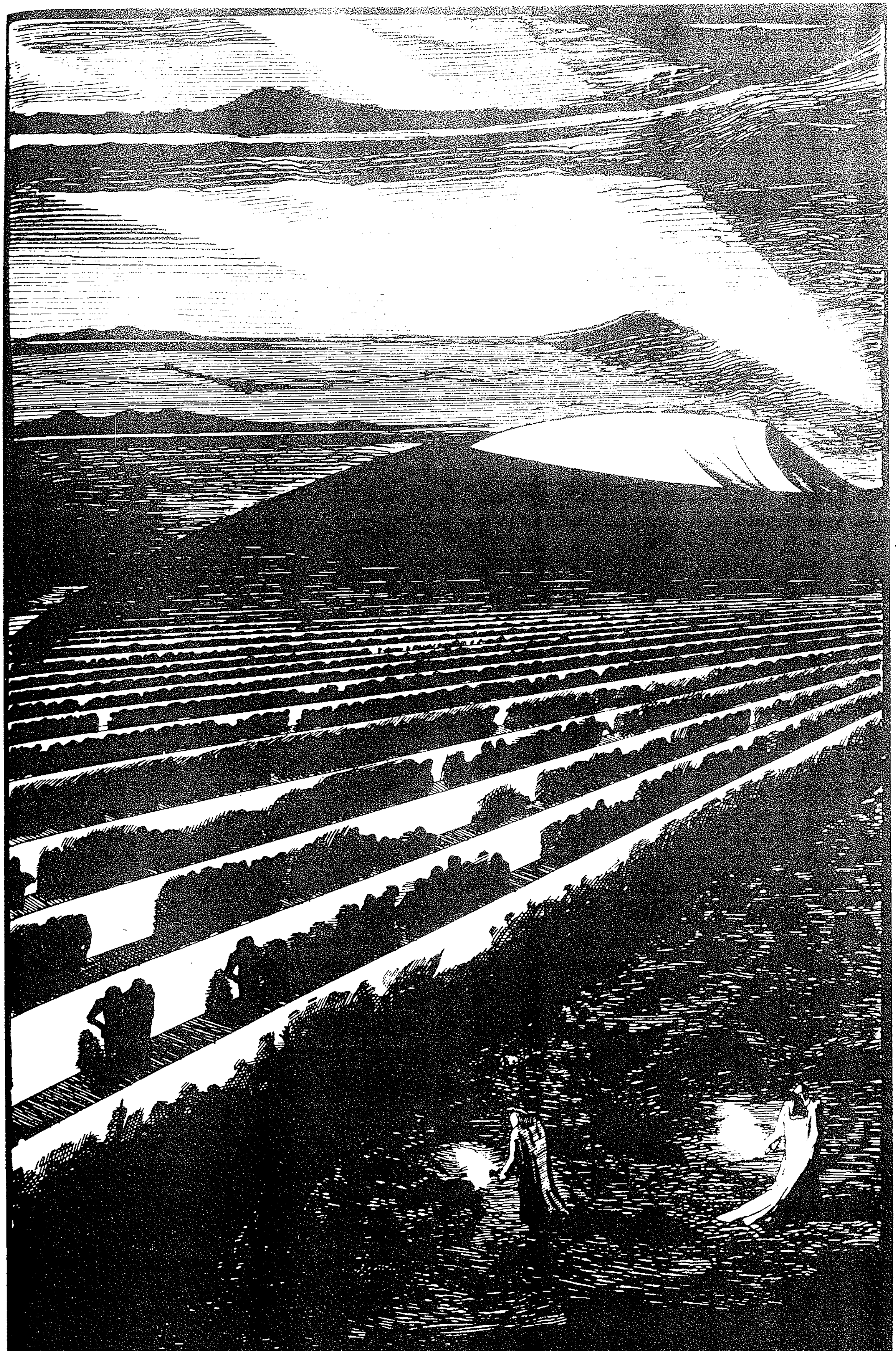
Let us return to the example above: $\pi_N(X)=\mathbf{Z}_2$, the generator of the group $H^N(X;\mathbf{Z}_2)$ is α, $Sq^{30}\alpha=0$, and in the cohomology of $X|_N$ there remain $f=Sq^{30}e$ and $Sq^{20}f=y$. If we employ the Serre procedure we shall not need to kill y: by that time it will have disappeared together with $f\in H^*(X|_N;\mathbf{Z}_2)$. Now following Adams we kill both elements at the very first step. Thus the latter requires more killings than the former. By calculating in each dimension $N+q$ the number of generators killed at all steps of the Adams procedure, we get an upper bound on the p-component of $\pi_{N+q}(X)$. This estimate is actually the first term of the Adams spectral sequence. The differentials here kill all the superfluous elements in the following way.

We observe that y is not the only element which was killed unnecessarily. The cohomology of $X(1)$ as well as of $X|_N$ contains the element f. Now in $X|_N$ we have $Sq^{20}f=$ $=y$ while in $X(1)$, $Sq^{20}f=0$, implying the occurrence of a useless element in $X(2)$. In fact let f be killed by some g, then there remains an element $Sq^{20}g$ that would not even appear if we applied the Serre method, and it has to be killed at the next step. The initial term of the Adams spectral sequence is:



and the second differential sends y to the element coming from $Sq^{20}g$ in $H^*(X(2);\mathbf{Z}_2)$ thus annihilating both useless elements.

The limit term E_∞ will be adjoint to the p-components of the stable homotopy groups of X.



§31. SOME AUXILIARY MATERIAL FROM ALGEBRA

Let A be an associative algebra with a unit element over a field k, and let it have a grading $A = \oplus_q A_q = \ldots \oplus A_{-1} \oplus A_0 \oplus A_1 \oplus \ldots$.

A left module over A (or an A-module) is a graded vector space T over k, i. e. a direct sum $T = \oplus_{k=-\infty}^{+\infty} T_k$ equipped with a mapping $T \times A \to T$ such that each element (x, a), $x \in T$, $a \in A$ is mapped onto some $ax \in T$, and the following axioms are satisfied:

(1) if $a \in A_k$ and $x \in T_l$, then $ax \in T_{k+l}$,

(2) $a(x_1 + x_2) = ax_1 + ax_2$,

(3) $(a_1 + a_2)x = a_1 x + a_2 x$,

(4) $b(ax) = (ba)x$ and $(ax)b = (ba)x$ for $a \in A$, $b \in k$.

The notion of a right A-module is defined similarly. A left A-module T is *free* if it contains a subset $T' \subset T$ such that each $x \in T$ can be written, in a unique way, as a (finite) sum $x = \sum_i a_i e_i$ with $a_i \in A$ and $e_i \in T'$. Such subset T' is called a *basis* of the free A-module T.

For example, the algebra A itself may be considered as a free A-module with the basis consisting of the unity even when A as an algebra has relations.

A homomorphism of an A-module T^1 into an A-module T^2 is a homomorphism $f\colon T^1 \to T^2$ such that $f(T^1_k) \subset T^2_k$ and $f(ax) = af(x)$ for every $a \in A$ and $x \in T^1$.

Clearly for every A-module T there exists an exact sequence

$$0 \longrightarrow I_T \longrightarrow F_T \xrightarrow{\ \pi\ } T \longrightarrow 0$$

such that F_T is a free A-module. (For F_T we may choose a vector space over k whose basis is the set of the pairs (a, x), $a \in A$, $x \in T$. The algebra A acts on F_T according to the formula $a'(a, x) = (a'a, x)$. The gradation of F is naturally defined. The epimorphism π is given by $\pi(a, x) = ax$. We write $I_T = \operatorname{Ker} \pi$.

An A-module P is *projective* if any diagram of the form

$$\begin{array}{ccccc} M & \longrightarrow & N & \longrightarrow & 0 \\ & & \uparrow & & \\ & & P & & \end{array}$$

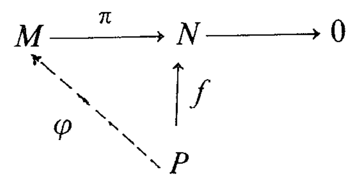
with the row exact, may be extended to a commutative diagram

$$\begin{array}{ccccc} M & \longrightarrow & N & \longrightarrow & 0 \\ & \nwarrow & \uparrow & & \\ & & P & & \end{array}$$

In other words, P is projective if any A-module homomorphism of P to any quotient module M/R is a composite $P \to M \to M/R$.

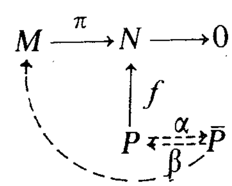
We claim that an A-module P is projective if and only if it is a direct summand in a free A-module.

Proof. Any free A-module P is projective. Indeed, if $P' = \{p_i\}$ is a basis of P and we are given a diagram



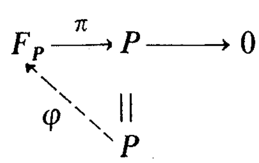
with exact row, we consider $n_i = f(p_i)$, and choose $m_i \in M$ such that $\pi(m_i) = n_i$. Let $\varphi\colon P \to M$ be defined by $\varphi(p_i) = m_i$.

Assume now that P is a direct summand in a projective module $\bar{P}$, i. e. there exist $\alpha\colon P \to \bar{P}$ and $\beta\colon \bar{P} \to P$ such that $\beta \circ \alpha\colon P \to P$ is identity. Then P is a projective module. Indeed, if



is a diagram with exact row then $f \circ \beta$ is a mapping of the projective module $\bar{P}$ to N, so there exists a $\varphi\colon \bar{P} \to M$ such that $\pi \circ \varphi = f \circ \beta$. The homomorphism $\psi = \varphi\alpha\colon P \to N$ is such that $\pi \circ \psi = \pi \circ \varphi \circ \alpha = f \circ \beta \circ \alpha = f$.

Finally, any projective module is a direct summand of some free module. Indeed, assume that P is projective. There exists an exact sequence $F_P \xrightarrow{\pi} P \longrightarrow 0$ with F_p free. Consider



Because P is projective, this diagram may be extended by $\varphi\colon P \to F_P$ so that $\pi \circ \varphi\colon P \to P$ is identity. Hence $F_P = P \oplus \operatorname{Ker} \pi$.

Exercise. Let A be the algebra of continuous (say, real) functions on a complex X, and T be the space of all (continuous) sections of a vector bundle ξ over X. Show that T is a projective A-module (with respect to the natural action of A in T), and that T is a free A-module if and only if the bundle ξ is trivial. Notice, that in all cases T is a summand in a free A-module, because ξ is a summand of a trivial bundle.

(This exercise illustrates the difference between projective and free modules. One may say that it is the same as the difference between vector bundles and trivial vector bundles.)

The only implication of this statement that we are going to use is that any free module is projective.

Let T be an arbitrary right A-module. Then there exists an exact sequence

$$\ldots \to A_k \to A_{k-1} \to \ldots \to A_1 \to A_0 \to T \to 0$$

where A_k ($k = 0, 1, 2, \ldots$) are right projective modules. It will be called a projective resolution of the module T (if all A_k are free we have a *free* resolution).

A free resolution may be constructed in the following way. For any module T we find an exact sequence

$$0 \to I_T \to F_T \to T \to 0$$

such that F_T is free. Let $T_1 = I_T$, $T_2 = I_{T_1}$, $T_3 = I_{T_3}$, ...

We have the exact sequences

$$0 \to T_1 \to F_T \to T \to 0$$

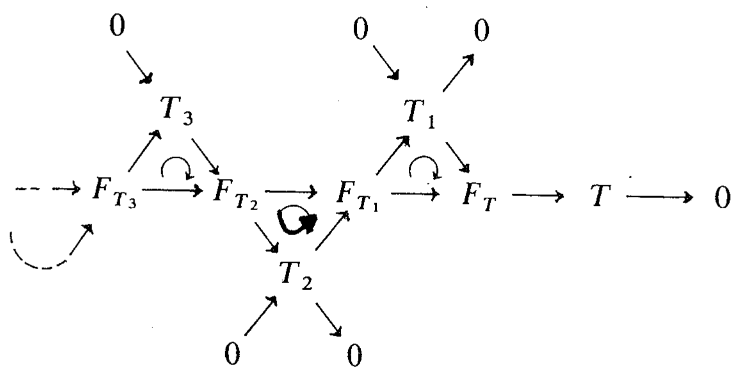
$$0 \to T_2 \to F_{T_1} \to T_1 \to 0$$

$$0 \to T_3 \to F_{T_2} \to T_2 \to 0$$

$$0 \to T_4 \to F_{T_3} \to T_3 \to 0$$

....

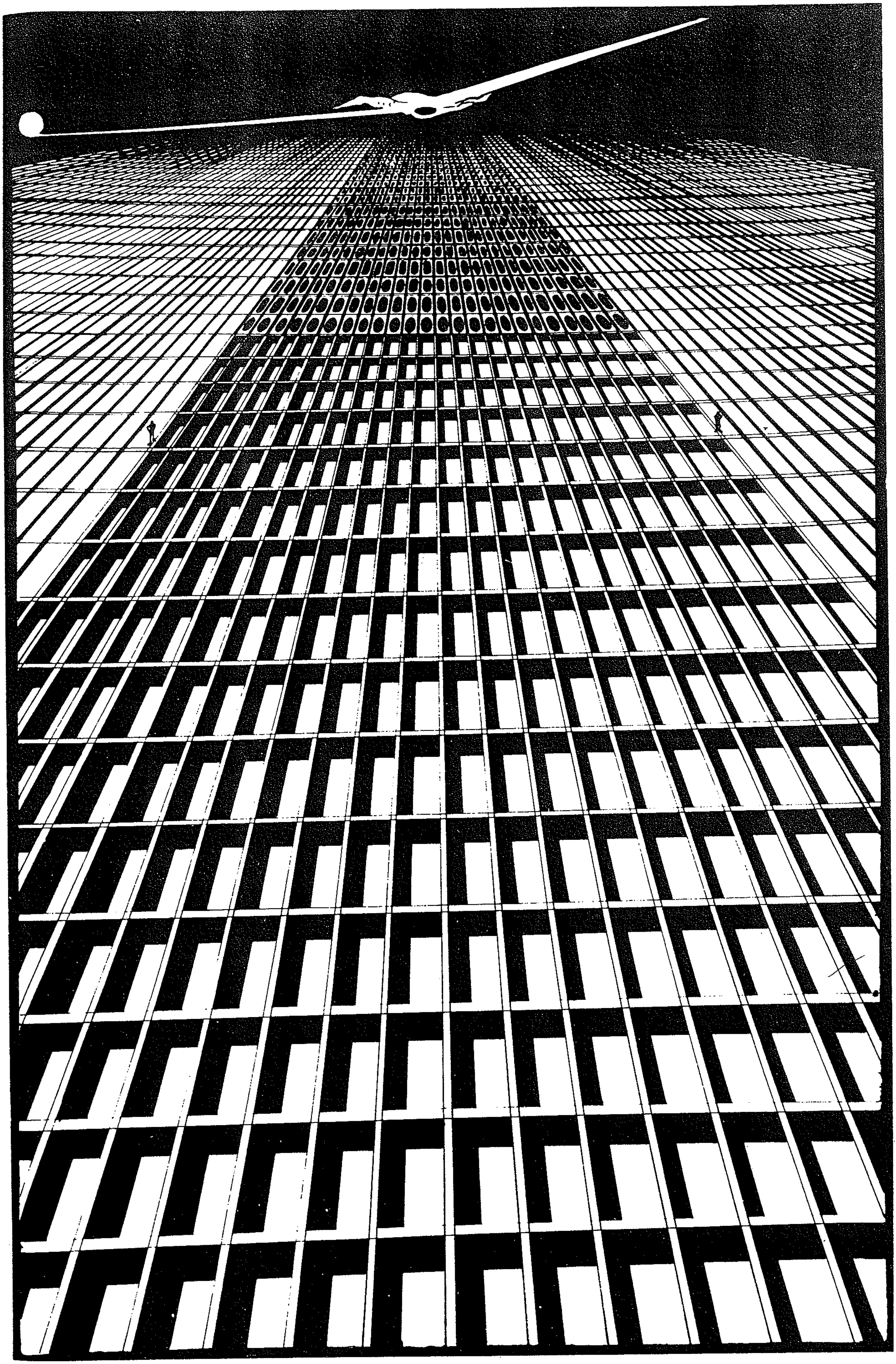
which constitute an exact sequence



It is true that the resolution of T is not uniquely determined, nevertheless the further constructions will not however depend on the particular choice of the resolution. (We are not going to prove this but the reader is advised to fill up the gaps.)

Consider the covariant functor of tensor multiplication by a fixed left A-module N and the contravariant functor $\mathrm{Hom}_A(\ldots, N)$.

The tensor product $M \otimes_A N$ of a right A-module M and a left A-module N is not necessarily an A-module, it is however naturally graded: the degree of $m \otimes n$, $m \in M_k$, $n \in N_l$, being $k+l$. If M and N are left A-modules, $[\mathrm{Hom}_A(M, N)]_s$ consists of homomorphisms "of degree $-s$", i. e. of homomorphisms commuting with the action of A and mapping M_k into N_{k-s} for each k. We remark that the group of the A-homomorphisms of M to N, in the above sense is $[\mathrm{Hom}_A(M, N)]_0$.



Let us now apply the functor $\otimes_A N$ to the projective resolution. The resulting sequence

$$\ldots \to A_k \otimes {}_A N \to \ldots \to A_1 \otimes {}_A N \to A_0 \otimes {}_A N$$

(notice the absence of $\to T \otimes_A N \to 0$ in this sequence!) is an algebraic complex (i. e. the composite of any pair of subsequent homomorphisms is trivial). Its deviation from exactness may be measured by the homology groups denoted in this case by $\mathrm{Tor}_n^A(T, N)$. We have $\mathrm{Tor}_0^A(T, N) = T \otimes_A N$ (prove this!).

$\mathrm{Tor}_n^A(T, N)$ is graded in an obvious way:

$$\mathrm{Tor}_n^A(T, N) = \bigoplus_k [\mathrm{Tor}_n^A(T, N)]_k = \bigoplus_k \mathrm{Tor}_{n,k}^A(T, N).$$

We still mention a further important property of Tor: for any projective A-module T, $\mathrm{Tor}_n^A(T, N) = 0$ for $n > 0$ (prove this!).

Next we consider the functor $\mathrm{Hom}_A(\cdot, N)$. By applying to the projective resolution we again get a complex

$$\ldots \leftarrow \mathrm{Hom}_A(A_k, N) \leftarrow \ldots \leftarrow \mathrm{Hom}_A(A_1, N) \leftarrow \mathrm{Hom}_A(A_1 N)$$

whose homology groups are denoted by $\mathrm{Ext}_A^n(T, N)$. We have $\mathrm{Ext}_A^0(T, N) = = \mathrm{Hom}_A(T, N)$ (prove this!).

If T is a projective A-module, $\mathrm{Ext}_A^n(T, N) = 0$ for $n > 0$ (prove this!). Note that if A is a graded algebra then each A-module $\mathrm{Tor}_n^A(T, N)$ and $\mathrm{Ext}_A^n(T, N)$ is graded as well.

Instead of $[\mathrm{Ext}_A^n(T, N)]_q$ we shall prefer the notation $\mathrm{Ext}_A^{n,q}(T, N)$. Thus

$$\mathrm{Ext}_A^{n,q}(T, N) = \frac{\mathrm{Ker}([\mathrm{Hom}_A(A_n, N)]_q \to [\mathrm{Hom}_A(A_{n+1}, N)]_q)}{\mathrm{Im}([\mathrm{Hom}_A(A_{n-1}, N)]_q \to [\mathrm{Hom}_A(A_n, N)]_q)}$$

Exercise. Let $A = \mathbf{Z}$. Prove that in this case

(1) $\mathrm{Tor}_n^A(T, N) = \mathrm{Ext}_A^n(T, N) = 0$ for any T and N, and $n \geq 2$;

(2) if T and N are finitely generated groups, then $\mathrm{Tor}_1^Z(T, N) = \mathrm{Tors}\, T \otimes \mathrm{Tors}\, N$.

(3) $\mathrm{Ext}_1^A(\mathbf{Z}, G) = 0$ for any G; $\mathrm{Ext}_1^A(\mathbf{Z}_n, G) = G/nG$, in particular $\mathrm{Ext}_1^A(\mathbf{Z}_n, \mathbf{Z}) \cong \mathbf{Z}_n$, $\mathrm{Ext}_1^A(\mathbf{Z}_n, \mathbf{Z}_m) \cong \mathbf{Z}_{(n,m)}$, and thus $\mathrm{Ext}_1^A(T, N) \cong (\mathrm{Tors}\, T) \otimes N$ for any finitely generated T and N.

§32. CONSTRUCTION OF THE SPECTRAL SEQUENCE

We are given a topological space X and we want to determine the stable homotopy groups $\pi_q^S(X)$, i. e. the groups $\pi_{N+q}(\Sigma^N X)$ for $N \gg q$. The principal case is $X = S^0$, the space consisting of two points. Then $\pi_q^S(X) = \pi_{N+q}(S^N))$.

As before, $A = A_{(p)}$ will stand for the Steenrod algebra. We shall write $H^*(X)$ for $H^*(X; \mathbf{Z}_p)$, where p is a prime, and $\tilde{H}^*(X)$ for $H^*(X, *)$ i. e. $\tilde{H}^0(X) = H^0(X)/\mathbf{Z}_p$ and $\tilde{H}^i(X) = H^i(X)$ for $i > 0$.

As A is acting on $\tilde{H}^*(X)$ it may as well be regarded as an A-module. (Here we note that we shall always have to deal with modules graded by non-negative degrees, i. e. the terms with negative indexes are trivial, as we have in the case of $\tilde{H}^*(X)$.)

Let us choose some generating system in this A-module. By this we define an epimorphism of a free B_1-module onto $\tilde{H}^*(X)$:

$$0 \leftarrow \tilde{H}^*(X) \leftarrow B_1.$$

For the sake of simplicity we assume X to be a complex of finite type (i. e. in each dimension the number of cells is finite) then all modules to be dealt with will have finitely many generators in every dimension.

Consider the kernel of the above epimorphism. In general it is not free so let the same procedure be repeated. This way we obtain an exact sequence (free resolution)

$$0 \leftarrow \tilde{H}^*(X) \leftarrow B_1 \leftarrow B_2 \leftarrow B_3 \leftarrow B_4 \leftarrow \dots$$

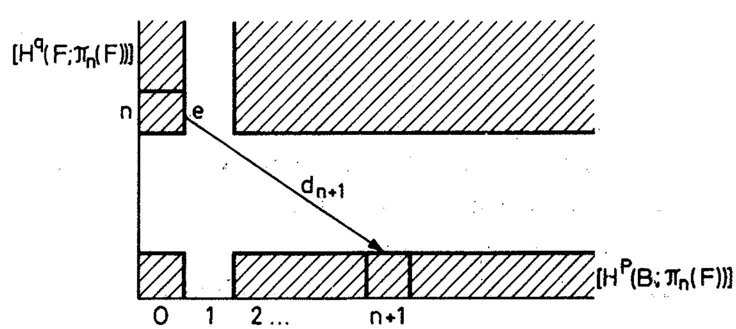
The A-modules B_i are certainly not the cohomology modules of any spaces (this would imply the relations $P_p^i(x) = 0$ for $n > (p-1)\dim x$, for example). If we want to "approximate" them by cohomology modules, it seems reasonable to choose the spaces $K(\mathbf{Z}_p, n)$, because the A-modules $H^*(K(\mathbf{Z}_p, n); \mathbf{Z}_p)$ have the least systems of relations.

Let N be a large number. The A-modules $H^*(\Sigma^N X)$ and $H^*(X)$ only differ in their gradings. Let $\alpha_i \in H^{q_i}(X)$ be the images of the free generators of B_1, i. e. the generators chosen in the A-module $H^*(X)$.

Consider the mappings $\Sigma^N(X) \to K(\mathbf{Z}_p, N+q_i)$ constructed along the elements $\Sigma^N \alpha_i \in H^{N+q_i}(\Sigma^N X)$ for which $q_i < N$. Together they define a mapping $\Sigma^N X \to Y_1 = \Pi_i K(\mathbf{Z}_p, N+q_i)$. Then the A-module $H^*(Y_1)$ coincides in the dimensions N through $2N$ with the A-module B_1 in the dimensions 0 through N, and the mapping $\Sigma^N X \to Y$ induces a homomorphism $H^*(Y_1) \to H^*(\Sigma^N X)$ which in the dimensions N through $2N$ coincides with the homomorphism $B_1 \to H^*(X)$ considered in the dimensions 0 through N.

The mapping $\Sigma^N X \to Y_1$ may be considered as a fibration. Let the fibre be denoted by $X(1)$; write $X(0) = X$.

It would be difficult to give full description of $H^*(X(1))$ in the general case, nevertheless in the dimensions $\leq 2N-3$ the A-module $H^*(X(1))$ is easily seen to be isomorphic, up to the shift of dimensions by one, to the A-module $\text{Ker}[\tilde{H}^*(Y_1) \to H^*(\Sigma^N X)]$. Indeed, we only have to consider the spectral sequence of the fibration



(The reason of the shift of dimensions is that the transgression τ increases the dimensions by 1.) Thus $\tilde{H}^*(X(1))$ in dimensions $N-1$ through $2N-3$ is isomorphic as an A-module to $\operatorname{Ker}[B_1 \to H^*(X)]$ in dimensions 0 through $N-2$ by the shift of dimensions by $N-1$.

Remark. Of course $X(1)$ could have been defined not only as fibre of the mapping $\Sigma^N X \to Y_1$ but as well as the total space of the fibration $X(1) \xrightarrow{\Pi_i K(\mathbf{Z}_p,\, N+q_i-1)} \Sigma^N X$

induced from the fibration $* \xrightarrow{\Pi_i K_{N+q_i-1}} \Pi_i K_{N+q_i} = Y_1$ by $\Sigma^N X \to Y_1$.

$$\begin{array}{ccc} X(1) & \longrightarrow & * \\ \Big\downarrow \Pi_i K_{N+q_i-1} & & \Big\downarrow \Pi_i K_{N+q_i-1} \\ \Sigma^N X & \xrightarrow{X(1)} & \Pi_i K_{N+q_i} = Y_1 \end{array}$$

Let it be emphasized that we have defined not only a space $X(1)$, but also a mapping $X(1) \to \Sigma^N X$ as well. Both $X(1)$ and the mapping are defined up to homotopy equivalence.

Next we repeat the procedure, previously applied to $\Sigma^N X$, with the space $X(1)$: select in he A-module $\tilde{H}^*(X(1))$ a system of generators which are (up to dimension $2N-3$) in one-to-one correspondence with the free generators of the A-module B_2 (we remind that there is an epimorphism $B_2 \to \operatorname{Ker}[B_1 \to \tilde{H}^*(X)]$) while the difference between the respective dimensions is $N-1$. Let $\beta_i \in \tilde{H}^{-1+r_i}(X(1))$ be these generators. We construct $X(1) \to Y_2 = \Pi_i K(\mathbf{Z}_p, N-1+r_i)$. The A-modules $\tilde{H}^*(Y_2)$ and B_2 coincide, with the dimension shift of $N-1$ up to $H^{2N-3}(Y_2)$. Let $X(2)$ denote the fibre of the fibration equivalent to $X(1) \to Y_2$. Thus we have obtained the next space $X(2)$ and mapping $X(2) \to X(1)$.

By repeating the construction we get the subsequent spaces

$$\ldots \to X(2) \to X(1) \to \Sigma^N X = X(0)$$

where each $X(i)$ is the fibre of some fibration whose total space is homotopy equivalent to $X(i-1)$ and whose base Y_i is a product of spaces of the type $K(\mathbf{Z}_p, m)$.

Let $n \ll N$ be fixed. The A-modules $\tilde{H}^*(Y_i)$ up to dimension $(N-i+1)+n$ and B_i up to dimension n coincide, except for a difference $N-i+1$ in their gradings. (Actually they coincide in higher dimensions as well, $\tilde{H}^*(Y_i)$ and B_i as far as up to dimensions $\sim 2N$ and $\sim N$ respectively. It has no significance as we are nevertheless going to consider N and n as tending to infinity.)

On the other hand the mapping $X(i) \to X(i-1)$, too, may be considered as a fibration. Its fibre is a space Z_i which is a product of as many $K(\mathbf{Z}_p, m')$ spaces as Y_i but each space is having a number m' smaller by one unit than the respective space in the latter product (we may assume that, in the permitted dimensions, $Z_i = \Omega Y_i$ and $\Sigma Z_i = Y_i$).

The A-module $\tilde{H}^*(Z_i)$ is isomorphic to B with a grading shift of $N-i$. Further, up to dimension $N-i+n$, $\tilde{H}^*(X(i))$ coincides with the dimension shift of $N-i$ with the kernel of the homomorphism $B(i) \to B(i-1)$ for $i \geq 2$, resp. $B_1 \to \tilde{H}^*(X)$ for $i=1$. Finally the composite $Z_i \subset X(i) \to Y_{i+1}$ induces a homomorphism $\tilde{H}^*(Y_{i+1}) \to \tilde{H}^*(Z_i)$ which coincides (up to some difference in the gradings) with the homomorphism $B_i \to B_{i-1}$ in the resolution.

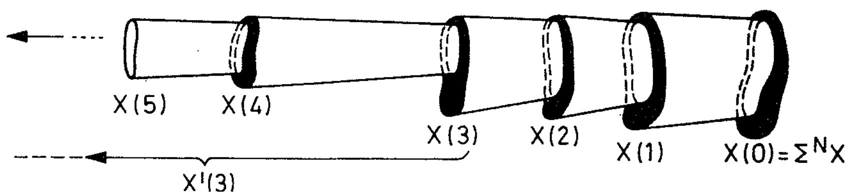
It is worth mentioning that every $X(q)$ is $(N-1)$-connected.

Indeed, let us verify it in the case of $X(1)$. The difference between the gradings of $\tilde{H}^*(X(1))$ and $\operatorname{Ker}(B_1 \to \tilde{H}^*(X))$ is $N-1$. Now the kernel is trivial in dimension 0 because there are no relations between elements of $\tilde{H}^0(X)$. Thus $\operatorname{Ker}[B_1 \to \tilde{H}^*(X)]_0 = 0$ and $\tilde{H}^{N-1}(X(1)) = 0$.

By analogously using that $\operatorname{Ker}[B_i \to B_{i-1}]_j = 0$ for $j < i$ we obtain that $X(i)$ is $(N-1)$-connected for any i.

Let us now transform the chain of mappings of $X(i)$ into a filtration. That is, let these mappings be transformed into imbeddings.

We construct the cylindre of each mapping $X(i) \to X(i-1)$, then attach them to each other as shown on the picture (the resulting space is called a "telescope"):



Let $X'(k)$ denote the part of the telescope to the left from $X(k)$ (on the picture). The chain of inclusions

$$\ldots \subset X'(k) \subset \ldots \subset X'(2) \subset X'(1) \subset X'(0)$$

is clearly homotopy equivalent to

$$\ldots \to X(k) \to \ldots \to X(2) \to X(1) \to X(0) = \Sigma^N X.$$

In the sequel let us write $X(k)$ instead of $X'(k)$. As all constructions are made up to homotopy equivalence, this may be done.

Later on it will be convenient to consider the filtration to be infinite in both directions with $\Sigma^N X = X(0) = X(-1) = X(-2) = \ldots$ (We notice that the notations here are not consistent with those in §18: the numeration of the filtration is in the opposite.)

The Adams spectral sequence is obtained by applying to the above filtration the same construction already used in §18 for the Leray spectral sequence. The main difference is in the use of homotopy rather than homology groups. It will be noted that homotopy groups in general are not applicable to spectral sequences as the formula $\pi_q(A, B) = \pi_q(B/B)$ is not valid. Nevertheless we shall have it as we may restrict ourselves to stable dimensions.

Let us consider the conclusion mapping of two pairs $(X(s), X(s+r)) \to (X(s+1-r), X(s+1))$ where $r \geq 1$, and introduce the groups $E_r^{s,t}$ by

$$E_r^{s,t} = \operatorname{Im}[\pi_{N+t-s}(X(s), X(s+r)) \to \pi_{N+t-s}(X(s+1-r), X(s+1))]$$

where the homomorphisms of the homotopy groups are induced by the inclusion mapping, and $t \leq n-r$.

Let us clarify the reason of the restriction on t. With N fixed, the formula for $E_r^{s,t}$ is correct for every r, s and t. How does the group depend on N? The space $X(m)$ has cohomology not depending on N in dimensions N through $N-m+n$. By substituting N by a larger number M we replace $X(m)$ by another space $\tilde{X}(m)$ which is homotopy equivalent with $\Sigma^{M-N}X(m)$ up to the dimension $M-m+n$. It follows then that under the restriction on t, $E_r^{s,t}$ is independent of N.

By tending with N and n to infinity we are thus able to define $E_r^{s,t}$ for any r, s and t in invariant way.

There can also be given the following equivalent definition. Set $E_r^{s,t} = G_r^{s,t}/D_r^{s,t}$ where

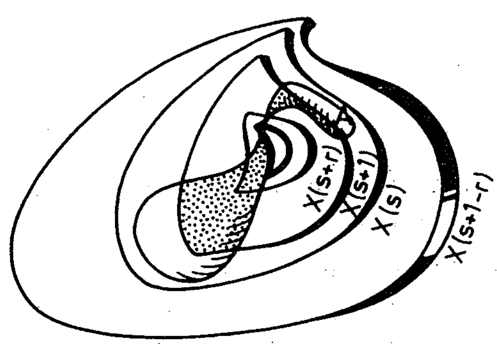
$$G_r^{s,t} = \operatorname{Im}[\pi_{N+t-s}(X(s), X(s+r)) \to \pi_{N+t-s}(X(s), X(s+1))],$$

$$D_r^{s,t} = \operatorname{Im}[\pi_{N+t-s}(X(s-1-r), X(s)) \to \pi_{N+t-s}(X(s), X(s+1))].$$

(This is the original definition given by Adams.)

The group $G_r^{s,t}$ is induced by the inclusion $(X(s), X(s+r)) \to (X(s), X(s+1))$ while $D_r^{s,t}$ is defined by using the boundary operator in the exact sequence of the triplet $(X(s+1-r), X(s), X(s+1))$.

To show the equivalence, we draw a picture for $X(s+1-r) \supset X(s) \supset X(s+1) \supset \supset X(s+r)$:



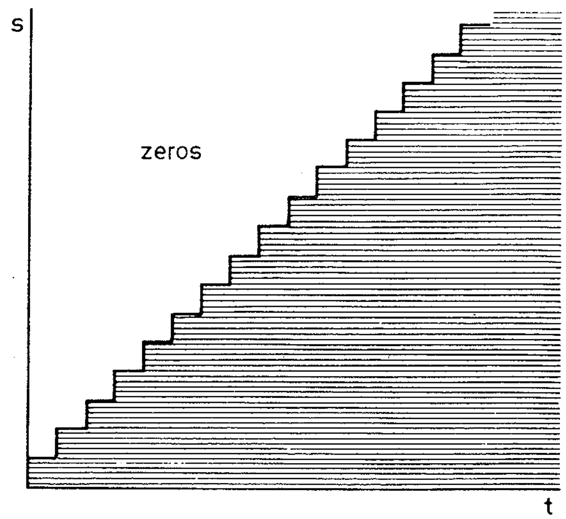
The elements of $D_r^{s,t}$ are classes represented by absolute spheroids of dimension $N+t-s$, lying in $X(s)$, regarded as relative ones modulo $X(s+1)$ (spheroid A). They are homotopic to zero if considered in the whole $X(s+1-r)$, as being boundaries of relative spheroids. The elements of $G_r^{s,t}$ are represented by relative spheroids of $X(s)$ mod $X(s+r)$ taken as relative spheroids in $X(s)$ mod $X(s+1)$, $(X(s+1) \supset X(s+r))$. Clearly $G_r^{s,t} \supset D_r^{s,t}$. Now to produce the quotient space $G_r^{s,t}/D_r^{s,t}$ we must consider the spheroids in $X(s+1-r)$ mod $X(s+1)$.

In the last step we have just taken the image by the homomorphism of $\pi_{N+t-s}((X(s), X(s+r))$ into $\pi_{N+t-s}(X(s+1-r), X(s+1))$ induced by the inclusion. That is exactly $E_r^{s,t}$ as given in the first definition.

Next we are going to study the groups $E_r^{s,t}$ more thoroughly.

They are defined for every $r \geq 1$, $s \geq 0$ and $t \geq 0$, moreover $E_r^{s,t} = 0$ for $t < s$ as clearly follows from the definition, because all $X(s)$ are $(N-1)$-connected

Write $E_r = \bigoplus_{s,t} E_r^{s,t}$. It is shown on the following picture:



We shall observe the behaviour of E_r as r is increasing. (We do not speak about spectral sequence because the differentials are not yet introduced.) So let r be increasing. With s and t fixed, the group $\pi_{N+t-s}(X(s+1-r)\ X(s+1))$ stabilizes at $r = s+$ $+1$ and, for every large r, is equal to $\pi_{N+t-s}(\Sigma^N X,\ X(s+1))$. We cannot say anything like this about $\pi_{N+t-s}(X(s), X(s+r))$ because we have no *a priori* information about the pair $(X(s),\ X(s+r))$; in general we have no reason to expect $E_r^{s,t}$ to stabilize and the question of convergence of a spectral sequence requires special investigation.

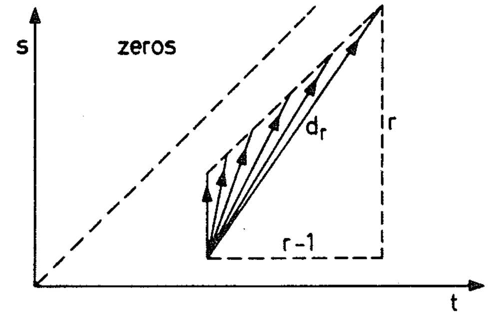
Clearly $\operatorname{Im}[\pi_{N+t-s}(X(s),\ X(s+r)) \to \pi_{N+t-s}(\Sigma^N X,\ X(s+1))]$ is a subgroup of $\operatorname{Im}[\pi_{N+t-s}(X(s), X(s+r-1)) \to \pi_{N+t-s}(\Sigma^N X, X(s+1))]$ because the latter mapping is the composite of the former and $\pi_{N+t-s}(X(s),\ X(s+r+1)) \to \pi_{N+t-s}(X(s),\ X(s+r))$.

Thus the limit group $E_\infty = \bigoplus_{s,t} E_\infty^{s,t}$ may be defined by $E_\infty^{s,t} = \bigcap_r E_r^{s,t}$.

We may also use the second definition $E_r^{s,t} = G_r^{s,t}/D_r^{s,t}$ for defining the limit, by taking $G_\infty^{s,t} = \bigcap G_r^{s,t}$ and

$$D_\infty^{s,t} = \operatorname{Im}[\pi_{N+t-s+1}(\Sigma^N X,\ X(s)) \to \pi_{N+t-s}(X(s),\ X(s+1))].$$

Let us now define the differentials $d_r^{s,t}\colon E_r^{s,t} \to E_r^{s+r,t+r-1}$. It will be recalled that indexing in the Adams spectral sequence is something entirely different from that in the Leray spectral sequence. The differentials $d_r^{s,t}$ act along directions near to the direction of the bisector of the first quadrant



Consider the triples $(X(s), X(s+r), X(s+2r))$ and $(X(s+1-r), X(s+1), X(s+r+1))$ and the boundary homomorphisms of their homotopy sequences. We have a diagram

$$\begin{array}{ccc}
\partial_1: \pi_{N+t-s}(X(s),\ X(s+r)) & \longrightarrow & \pi_{N+(t+r-1)-(s+r)}(X(s+r),\ X(s+r+r)) \\
\Big\downarrow f & & \Big\downarrow g \\
\partial_2: \pi_{N+t-s}(X(s+1-r),\ X(s+1)) & \longrightarrow & \pi_{N+(t+r-1)-(s+r)}(X(s+1),\ X(s+r-1))
\end{array}$$

By definition $d_r^{s,t}$ is restriction of ∂_2 to $E_r^{s,t} = \operatorname{Im} f$ which is a subgroup of $\pi_{N+t-s}(X(s+1-r),\ X(s+1))$.

It takes its values in $E_r^{s+r,\ t+r-1} = \operatorname{Im} g$.

Obviously $d_r^{s+r,t+r-1} \circ d_r^{s,t} = 0$.

This is now the best time to formulate the main theorem, that has partly been dealt with.

The Adams theorem

Theorem. Let X be a CW complex of finite type and p be a prime. Then there exists a spectral sequence $\{E_r^{s,\ t} = E_r^{s,\ t}(X)\}$, where $E_r^{s,\ t} = 0$ if $s<0$ and $t<s$ (and in particular $E_r^{s,\ t} = 0$ for $t<0$) with differentials

$$d_r^{s,t}: E_r^{s,t} \to E_r^{s+r,t+r-1}$$

such that

(1) there is a canonical isomorphism

$$E_2^{s,t} \cong \operatorname{Ext}_A^{s,t}(\tilde{H}^*(X); \mathbf{Z}_p)$$

(here $\mathbf{Z}_p$ is considered as an A-module with trivial action of A and with a single generator of dimension 0);

(2) there is a canonical isomorphism

$$E_{r+1}^{s,t} = \operatorname{Ker} d_r^{s,t}/\operatorname{Im} d_r^{s-r,\, t-r+1};$$

(3) for $r>s$, $\operatorname{Im} d_r^{s-r,t-r+1} = 0$ and so $E_k^{s,t} \subset E_r^{s,t}$ $(s<r<k)$; let $E_\infty^{s,t} = \bigcap_{s<r<\infty} E_r^{s,t}$, then there exist groups $B^{s,t} \subset \pi_{t-s}^S(X)$ such that

$$B^{s,t} \subset B^{s-1,t-1} \subset \ldots \subset B^{0,t-s} = \pi_{t-s}^S(X)$$

and $E_\infty^{s,t} \cong B^{s,t}/B^{s+1,t+1}$;

(4) $\bigcap_{t-s=m} B^{s,t} = K^m$ is the subgroup of all elements of $\pi_{t-s}^S(X)$ whose order is finite and relative prime to p.

Proof. The groups $E_r^{s,t}$ and the differentials $d_r^{s,t}$ are already defined. Let us clarify the structure of E_1 and E_2. By definition $E_1^{s,t} = \pi_{N+t-s}(X(s), X(s+1))$. As follows from the construction of the spaces $X(k)$ there exists a fibration $X(s) \xrightarrow{X(s+1)} \Pi K_m$, hence $\pi_q(X(s), X(s+1)) = \pi_q(\Pi K_m)$ for all q. Then

$$E_1 = \oplus_{s,\ t} E_1^{s,t} = \oplus_{s,\ t}\pi_{N+t-s}(X(s), X(s+1)) = \oplus_{s,\ t}\pi_{N+t-s}(\Pi K_m).$$

(We recall that $N \gg t-r=t-1$. Thus N does depend on t and so do the spaces $X(s)$ and $X(s+1)$ whose definition includes N. Nevertheless the group $\pi_{N+t-s}(X(s), X(s+1))$ will not depend on anything, if N is sufficiently large, thus all terms are correctly defined.)

Consider the group $E_1^s = \oplus_t E_1^{s,t}$. It is further equal to $\oplus_t \pi_{N+t-s}(\Pi K_m) = \oplus_t \mathbf{Z}_p$ where $\mathbf{Z}_p$ appears every time t is such that $N+t-s$ coincides with one of the dimensions taking part in the direct product.

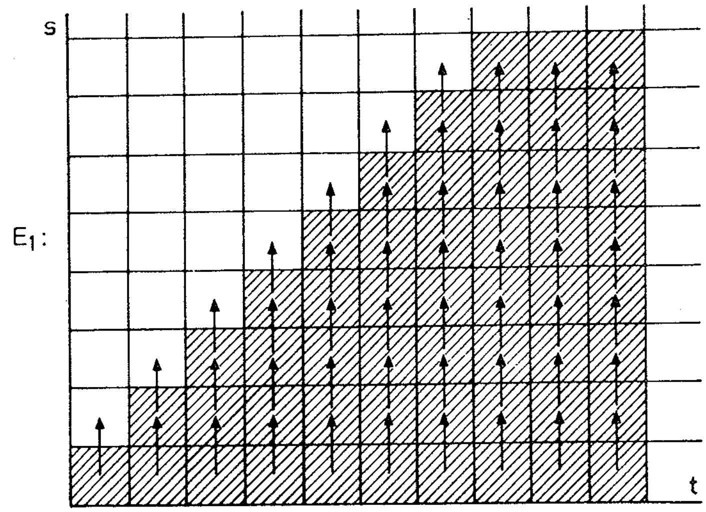
Consequently for $t \ll N$ the terms $\mathbf{Z}_p$ in the sum are in a one-to-one correspondence with the generators of the A-module $H^*(\Pi K_m)$ and have the same dimensions. Hence they are in a one-to-one correspondence with the generators of the A-module B_s and have dimensions larger by $N-s$ units.

By other words, $\pi_{N+t-s}(\Pi K_m) = [\operatorname{Hom}_A(B_s, \mathbf{Z}_p)]_t$.

A homomorphism $B_s \to \mathbf{Z}_p$ may send any generator of the A-module B_s into any element of $\mathbf{Z}_p$; on the other hand any element of the form $\varphi\alpha$, where $\varphi \in A^q$, $q>0$, is necessarily sent into 0. Thus $\operatorname{Hom}_A(B_s, \mathbf{Z}_p)$ as a vector space over $\mathbf{Z}_p$ is generated by homomorphisms $B_s \to \mathbf{Z}_p$ that send one of the generators of the A-module B_s into $1 \in \mathbf{Z}_p$ and the rest into 0. Any such homomorphism has the same degree in the graded module $\operatorname{Hom}_A(B_s; \mathbf{Z}_p)$ as the generator itself.

We conclude that $E_1^{s,t} = [\operatorname{Hom}_A(B_s, \mathbf{Z}_p)]_t$ and $\oplus_t E_1^{s,t} = \operatorname{Hom}_A(B_s, \mathbf{Z}_p)$ with regard to gradings.

Consider the homomorphism d_1; $d_1^{s,\ t}\colon E_1^{s,\ t} \to E_1^{s+1,\ t}$, i. e. $[\operatorname{Hom}_A(B_s, \mathbf{Z}_p)]_t \to$ $\to [\operatorname{Hom}_A(B_{s+1}, \mathbf{Z}_p)]_t$



which, as it may easily be verified by the reader, coincides with the homomorphism induced by the mapping $B_{s+1} \to B_s$. As soon as the statement (2) of the Adams theorem is proved it will follow that

$$E_2^{s,t} = \operatorname{Ext}_A^{s,t}(\tilde{H}^*(X), \mathbf{Z}_p).$$

Next we prove statement (2), that is

$$\operatorname{Ker} d_r^{s,t}/\operatorname{Im} d_r^{s-r,t-r+1} = E_{r+1}^{s,t} =$$

$$= \operatorname{Im}[\pi_{N+t-s}(X(s), X(s+r-1)) \to \pi_{N+t-s}(X(s-r), X(s+1))].$$

Let us examine $\operatorname{Ker} d_r^{s,t}/\operatorname{Im} d_r^{s-r,t-r+1}$ where

$$d_r^{s,t}\colon E_r^{s,t} \to E_r^{s+r,t+r-1},\ d_r^{s-r,t-r+1}\colon E_r^{s-r,t-r+1} \to E_r^{s,t}.$$

By the definition of $d_r^{s,t}$ we have

$$\begin{array}{ccc}
\partial_1\colon \pi_{N+t-s}(X(s), X(s+r)) & \longrightarrow & \pi_{N+t-s-1}(X(s+r), X(s+2r)) \\
\downarrow f & & \downarrow g \\
\partial_2\colon \pi_{N+t-s}(X(s+1-r), X(s+1)) & \longrightarrow & \pi_{N+t-s-1}(X(s+1), X(s+r+1))
\end{array}$$

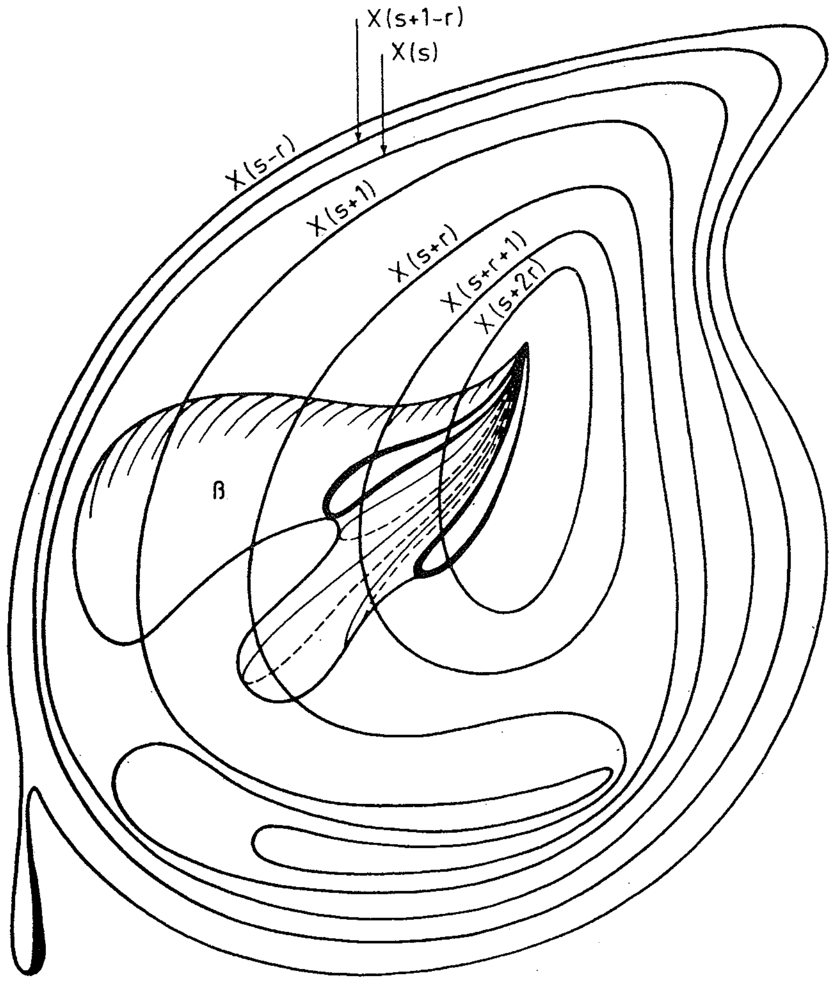
$$d_r^{s,t} = \partial_2|_{\operatorname{Im} f}$$

Consider the following chain of spaces

$$X(s-r) \supset X(s+1-r) \supset X(s) \supset X(s+1) \supset$$

$$\supset X(s+r) \supset X(s+r+1) \supset X(s+2r)$$

An element $\alpha \in E_r^{s,t} = \operatorname{Im}[\pi_{N+t-s}(X(s), X(s+r)) \to \pi_{N+t-s}(X(s+1+r), X(s+1))]$ is the image of some $\beta \in \pi_{N+t-s}(X(s), X(s+r))$ by the natural homomorphism (a spheroid representing β is shown on the picture below).



Suppose $d_r^{s,t}\alpha = 0$. Then $g\partial_1(\beta) = 0$, implying that the boundary of β, as an $(n+t-s-1)$-dimensional spheroid in $X(s+1)$, is homotopic to a spheroid lying in $X(s+r+1)$. Then β, considered as a spheroid modulo $X(s+1)$, is homotopic modulo $X(s+r+1)$ to a spheroid of $X(s)$. Consequently α belongs not only to $E_r^{s,t}$ but to the smaller group $\operatorname{Im}[\pi_{N+t-s}(X(s), X(s+r+1)) \to \pi_{N+t-s}(X(s+1-r), X(s+1))]$ as well.

Conversely, if α belongs to this subgroup, then $d_r^{s,t}\alpha = 0$, i. e. $\operatorname{Ker} d_r^{s,t} = \operatorname{Im}[\pi_{N+t-s}(X(s), X(s+r+1)) \to \pi_{N+t-s}(X(s+1-r), X(s+1))]$.

The homomorphism $\pi_{N+t-s}(X(s+1-r), X(s+1)) \to \pi_{N+t-s}(X(s-r), X(s+1))$ induces a homomorphism

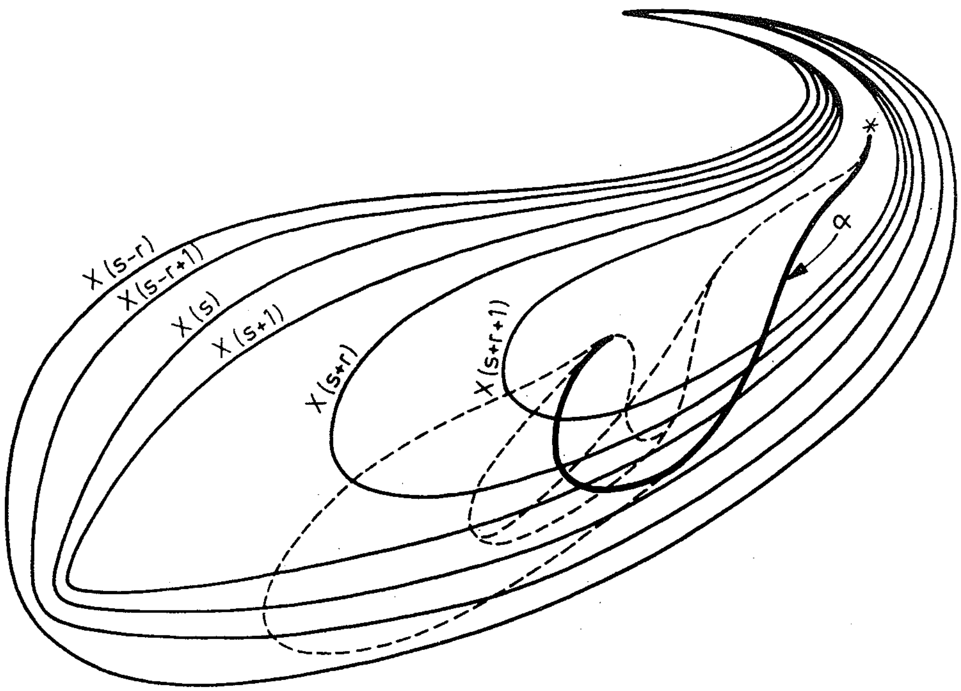
$$\operatorname{Ker} d_r^{s,t} = \operatorname{Im}[\pi_{N+t-s}(X(s), X(s+r+1)) \to \pi_{N+t-s}(X(s+1-r), X(s+1))] \to$$

$$\to \operatorname{Im}[\pi_{N+t-s}(X(s), X(s+r+1)) \to \pi_{N+t-s}(X(s-r), X(s+1))] = E_{r+1}^{s,t}.$$

Its kernel will be shown to be $\operatorname{Im} d_r^{s-r,t-r+1}$. Indeed, by the definition of $d_r^{s-r,t-r+1}$ we have

$$\begin{array}{ccc} \partial_3 : \pi_{N+t-s+1}(X(s-r), X(s)) & \longrightarrow & \pi_{N+t-s}(X(s), X(s+r)) \\ \downarrow f' & & \downarrow g' \\ \partial_4 : \pi_{N+t-s+1}(X(s+1-2r), X(s+1-r)) & \longrightarrow & \pi_{N+t-s}(X(s+1-r), X(s+1)) \end{array}$$

$$d_r^{s-r,t-r+1} = \partial_4|_{\operatorname{Im} f'}$$



If $\alpha \in \operatorname{Ker} d_r^{s,t} \subset \pi_{N+t-s}(X(s+1-r), X(s+1))$ is sent into zero by the homomorphism to $\pi_{N+t-s}(X(s-r), X(s+1))$ then the relative spheroid representing α (the continuous line on the picture) is homotopic to zero in the pair $(X(s-r), X(s+1))$. The homotopy φ_t: $(D^{N+t-s}, S^{N+t-s-1}) \to (X(s-r), X(s+1))$ may be considered as a mapping $D^{N+t-s} \times I = D^{N+t-s+1} \to X(s-r)$ such that the bottom $D^{N+t-s} \times \{0\}$, the side surface $S^{N+t-s-1} \times I$ and the upper face $D^{N+t-s} \times \{1\}$ are sent into the spheroid α, the space $X(s+1)$, and the base point, respectively. Thus we obtain an $(N+t-s+1)$-dimensional spheroid in $X(s-r)$ mod $X(s-r+1)$ whose boundary is obtained from α by adding to i some part lying in $X(s+1)$. Finally we take into account that α belongs to the image of $\pi_{N+t-s}(X(s+r+1))$, i. e. we may consider α as lying in $(X(s), X(s+r+1))$, and the spheroids constructed as a relative spheroid $X(s-r)$ mod $X(s)$.

Let us look at the picture once more. We have a spheroid $\gamma \in \pi_{N+t-s+1}(X(s-r), X(s))$ whose boundary coincides with α as a relative spheroid of $X(s-r+1)$ mod $X(s+1)$, i. e. $\alpha = g'\partial_3\alpha \in \operatorname{Im} d_r^{s-r,t-r+1}$.

By repeating the argumentation in the opposite direction we get that, conversely, if α belongs to the image of $d_r^{s-r,\ t-r+1}$ then it is mapped into zero by $\operatorname{Ker} d_r^{s,\ t} \to E_{r+1}^{s,\ t}$.

Statement (2) is proved, and so is (1).

Let us now make a remark concerning (1), which will have significance in practical applications of the theorem.

Take a free resolution of the A-module $\tilde{H}^*(X)$:

$$\ldots \to B_3 \to B_2 \to B_1 \to \tilde{H}^*(X) \to 0$$

Suppose that we have to compute $\mathrm{Ext}_A^{s,t}(\tilde{H}^*(X); \mathbf{Z}_p)$. First we have to apply the functor Hom to the resolution and then to take the homology of the complex obtained. Since the choice of the resolution does not alter the final result it is worth looking for the most convenient resolution.

Let us choose in $\tilde{H}^*(X)$ a minimal generating system. This may be done by first taking a system of additive generators in the first non-trivial group $\tilde{H}^q(X)$, then adding to it those elements of an additive generating system of the second non-trivial group that are independent over the elements obtained by any cohomology operation from elements of the previous group $\tilde{H}^q(X)$, etc.

The result is some minimal generating system $a_1, a_2, a_3, \ldots$ such that for any a_k any decomposition $a_k = \sum_{i \neq k} \varphi_i a_i$ with $\varphi_i \in A$, $\deg \varphi_i > 0$ is impossible.

Next a free A-module B_1 is spanned on the selected generators. The generators in the kernel of $B_1 \to \tilde{H}^*(X)$ are then chosen in the same way as in $\tilde{H}^*(X)$. The subsequent steps are similar.

For every k the homomorphism $\mathrm{Hom}_A(B_k, \mathbf{Z}_p) \to \mathrm{Hom}_A(B_{k+1}, \mathbf{Z}_p)$ is clearly trivial. (Indeed any homomorphism $B_k \to \mathbf{Z}_p$ sends any element $\sum_i \varphi_i a_i^{(k)}$, where $a_i^{(k)}$ are generators of B_k and $\deg \varphi_i > 0$, into zero. Now the homomorphism $B_{k+1} \to B_k$ in question sends all generators of B_{k+1} into elements of this form.)

Consequently for this resolution the complex $\{\mathrm{Hom}_A(B_k, \mathbf{Z}_p)\}$ has trivial differential and so

$$\mathrm{Ext}_A^k(\tilde{H}^*(X), \mathbf{Z}_p) = \mathrm{Hom}_A(B_k, \mathbf{Z}_p).$$

Let us now prove statements (3) and (4) of the Adams theorem. Write

$$B^{s,t} = \mathrm{Im}\{\pi_{N+t-s}(X(s)) \to \pi_{N+t-s}(X(0)) = \pi_{N+t-s}(\Sigma^N X) = \pi_{t-s}^S(X)\}$$

where $X(s) \to X(0)$ is inclusion. We obtain a chain of inclusions

$$\ldots \subset B^{s,t} \subset B^{s-1,t-1} \subset \ldots \subset B^{0,t-s}$$

where $B^{0,t-s} = \mathrm{Im}[\pi_{N+t-s}(X(0)) \to \pi_{N+t-s}(X(0))] = \pi_{N+t-s}(\Sigma^N X) = \pi_{t-s}^S(X)$.

This filtration is essentially infinite for every complex X.

We must prove $E_\infty^{s,t} \cong B^{s,t}/B^{s+1,t+1}$ and $\bigcap_{t-s=m} B^{s,t} = K^m$; here $K^m \subset \pi_m^S(X)$ denotes the subgroup of all elements whose order is finite and relatively prime to p.

Remark 1. The E_2 term, and, in consequence, E_∞ as well, only contains elements of order p as immediately follows from the remark about the choice of the resolution.

Remark 2. The statement is obvious for $t<s$. Indeed, for $t<s$, $B^{s,t} = = \operatorname{Im}[\pi_{N+t-s}(X(s)) \to \pi^S_{N+t-s}(X)] = \operatorname{Im}[\pi_{N+t-s}(X(s)) \to 0] = 0$, and $E^{s,t}_\infty =$ Analogously, if $s<0$ then again $E^{s,t}_\infty = 0$ and $B^{s,t} = \pi_{N+t-s}(X(0))$ while $B^{s+1,t+1} = = \operatorname{Im}[\pi_{N+t-s}(X(s+1)) \to \pi_{N+t-s}(X)] = \pi_{N+t-s}(X(0))$, as $s+1 \leq 0$.

A particular case: All the stable homotopy groups of the space are finite

An algebraic lemma

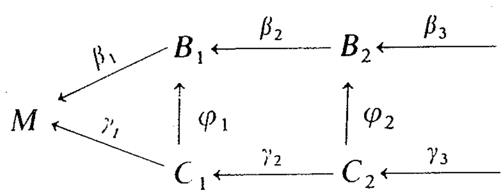
Let M be an arbitrary A-module and

$$M \xleftarrow{\beta_1} B_1 \xleftarrow{\beta_2} B_2 \xleftarrow{\beta_3} \ldots$$

its projective resolution. Assume that

$$M \xleftarrow{\gamma_1} C_1 \xleftarrow{\gamma_2} C_2 \xleftarrow{\gamma_3} \ldots$$

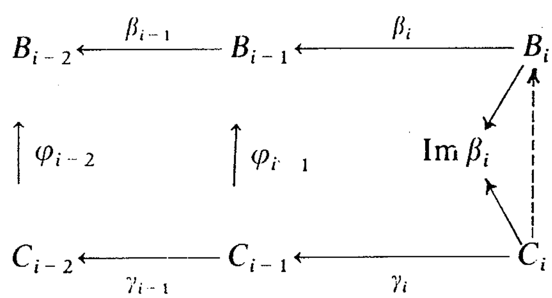
is a sequence of projective A-modules such that the composite of any pair of subsequent homomorphisms is trivial. Then there exist A-homomorphisms $\varphi_i \colon C_i \to B_i$ such that the diagram



is commutative.

(The lemma is true without assuming projectiveness of B_i.)

Suppose we are given the homomorphisms $\varphi_0 \colon M \to M$ (identity), $\varphi_1, \varphi_2, \ldots, \varphi_{i-1}$. In the diagram



the homomorphism $\beta_{i-1}\varphi_{i-1}\gamma_i \colon C_i \to B_{i-2}$ is trivial because $\beta_{i-1}\varphi_{i-1}\gamma_i = \varphi_{i-2}\gamma_{i-1}\gamma_i$. Therefore $\operatorname{Im}\varphi_{i-1}\gamma_i \subset \operatorname{Ker}\beta_{i-1} = \operatorname{Im}\beta_i$ i.e. $\varphi_{i-1}\gamma_i \colon C_i \to B_{i-1}$ may be considered to be a homomorphism $C_i \to \operatorname{Im} \beta_i$. Because C_i is a projective A-module, there exists a homomorphism $C_i \to B_i$ whose composite with the epimorphism $B_i \to \operatorname{Im} \beta_i$ coincides with $\varphi_{i-1}\gamma_i$. Let us choose $\varphi_i = \varphi_{i-1}\gamma_i$.

The Serre filtration

Together with the Adams filtration

$$\ldots \to X(2) \to X(1) \to \Sigma^N X$$

we have a similar filtration that arises as homotopy groups are being killed by Serre's procedure. Let m be the dimension of the first non-trivial cohomology group of X. (That is, $\tilde{H}^j(X; \mathbf{Z}_p) = 0$ for $j < m$. We mention that $m > 0$ as $\tilde{H}^0(X; \mathbf{Z}_p) \neq 0$ would imply that X is not connected and so $\pi_0^S(X) = \mathbf{Z} \oplus \ldots \oplus \mathbf{Z}$ where the number of the summands is equal to the number of connected components minus one, in obvious contradiction with the assumption that the homotopy groups are finite.)

Denote by $\tilde{Y}_1$ the product space $K(H^m(X; \mathbf{Z}_p), N+m) = K(\mathbf{Z}_p, N+m) \times \ldots \times K(\mathbf{Z}_p, N+m)$. Let $\Sigma^N X \to \tilde{Y}_1$ be a mapping (defined uniquely up to homotopy) that induces isomorphism of the groups $H^{N+m}(\cdot\,; \mathbf{Z}_p)$ and $\tilde{X}_1$ be the fibre of the homotopy equivalent fibration.

The mapping $\pi_{N+m}(\Sigma^N X) \to \pi_{N+m}(\tilde{Y}_1)$ is clearly an epimorphism with kernel $\pi_{N+m}(\tilde{X}_1)$ as follows from the exact sequence of the fibration. Hence the order of $\pi_{N+m}(\tilde{X}_1)$ is less than that of $\pi_{N+m}(\Sigma^N X)$.

By repeating the construction we get a sequence of killing spaces and mappings

$$\ldots \to \tilde{X}_2 \to \tilde{X}_1 \to \Sigma^N X.$$

Because the homotopy groups of $\Sigma^N X$ (up to $N+n$) are finite for any $q < n$ there exists an s_0 such that for $s > s_0$ the order of $\pi_{N+q}(X(s))$ is not divisible by p.

The resulted sequence will be called the Serre filtration.

Our aim is to find such mappings f_s: $X(s) \to \tilde{X}_s$ that the diagram

$$\begin{array}{ccccccc}
 & X(1) & \longleftarrow & X(2) & \longleftarrow & X(3) & \longleftarrow \\
\swarrow & \uparrow f_1 & & \uparrow f_2 & & \uparrow f_3 & \\
\Sigma^N X & & & & & & \\
\nwarrow & \tilde{X}_1 & \longleftarrow & \tilde{X}_2 & \longleftarrow & \tilde{X}_3 & \longleftarrow
\end{array}$$

is commutative.

Mapping the Adams filtration into the Serre filtration

Let $\tilde{Y}_i$ be the space used in the definition of the Serre filtration. We have the fibrations $\tilde{X}_i \xrightarrow{\tilde{X}_{i+1}} \tilde{Y}_{i+1}$ and $\tilde{X}_{i+1} \xrightarrow{\tilde{Z}_{i+1}} \tilde{X}_i$ where $\tilde{Y}_{i+1} = \Sigma\tilde{Z}_{i+1}$ (in the stable dimensions). The second fibration induces two homomorphisms, one which preserves the dimensions, $\tilde{H}^*(\tilde{X}_{i+1}) \to H^*(\tilde{Z}_{i+1})$ and another which increases dimensions by one, $\tilde{H}^*(\tilde{Z}_{i+1}) \to \tilde{H}^*(\tilde{X}_i)$. The latter is the transgression and is defined only in stable dimensions. In stable dimensions together they define an A-homomorphism $\tilde{H}^*(\tilde{Z}_{i+1}) \to$ $\to \tilde{H}^*(\tilde{X}_i) \to \tilde{H}^*(\tilde{Z}_i)$ which increases the dimensions by one. Let C_i denote the free A-module with $[C_i]_q = H^{N-i+q}(\tilde{Z}_i; \mathbf{Z}_p)$ for $q < n$. The mapping $\tilde{H}^*(\tilde{Z}_{i+1}) \to \tilde{H}^*(\tilde{Z}_i)$ defines a grading preserving A-homomorphism $C_{i+1} \to C_i$. A mapping $C_1 \to \tilde{H}^*(X)$ is defined by $\Sigma^N X \to \tilde{Y}_1 = \Sigma\tilde{Z}_i$. The result is a sequence

$$\ldots \to C_2 \to C_1 \to \tilde{H}^*(X).$$

Here the composite of any subsequent homomorphisms is clearly zero. (Already the composite $\tilde{H}^*(\tilde{X}_{i+1}) \to \tilde{H}^*(\tilde{Z}_{i+1}) \to \tilde{H}^*(X_i)$ is trivial.)

According to the lemma there is a diagram

$$\begin{array}{ccccccc}
 & \swarrow & B_1 & \longleftarrow & B_2 & \longleftarrow & B_3 \longleftarrow \\
\tilde{H}^*(X) & & \uparrow \varphi_1 & & \uparrow \varphi_2 & & \uparrow \varphi_3 \\
 & \nwarrow & C_1 & \longleftarrow & C_2 & \longleftarrow & C_3 \longleftarrow
\end{array}$$

The mapping $\varphi_1 : C_1 \to B_1$ defines $g_1 : Y_1 \to \tilde{Y}_1$ such that $g_1^* = \varphi_1$ (up to grading and in stable dimensions). Moreover the diagram

$$\begin{array}{ccc}
 & \nearrow & \tilde{Y}_1 \\
\Sigma^N X & & \uparrow g_1 \\
 & \searrow & Y_1
\end{array}$$

is homotopy commutative (as implied by $\tilde{Y}_1 = K(\pi, N+m)$ and the theorem about mappings into Eilenberg–MacLane spaces). If the mappings of $\Sigma^N X$ into Y_1 and $\tilde{Y}_1$ are fibrations, the fibre $X(1)$ of the former is contained in the fibre $\tilde{X}_1$ of the latter, hence there is a mapping $f_1 : X(1) \to \tilde{X}_1$.

Again the diagram

$$\begin{array}{ccc}
H^*(X(1)) & \xleftarrow{f_1^*} & H^*(\tilde{X}_1) \\
\downarrow \tau & & \downarrow \\
H^*(Y_1) & \xleftarrow{g_1^*} & H^*(\tilde{Y}_1)
\end{array}$$

is commutative. Let us construct a mapping $g_1 = Y_1 \to \tilde{Y}_2$ such that $g_2^* = \varphi_2$ (up to gradation, in stable dimensions) and consider the diagram

$$\begin{array}{ccc}
H^*(Y_2) & \xleftarrow{g_2^*} & H^*(\tilde{Y}_2) \\
\downarrow & & \downarrow \\
H^*(X(1)) & \xleftarrow{f} & H^*(\tilde{X}_1) \\
\downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_2} \\
H^*(Y_1) & \xleftarrow{g_1^*} & H^*(\tilde{Y}_1)
\end{array}$$

The small rectangle below and the large rectangle are commutative and τ_1 is a monomorphism (again in the stable dimensions). Hence the commutativity of the rectangle on the top (by the theorem on mappings into Eilenberg–MacLane spaces) and homotopy commutativity of the diagram follow.

$$\begin{array}{ccc}
Y_2 & \longrightarrow & \tilde{Y}_2 \\
\uparrow{\scriptstyle X(2)} & & \uparrow{\scriptstyle \tilde{X}_2} \\
X(1) & \longrightarrow & \tilde{X}_1
\end{array}$$

This makes it possible to define a mapping of fibres $f_2 \cdot X(2) \to \tilde{X}_2$ and the diagram

$$\begin{array}{ccc}
X(2) & \longrightarrow & X(1) \\
\downarrow & & \downarrow \\
\tilde{X}_2 & \longrightarrow & \tilde{X}_1
\end{array}$$

is homotopy commutative, too. Further the construction is carried on similarly.

The Basic Lemma

For any s and $q<n$ and for sufficiently large M, the homomorphism $\pi_{N+q}(X(s+M)) \to \to \pi_{N+q}(X(s))$ is trivial on the p-components.

In view of the preceding construction the homomorphism $\pi_{N+q}(X(M)) \to \to \pi_{N+q}(\Sigma^N X)$ is clearly trivial on the p-components if M is sufficiently large, as immediately follows from the diagram

$$\begin{array}{ccccc} & \swarrow & X(1) & \longleftarrow \cdots \longleftarrow & X(M) \\ \Sigma^N X & & \downarrow f_1 & & \downarrow f_M \\ & \nwarrow & X_1 & \longleftarrow \cdots \longleftarrow & \tilde{X}_M \end{array}$$

and the triviality of the p-component $\pi_{N+q}(X(M))$ for large M.

To finish the proof of the lemma it remained to notice that the part

$$\ldots \to X(s+1) \to X(s)$$

of the Adams filtration is itself the Adams filtration of the space $X(s)$.

Remark 1. For any m, s and t the order of the group $\pi_m(X(s), X(t))$ is a power of p.

Remark 2. For any m and any prime number $p' \neq p$, the p'-components of the groups $\pi_m(X(s))$ are independent of s and are isomorphically mapped onto each other by the homomorphisms induced by the inclusions $X(s+r) \subset X(s)$.

Both remarks follow from the exact sequences for triples and fibrations

$$X(s) \xrightarrow{Y_s} X(s-1)$$

and from the fact that the homotopy groups of the spaces Y_s are p-groups.

Deducing the statements (3) *and* (4) *of the Adams theorem from the lemma and the remarks.* By definition

$$\begin{aligned} B^{s,t} &= \operatorname{Im}[\pi_{N+t-s}(X(s)) \to \pi_{N+t-s}(\Sigma^N X)], \\ B^{s+1,t+1} &= \operatorname{Im}[\pi_{N+t-s}(X(s+1)) \to \pi_{N+t-s}(\Sigma^N X)] = \\ &= \operatorname{Ker}[\pi_{N+t-s}(\Sigma^N X) \to \pi_{N+t-s}(\Sigma^N X, X(s+1))] \end{aligned}$$

hence

$$B^{s,t}/B^{s+1,t+1} = \operatorname{Im}[\pi_{N+t-s}(X(s)) \to \pi_{N+t-s}(\Sigma^N X, X(s+1))].$$

Further

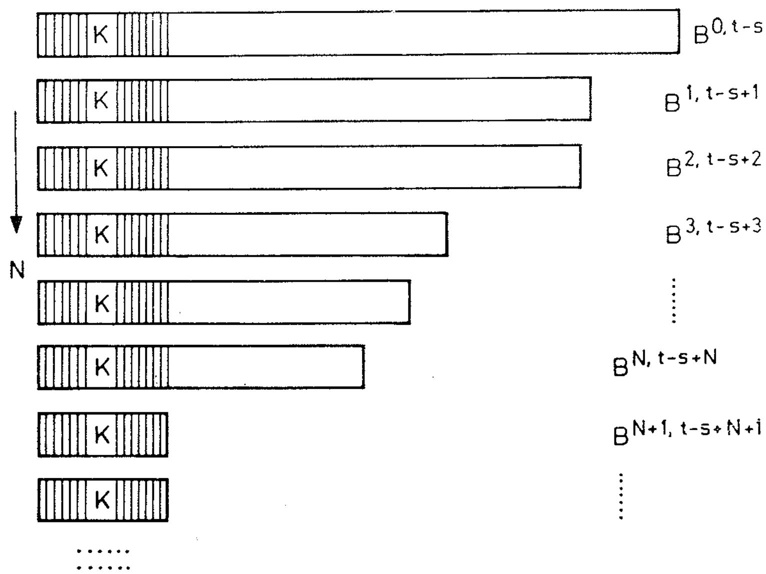
$$E_M^{s,t} = \operatorname{Im}[\pi_{N+t-s}(X(s), X(s+M)) \to \pi_{N+t-s}(\Sigma^N X, X(s+1))].$$

Consider the diagram

$$\begin{array}{ccccc} & & (\beta) & & (\delta) \\ (\varepsilon) & \overset{\xi_2}{\nearrow} & \pi_{N+t-s}(X(s), X(s+M)) & \xrightarrow{\xi_7} & \pi_{N+t-s-1}(X(s+M)) \\ \pi_{N+t-s}(X(s)) & & \downarrow \xi_5 \quad (\gamma) & & \downarrow \xi_{10} \\ \xi_1 \downarrow & \underset{\xi_3}{\searrow} & \pi_{N+t-s}(X(s), X(s+1)) & \overset{\xi_8}{\searrow} & \\ & & \downarrow \xi_6 & & \pi_{N+t-s-1}(X(s+1)) \\ \pi_{N+t-s}(\Sigma^N X) & \xrightarrow{\xi_4} & \pi_{N+t-s}(\Sigma^N X, X(s+1)) & \underset{\xi_9}{\nearrow} & \\ & & (\alpha) & & \end{array}$$

It is commutative and the three "horizontal" lines are exact. Let $\alpha \in E_M^{s,t}$, i. e. $\alpha \in \pi_{N+t-s}(\Sigma^N X, X(s+1))$, and $\alpha = \xi_6 \xi_5(\beta)$ where $\beta \in \pi_{N+t-s}(X(s+M))$. Write $\gamma = \xi_5(\beta)$ and $\delta = \xi_7(\beta)$. Then, by remark 1, β is of order p^h and so is δ, hence $\xi_{10}(\delta) = 0$ and $\xi_8(\gamma) = 0$, i. e. $\gamma = \xi_3(\varepsilon)$, $\varepsilon \in \pi(X(s))$. Then $\alpha = \xi_6 \xi_3(\varepsilon)$, i. e. $\alpha \in B^{s,t}/B^{s+1,t+1}$. For every M we clearly have the inclusion $B^{s,t}/B^{s+1,t+1} \subset E_M^{s,t}$, thus $B^{s,t}/B^{s+1,t+1} = \bigcap_M E_M^{s,t} = E_\infty^{s,t}$, proving statement (3).

Statement (4) is the direct consequence of remark 2 and the basic lemma.



Some further properties of the Adams spectral sequence

Before proceeding to prove statements (3) and (4) of the Adams theorem in the general case let us examine the behaviour of the spectral sequence under mappings.

Suppose that we have constructed Adams spectral sequences for the spaces X and X' and we are given some mapping $f: X \to X'$. It induces a homomorphism between the A-modules $\tilde{H}^*(X')$ and $\tilde{H}^*(X)$. A construction, analogous to that used in the course of proving the last algebraic lemma, gives a "homomorphism of projective resolutions"

$$\begin{array}{ccccccc} \tilde{H}^*(X) & \longleftarrow & B_1 & \longleftarrow & B_2 & \longleftarrow & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \\ \tilde{H}^*(X') & \longleftarrow & B'_1 & \longleftarrow & B'_2 & \longleftarrow & \cdots \end{array}$$

which induces a mapping of filtrations

$$\begin{array}{ccccccc}\Sigma^N X & \longleftarrow & X(1) & \longleftarrow & X(2) & \longleftarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ \Sigma^N X' & \longleftarrow & X'(1) & \longleftarrow & X'(2) & \longleftarrow & \dots \end{array}$$

(The construction is similar to that as Adams filtrations are mapped into Serre filtrations.) That induces, on its turn, mappings of the relative homotopy groups that have taken part in the construction of the Adams spectral sequence. The family of these mappings induces a *homomorphism of the Adams spectral sequences.*

Theorem. The mapping $f: X \to X'$ induces a homomorphism of the Adams spectral sequence $\{E_r^{s,t}, d_r^{s,t}\}$ of X to the Adams spectral sequence $\{'E_r^{s,t}, 'd_r^{s,t}\}$ of X', i. e. a set of homomorphisms $f_r^{s,t}: E_r^{s,t} \to 'E_r^{s,t}$ such that:

(i) the homomorphisms commute with the differentials, i. e. the diagram

$$\begin{array}{ccc} E_r^{s,t} & \xrightarrow{f_r^{s,t}} & 'E_r^{s,t} \\ {\scriptstyle d_r^{s,t}}\downarrow & & \downarrow{\scriptstyle 'd_r^{s,t}} \\ E_r^{s+r,t+r} & \xrightarrow[f_r^{s+r,t+r-1}]{} & 'E_r^{s+r,t+r-1} \end{array}$$

is commutative;

(ii) the homomorphism $f_{r+1}: E_{r+1} \to 'E_{r+1}$ is the mapping induced by $f_r: E_r \to 'E_r$ between the homology of the complexes (E_r, d_r) and $('E_r, 'd_r)$;

(iii) the homomorphism $f_2^{s,t}: E_2^{s,t} = \mathrm{Ext}_A^{s,t}(\tilde{H}^*(X), \mathbf{Z}_p) \to 'E_2^{s,t} = \mathrm{Ext}_A^{s,t}(\tilde{H}^*(X'), \mathbf{Z}_p)$ is induced by $f^*: \tilde{H}^*(X') \to \tilde{H}^*(X)$.

(Explanation. The functor Ext is known from homological algebra to be contravariant in the first variable and covariant in the second. The mapping $\mathrm{Ext}_A^{**}(M_1, N) \to \mathrm{Ext}_A^{**}(M_2, N)$ induced by $M_2 \to M_1$, is constructed in the following way. We choose a mapping between the projective resolutions of M_2 and M_1 to yield a mapping in the opposite direction on the level of Hom, which again induces a mapping between the homology of these complexes.)

(iv) The limit mapping $f_\infty^{s,t}: E_\infty^{s,t} \to 'E_\infty^{s,t}$ is induced by $\pi_*^S(X) \to \pi_*^S(X')$.

We do not prove this theorem because it is obvious. We notice that, however trivial the last statement may be, we cannot consider it as proved as it is based on a statement of the Adams theorem which is not proved as yet. Of course we are not going to use this theorem in the proof of the missing statement. What we shall only need is only the existence of a mapping between the Adams filtrations when a space is mapped into another.

An important corollary. Starting at the second term the Adams spectral sequence only depends on the stable homotopy type of the space.

By other words, if X and X' are stable homotopy equivalent (i. e. their multiple suspensions are homotopy equivalent in the ordinary sense) then for every $r \geq 2$, s and t there exists an isomorphism $E_r^{s,t} \cong {}'E_r^{s,t}$ commuting with the differentials, such that the isomorphic groups $\pi_{t-s}^S(X)$ and $\pi_{t-s}^S(X')$ have the identical filtration and the terms E_∞ are associated with the respective homotopy groups in the same way.

Finishing the proof of the Adams theorem in the general case

The basic lemma, applied successfully in the case of finite stable homotopy groups, is of no use in general. (The proof of the lemma, as given above, would neither do in the general case.) We should like to have the lemma to say: "For sufficiently large M the homomorphism $\pi_{N+q}(X(s+M)) \to \pi_{N+q}(X(s))$ is trivial on the p-component and on the free summands", which is obviously not true, unfortunately. (If $\pi_{N+q}(X(s))$ contains $\mathbf{Z}$ as a free summand then so does $\pi_{N+q}(X(s+M))$, therefore the kernel of $\pi_{N+q}(X(s+M)) \to$ $\to \pi_{N+q}(X(s))$ is finite, because the difference between the homotopy of $X(s)$ and $X(s+M)$ is measured by the homotopies of $Y_{s+1}, Y_{s+2}, \ldots, Y_{s+M}$ which are finite p-groups.

The role of the basic lemma will be played by the following statement.

The Generalized Basic Lemma. Let $\alpha \in \pi_{N+q}(X(s))$ have order ∞ or p^k. Then, for sufficiently large M, α does not belong to the image of the homomorphism $\pi_{N+q}(X(s+M)) \to \pi_{N+q}(X(s))$.

This implies among others the triviality on the p-components of the homomorphism $\pi_{N+q}(X(s+M)) \to \pi_{N+q}(X(s))$ for sufficiently large M, as the p-component of $\pi_{N+q}(X(s))$ is finite.

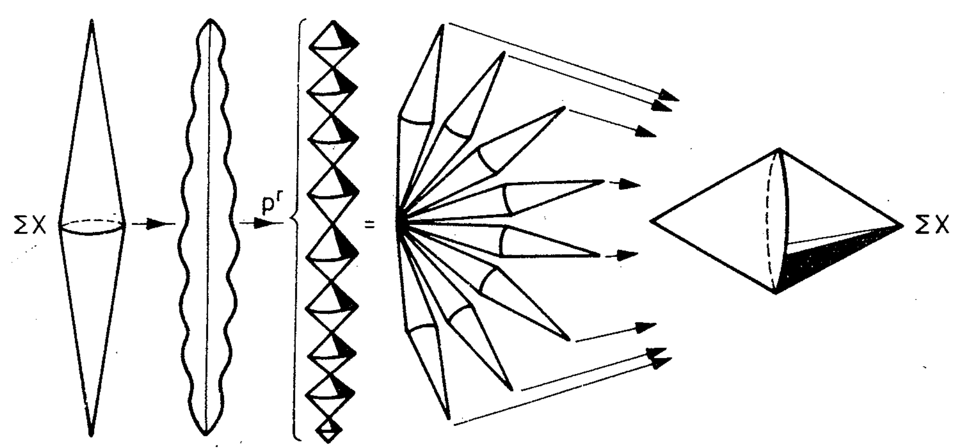
Proof. We may assume without loss of generality that $s=0$, i. e. $X(s) = \Sigma^N X$. We recall that $\ldots \to X(s+1) \to X(s)$ is the Adams filtration for $X(s)$.

By assumption the element α is not infinitely divisible by p i. e. there exists a number r such tha $\alpha \neq p^r \alpha'$ for any $\alpha' \in \pi_{N+q}(\Sigma^N X)$.

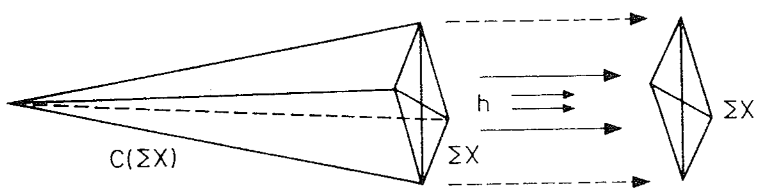
Let a mapping $h: \Sigma X \to \Sigma X$ be defined as follows. There is the well-known mapping $\Sigma X \to \Sigma X \vee \Sigma X$. By composing it with itself we get

$$\Sigma X \to \underbrace{\Sigma X \vee \Sigma X \vee \ldots \vee \Sigma X}_{p^r}.$$

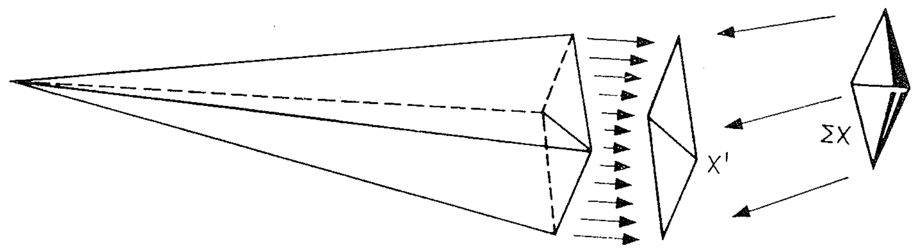
This wedge may be mapped into ΣX by folding its components together.



The result is a mapping $\Sigma X \to \Sigma X$ that may also be described in the following way. As it is known, we have $\Sigma X = X \otimes S^1$. The mapping in view is $X \otimes S^1 \to X \otimes S^1$ induced by the identity mapping $X \to X$ and a mapping $S^1 \to S^1$ of degree p^r. We attach to ΣX the cone over ΣX along this mapping



to obtain a space X'. Next a mapping $\Sigma X \to X'$ is defined in the obvious way



which induces mappings of the $(N-1)$-th suspensions and the Adams filtrations (by the above remark). We have a diagram

$$\begin{array}{ccccccc} \Sigma^N X & \longleftarrow & X(1) & \longleftarrow & X(2) & \longleftarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ \Sigma^{N-1} X' & \longleftarrow & X'(1) & \longleftarrow & X'(2) & \longleftarrow & \cdots \end{array}$$

Now the lemma follows from the old basic lemma and the following statements:
(a) the stable homotopy groups of X' are finite;
(b) $\alpha \notin \operatorname{Ker}[\pi_{N+q}(\Sigma^N X) \to \pi_{N+q}(\Sigma^{N-1} X')]$.
Proof of (a). Examine the exact sequence of the pair $1\Sigma^{N-1}X', \Sigma^N X)$:

$$\pi_{N+r+1}(.,.) \xrightarrow{\partial} \pi_{N+r}(\Sigma^N X) \to \pi_{N+r}(\Sigma^{N-1} X') \to \pi_{N+r}(.,.) \xrightarrow{\partial} \pi_{N+r-1}(\Sigma^N X)$$

Now in the stable dimensions we have

$$\pi_{N+r+1}(\Sigma^{N-1}X', \Sigma^N X) = \pi_{N+r+1}(\Sigma^{N-1}X'/\Sigma^N X) = \pi_{N+r+1}(\Sigma^{N+1}X) = \pi_{N+r}(\Sigma^N X);$$

further the mapping $\partial\colon \pi_{N+r+1}(.,.) = \pi_{N+r}(\Sigma^N X) \to \pi_{N+r}(\Sigma^N X)$ is multiplying by p^r. Thus the kernel and cokernel of the homomorphism are finite as well as the groups $\pi_{N+r}(\Sigma^{N-1}X')$.

Proof of (b). As it can easily be seen on the same exact sequence, the kernel of $\pi_{N+q}(\Sigma^N X) \to \pi_{N+q}(\Sigma^{N-1} X')$ consists of the elements divisible by p^r, so α does not belong to it.

It remains to notice that in view of the old basic lemma the image of α, which is a non-zero element of order p^s, will not belong to the image of the group $\pi_{N+q}(X'(M))$ if M is large enough, so α will not belong to that of $\pi_{N+q}(X(M))$, either, which ends the proof of the generalized basic lemma.

Proof of statements (3) *and* (4). In order to prove that

$$B^{s,t}/B^{s+1,t+1} = \mathrm{Im}\,[\pi_{N+t-s}(X(s)) \to \pi_{N+t-s}(\Sigma^N X, X(s+1))]$$

and

$$E_\infty^{s,t} = \bigcap_M \mathrm{Im}\,[\pi_{N+t-s}(X(s+M)) \to \pi_{N+t-s}(\Sigma^N X, X(s+1))]$$

$$\begin{array}{ccccc}
 & \overset{\xi_2}{\nearrow} & \pi_{N+t-s}(X(s),\ X(s+M)) & \xrightarrow{\xi_7} & \pi_{N+t-s-1}(X(s+M)) \\
\pi_{N+t-s}(X(s)) & & \downarrow \xi_5 \qquad \textcircled{δ} & & \downarrow \xi_{10} \\
 & \underset{\xi_3}{\searrow} & \pi_{N+t-s}(X(s),\ X(s+1)) & \overset{\xi_8}{\searrow} & \\
\downarrow \xi_1 & & \downarrow \xi_6 \qquad \textcircled{γ} & & \pi_{N+t-s-1}(X(s+1)) \\
\pi_{N+t-s}(\Pi^N x) & \xrightarrow{\xi_4} & \pi_{N+t-s}(\Pi^N x,\ X(s+1)) & \underset{\xi_9}{\nearrow} & \textcircled{β} \\
 & & \textcircled{α} & &
\end{array}$$

be considered once again. It is commutative and exact along the "horizontal" arrows. Suppose that $\alpha \in \pi_{N+t-s}(\Sigma^N X,\ X(s+1))$ does not belong to the image of the homomorphism $\xi_6\xi_3 : \pi_{N+t-s}(X(s)) \to \pi_{N+t-s}(\Sigma^N X,\ X(s+1))$. We have to prove that, for sufficiently large M, it does not belong to the image of $\xi_6\xi_5$ either.

The element $\beta = \xi_9\alpha \in \pi_{N+t-s-1}(X(s+1))$ has finite order equal to a power of p (as have all elements of $\pi_{N+t-s}(\Pi^N X, X(s+1))$, including α). Thus either $\beta = 0$ or β does not belong to the image of ξ_{10} with large M. If $\beta = 0$ and $\alpha = \xi_6\xi_5(\delta)$ then $\alpha = \xi_6(\gamma)$ where $\gamma = \xi_5(\delta)$, $\gamma \in \pi_{N+t-s}(X(s+1))$. Now $\xi_8(\gamma) = \xi_9(\alpha) = \beta = 0$, hence $\gamma = \xi_3(\varepsilon)$ where $\varepsilon \in \pi_{N+t-s}(X(s))$ and $\alpha = \xi_6\xi_3(\varepsilon)$ contradict to the assumption. If $\beta \neq 0$ then $\beta \in \mathrm{Im}\,\xi_{10}$ if M is sufficiently large. Now $\alpha = \xi_6\xi_5\delta$ with $\delta \in \pi_{N+t-s}(X(s), X(s+M))$ would imply $\beta = \xi_{10}\xi_7\delta$ contrary to the assumption. Statement (3) is proved. Statement (4) immediately follows from the generalized basic theorem. Q. e. d.

§33. MULTIPLICATIVE STRUCTURES

The multiplicative structure in the spectral sequence of Leray comes from cohomology multiplication, a fact completely natural as all groups in question are either cohomology groups or subgroups of cohomology groups. Now in the case of an Adams spectral sequence we have to deal with homotopy groups and their subgroups. They have no multiplicative structure of any use (there is the Whitehead product, but it is applicable only to non-stable homotopy groups) and so we cannot define multiplication in the spectral sequence either, at least anything resembling in usefulness to the Leray's case. Nevertheless under certain assumptions we may construct some analogue of a multiplicative structure that turns the Adams spectral sequence into a sequence of rings in the single but important case when X is the sphere.

Let us begin with this particular case. Suppose that we already have the promised ring structure on the terms of the spectral sequence. Then $\oplus_q \pi_q^S(S^0)$ is a ring, too. The multiplication in this ring is surely adjoint to something which we want to find.

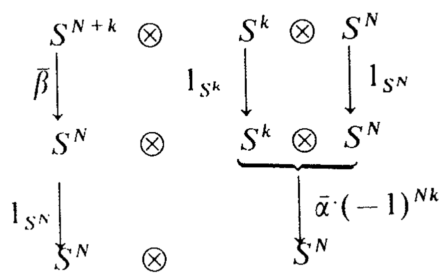
First we are going to show that the direct sum has a natural ring structure.

Composition product in stable homotopy groups of the sphere

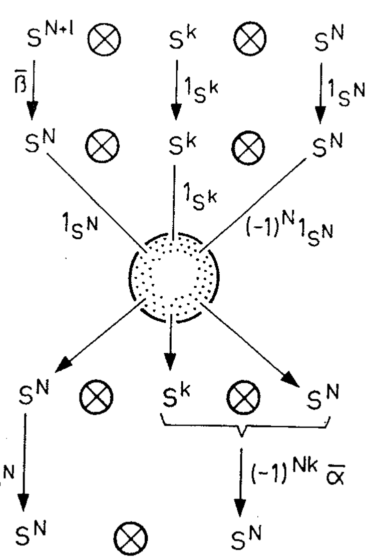
Let $\alpha \in \pi_{N+k}(S^N)$ and $\beta \in \pi_{N+l}(S^N)$ (k and $l \ll N$). Then β may be regarded as an element of $\pi_{N+k+l}(S^{N+k})$. Let the mappings $\bar{\beta}$: $S^{N+k+l} \to S^{N+k}$ and $\bar{\alpha}$: $S^{N+k} \to S^N$ represent β and α. The composite $\bar{\alpha}\bar{\beta}: S^{N+k+l} \to S^N$, represents an element of $\pi_{N+k+l}(S^N)$ called the composite of α and β and denoted by $\alpha \circ \beta$.

An alternative definition of the multiplication is the following. Let α, β and $\bar{\alpha}$ be as above, $\bar{\beta}$: $S^{N+l} \to S^N$; define $\alpha \circ \beta$ by setting $(-1)^{Nk}\overline{\alpha \circ \beta} = \bar{\alpha} \otimes \bar{\beta}$: $S^{N+k} \otimes S^{N+l} \to$ $\to S^N \otimes S^N$ (i. e. the first multiplier is mapped by $\bar{\alpha}$, the second—by $\bar{\beta}$) thus we have a mapping $S^{N+N+k+l} \to S^{N+N}$ or $S^{N+k+l} \to S^N$, as $k+l \ll N$.

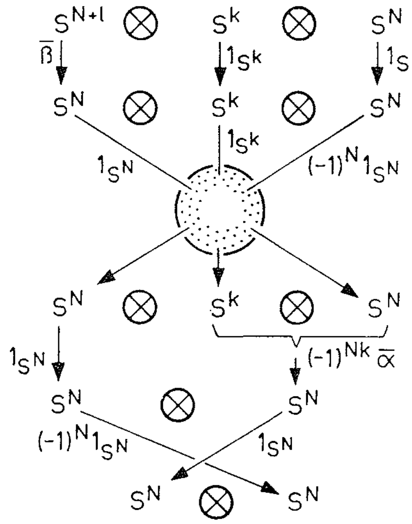
The two definitions are equivalent. Indeed, the mapping $(-1)^{Nk}(\bar{\alpha} \otimes \bar{\beta}) =$ $= ((-1)^{Nk}\bar{\alpha}) \otimes \bar{\beta}$: $S^{N+k} \otimes S^{N+l} \to S^N \otimes S^N$ is a composite of two mappings



We add a mapping to the diagram which will interchange the outside factors (it preserves orientation if N is even and turns it to the opposite if N is odd). In order to ensure it to be homotopic to the identity mapping, we prefer to map the third factor by applying $(-1)^{Nk}1_{S^N}$ rather than the identity mapping.



Finally we complete the diagram with the mapping of changing the order of multiplications in the last product (also rectifying the sign as above).



We have obtained a mapping $S^{N+k+l} \otimes S^N \to S^N \otimes S^N$ that is identity on the second factor while on the first it coincides with the composite

$$S^{N+k+l} \xrightarrow{\bar{\beta}} S^{N+k} \xrightarrow[\left(\substack{\text{interchanging} \\ \text{the factors}}\right)]{(-1)^{Nk}} S^{N+k} \xrightarrow{(-1)^{Nk}\bar{\alpha}} S^N$$

i. e. $\bar{\alpha} \circ \bar{\beta}$. The statement is proved.

The composition product is anticommutative, i. e. $\alpha \circ \beta = (-1)^{kl} \beta \circ \alpha$. Obviously the element $\alpha \circ \beta$ does not depend on the number N used in the definition. So we may assume N even. Then $\overline{\alpha \circ \beta} = \bar{\alpha} \otimes \bar{\beta}$ and $\overline{\beta \circ \alpha} = \bar{\beta} \otimes \bar{\alpha}$. Further, the elements of $\pi_{2N+k+l}(S^{2N})$ defined by the mappings $\bar{\alpha} \otimes \bar{\beta}$ and $\bar{\beta} \otimes \bar{\alpha}$ differ in a multiplier $(-1)^{kl}$. Indeed, $\bar{\alpha} \otimes \bar{\beta}$ is the composite mapping

$$S^{N+k} \otimes S^{N+l} \to S^{N+l} \otimes S^{N+k} \to S^N \otimes S^N \to S^N \otimes S^N$$

where the outside mappings are interchanging the factors.

The first mapping here either changes or preserves the orientation depending on the sign of the number $(-1)^{(N+k)(N+l)}$; the second is homotopic to the identity (N is even) and may be neglected. As it is well known, by reversing the orientation of the sphere *to be mapped* we reverse the sign of the homotopy class of the spheroid (as division is defined in the homotopy groups). Thus the signs of $\alpha \circ \beta$ and $\beta \circ \alpha$ really differ in $(-1)^{(N+k)(N+l)} = (-1)^{kl}$ as stated.

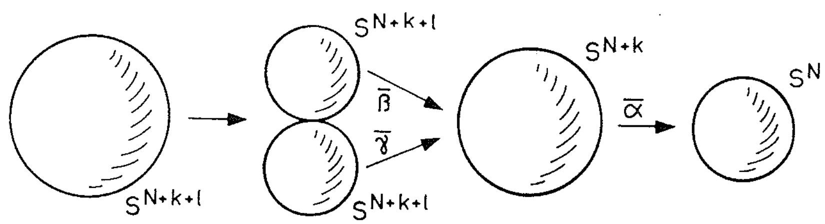
Remark. The reader may now be wondering why did not we make use of the simple fact that if $\varphi: S^q \to S^r$ is an arbitrary mapping, $\psi_1 : S^q \to S^q$ and $\psi_2 : S^r \to S^r$ are mappings of degree $\varepsilon_1 = \pm 1$ and $\varepsilon_2 = \pm 1$, respectively, then the spheroids

$$S^q \xrightarrow{\psi_1} S^q \xrightarrow{\varphi} S^r \xrightarrow{\psi_2} S^r$$

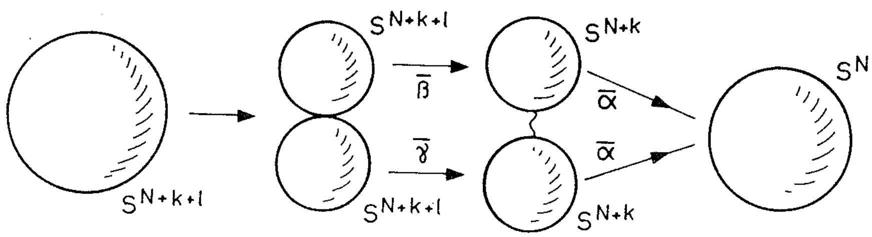
has the homotopy class of $\varphi\colon S^q \to S^r$ multiplied by $\varepsilon_1\varepsilon_2$. We could have spared the difficulties in the proofs of the last two theorems Unfortunately this fact is too good to be true. For example, if $\chi\colon S^3 \to S^2$ is the Hopf mapping and $\psi\colon S^2 \to S^2$ is a mapping of degree -1, the composite $S^3 \xrightarrow{\chi} S^2 \xrightarrow{\psi} S^2$ is homotopic to χ (instead of $-\chi$). If the spheroid φ is in a stable dimension, the statement is true as proposed and will follow from the last theorem.

We are going now to prove that the multiplication is *distributive* from both sides: $(\beta+\gamma)\circ\alpha = \beta\circ\alpha + \gamma\circ\alpha$ and $\alpha\circ(\beta+\gamma) = \alpha\circ\beta + \alpha\circ\gamma$.

As the anticommutativity law is already at our disposal it suffices to prove one of the formulas. Consider the mapping on the left:



and on the right:



It is quite obvious that the two mappings are in fact coincide.

The distributivity of the multiplication is proved.

Remark. Had we tried to prove directly the analogous statement $1\beta+\gamma)\circ\alpha = \beta\circ\alpha + {}$ $+\gamma\circ\alpha$ hardly would we have succeeded because the formula does not hold in unstable dimensions (and the difficulty of making a geometric construction that operates on stability of dimensions is obvious). As a counterexample to the left distributivity law in unstable dimensions we mention that the composite of χ: $S^3 \to S^2$ and $2\cdot 1_{S^2} =$ $= +1_{S^2}$: $S^2 \to S^2$ is equal to 4χ rather than $2\chi = 1_{S^2}\circ\chi + 1_{S^2}\circ\chi$.

Associativity of the multiplication is obvious.

Thus we have shown that $\pi_*^S(S^0) = \oplus_q \pi_q^S(S^0)$ is an anticommutative associative graded ring.

An algebraic digression

A graded algebra A with unit element over a field k is called a *Hopf algebra* if

(1) $A^k=0$ for negative k and $A^0=k$.

(2) there is a "diagonal mapping" or "comultiplication" $\Delta: A\to A\otimes_k A$ which is an algebra homomorphism (we recall that multiplication in $A\otimes_k A$ is given by $(\alpha'\otimes\beta')(\alpha''\otimes\beta'')=(-1)^{\deg\beta'\deg\alpha'}\alpha'\alpha\otimes\beta'\beta$); further for any $a\in A$, $\Delta(a)=\rho(a)\otimes 1+$ $+1\otimes\sigma(a)+\ldots$ where ρ and σ are automorphisms of the algebra A and the terms "..." are tensor products of elements of positive degree.

We shall have Hopf algebras where ρ is identity and σ is multiplying by $(-1)^{\dim a}$.

An example of a Hopf algebra is the Steenrod algebra, with the diagonal mapping defined by $\Delta(\beta)=\beta\otimes 1-1\otimes\beta$, $\Delta(P^i_p)=\sum_{k+l=i} P^k_p\otimes P^l_p$ (for $p=2$, $\Delta Sq^i=$ $=\sum_{k+l=i} Sq^k\otimes Sq^l$).

(Another important example is the cohomology algebra of a H-space.)

If A is a Hopf algebra and B and C are A-modules, their tensor product, considered as a vector space over k, is also an A-module. (Clearly the product is an $A\otimes_k A$-module. The homomorphism $\Delta: A\to A\otimes_k A$ makes it an A-module as well.)

An important remark. If A is the Steenrod algebra, the above construction is compatible with the Künneth formula: for any pair X, Y of spaces we have $\tilde H^*(X\otimes Y;\mathbf{Z}_p)=\tilde H^*(X;\mathbf{Z}_p)\otimes\tilde H^*(Y;\mathbf{Z}_p)$. Thus $\tilde H^*(X\otimes Y;\mathbf{Z}_p)$ is an A-module by two reasons: first as cohomology of a space and second because A is a Hopf algebra. The Cartan theorem shows that the two structures coincide.

If B and C are free A-modules, then such is $B\otimes_k C$, too. The proof is left to the reader.

Let us now consider the case of the Steenrod algebra A, with $p=2$. The case $p>2$ is left to the reader with the remark that the only difference is the appearence of a multiplier $(-1)^{\cdots}$ at certain places.

For any A-modules M', M'', N' and N'' a multiplication

$$\operatorname{Ext}_A^{s',t'}(M',N')\otimes_{Z_2}\operatorname{Ext}_A^{s'',t''}(M'',N'')\to\operatorname{Ext}_A^{s'+s'',t'+t''}(M'\otimes_{Z_2}M'',N'\otimes_{Z_2}N'')$$

is defined in the following way. We take free resolutions for M' and M''

$$M'\xleftarrow{\partial'_1}B'_1\xleftarrow{\partial'_2}B'_2\xleftarrow{\partial'_3}\ldots$$
$$M''\xleftarrow{\partial''_1}B''_1\xleftarrow{\partial''_2}B''_2\xleftarrow{\partial''_3}\ldots$$

Then we have a free resolution of the A-module $M'\otimes M''$

$$M'\otimes_{Z_2}M''\xleftarrow{\partial_1}B'_1\otimes B''_1\xleftarrow{\partial_2}B'_2\otimes B''_1+B'_1\otimes B''_2\xleftarrow{\partial_3}B'_3\otimes B''_1+$$
$$+B'_2\otimes B''_2+B'_1\otimes B''_2\xleftarrow{\partial_4}\ldots$$

(the tensor products are over $\mathbf{Z}_2$) defining ∂_i by the formula

$$\partial_{p+q-1}(\alpha' \otimes \alpha'') = \partial'_p\alpha' \otimes \alpha'' + (-1)^p \alpha' \otimes \partial''_q\alpha'', \quad \alpha' \in B'_p, \quad \alpha'' \in B''_q.$$

There is a natural homomorphism

$$\mathrm{Hom}_A(B'_p, N') \otimes \mathrm{Hom}_A(B''_q, N'') \to \mathrm{Hom}_A(B'_p, B''_q, N' \otimes N'')$$

which by transition to homology yields the needed homomorphism in Ext.

If $M' = M'' = N' = N'' = M' \otimes_{Z_2} M'' = N' \otimes_{Z_2} N'' = \mathbf{Z}_2$, the procedure defines a ring structure on $\bigoplus_{s,t} \mathrm{Ext}_A^{s,t}(\mathbf{Z}_2, \mathbf{Z}_2)$.

Definition. The groups $\mathrm{Ext}_A^{s,t}(\mathbf{Z}_2, \mathbf{Z}_2)$ are called the *homology groups of the algebra A* and denoted by $H^{s,t}(A)$.

Thus we have defined the *homology ring of a Steenrod algebra.*

The reader will show it to be associative and commutative.

If $M'' \neq \mathbf{Z}_2$ in the above construction we obtain a homomorphism

$$\mathrm{Ext}_A^{s',t'}(\mathbf{Z}_2, \mathbf{Z}_2) \otimes \mathrm{Ext}_A^{s'',t''}(M'', \mathbf{Z}_2) \to \mathrm{Ext}_A^{s'+s'',t'+t''}(M'', \mathbf{Z}_2)$$

in short: $\mathrm{Ext}_A^{**}(M, \mathbf{Z}_2)$ is an $H^{**}(A)$-module for any A-module M. If $M \to N$ is any A-module homomorphism, the induced mapping $\mathrm{Ext}_A^{**}(N, \mathbf{Z}_2) \to \mathrm{Ext}_A^{**}(M, \mathbf{Z}_2)$ is a $H^{**}(A)$-homomorphism.

Theorem (Adams). If $X = S^0$, the Adams spectral sequence may be equipped with a multiplication $E_r^{s,t} \otimes E_r^{s',t'} \to E_r^{s+s',t+t'}$ such that

(i) it is commutative and associative;

(ii) it coincides with the multiplication

$$H^{s,t}(A) \otimes H^{s',t'}(A) \to H^{s+s',t+t'}(A)$$

in the homology of the Steenrod algebra;

(iii) $d_r(uv) = (d_r u)v + u(d_r v)$;

(iv) it commutes with the isomorphism $E_{r+1}^{s,t} \cong H(E_r^{s,t}; d_r^{s,t})$ and the monomorphism $E_k^{s,t} \to E_r^{s,t}$ (for $s < r < k \leq \infty$);

(v) the multiplication in E_∞ is adjoint to the composition product

$$\pi_k^S(S^0) \otimes \pi_l^S(S^0) \to \pi_{k+l}^S(S^0).$$

Proof. As we actually wish to prove a statement which is somewhat more general than the theorem we start with two spaces X' and X''. Let

$$\tilde{H}^*(X'; \mathbf{Z}_2) \leftarrow B'_1 \leftarrow B'_2 \leftarrow \ldots$$

and

$$\tilde{H}^*(X''; \mathbf{Z}_2) \leftarrow B''_1 \leftarrow B''_2 \leftarrow \ldots$$

be free A-resolutions of the A-modules $\tilde{H}^*(X'; \mathbf{Z}_2)$ and $\tilde{H}^*(X''; \mathbf{Z}_2)$, and

$$\Sigma^{N'} X' \leftarrow X'(1) \leftarrow X'(2) \leftarrow \ldots$$

and

$$\Sigma^{N''} X'' \leftarrow X''(1) \leftarrow X''(2) \leftarrow \ldots$$

be the corresponding Adams filtrations. Let us define a filtration in the space $\Sigma^{N'}X' \otimes \Sigma^{N''}X'' = \Sigma^{N'+N''}(X' \otimes X'')$ by writing $X(n) = \bigcup_{i+j=n} X'_i \otimes X''_j$; here $X'(0) = \Sigma^{N'}X'$ and $X''(0) = \Sigma^{N''}X''$.

Obviously

$$Y_n = X(n)/X(n+1) = \bigvee_{i+j=n} (X'_i/X'_{i+1}) \otimes (X''_j/X''_{j+1}).$$

On the other hand the spaces $X'(i)/X'(i+1)$ and $X''(j+1)$ are equivalent to Y'_i and Y''_j, respectively, in the stable dimensions. Hence

$$Y_n = \bigvee_{i+j=n} Y'_i \otimes Y''_j$$

and

$$\tilde{H}^*(Y_n) = \oplus_{i+j=n} \tilde{H}^*(Y'_i) \otimes \tilde{H}^*(Y''_j).$$

(Here Y'_i and Y''_j stand for the products of Eilenberg-MacLane spaces applied to constructing the Adams filtrations $\{X'(i)\}$ and $\{X''(j)\}$ and $Y_n = X(n)/X(n+1)$.)

Clearly the filtration

$$\Sigma^{N'+N''}(X' \otimes X'') \leftarrow X(1) \leftarrow X(2) \leftarrow \ldots$$

is an Adams filtration for the resolution

$$\tilde{H}^*(X' \otimes X''; \mathbf{Z}_2) \leftarrow B'_1 \otimes B''_1 \leftarrow B'_2 \otimes B''_1 + B'_1 \otimes B''_2 \leftarrow \ldots$$

Let us construct the Adams spectral sequences for X', X'' and $X' \otimes X''$ by using the resolutions and filtrations in view. The multiplication

$$E_r^{s',t'}(X') \otimes E_r^{s'',t''}(X'') \rightarrow E_r^{s'+s'',t'+t''}(X' \otimes X'')$$

is then defined as follows. For any pair

$$\alpha' \in E_r^{s',t'}(X') = \operatorname{Im}[\pi_{N'+t'-s'}(X'(s'), X'(s'+r)) \rightarrow \pi_{N'+t'-s'}(X'(s'+1-r), X'(s'+1))]$$

$$\alpha'' \in E_r^{s'',t''}(X'') = \operatorname{Im}[\pi_{N''+t''-s''}(X''(s''), X''(s''+r)) \rightarrow \pi_{N''+t''-s''}(X''(s''+1-r), X''(s''+1))]$$

we take the corresponding elements of

$$\pi_{N'+t'-s'}(X'(s'), X'(s'+r))$$

and

$$\pi_{N''+t''+s''}(X''(s''), X''(s''+r))$$

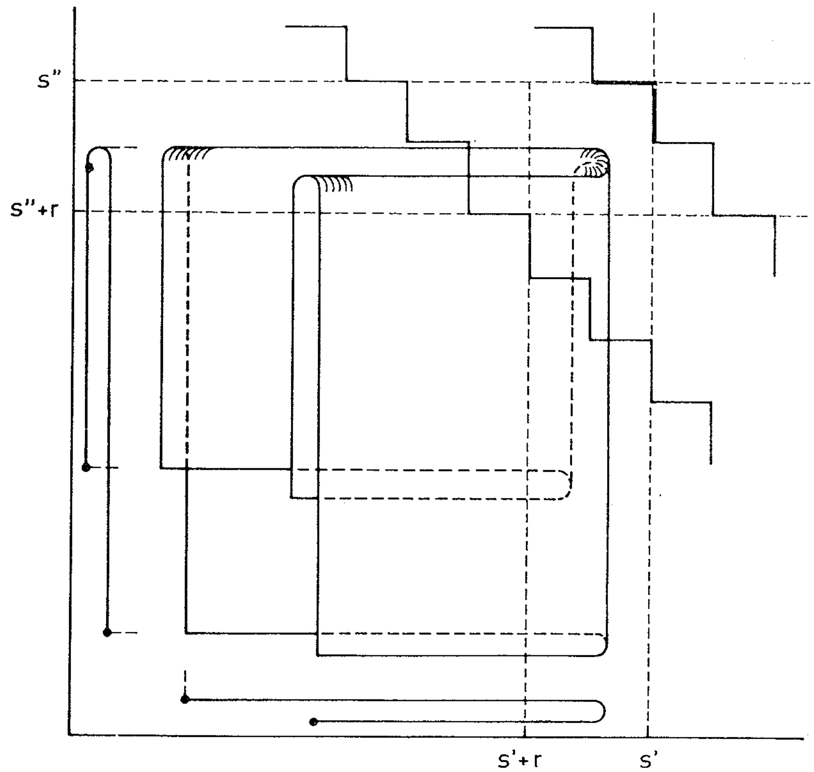
whose images are α and β, i. e. mappings of the cubes $I^{N'+t'-s'}$ and $I^{N''+t''-s''}$ into X' and X'', which map the cubes themselves into $X'(s')$ and $X''(s'')$ and their boundaries into $X'(s'+r)$ and $X''(s''+r)$. These mappings may be naturally "multiplied" by taking the mapping

$$I^{N'+t'-s'} \times I^{N''+t''-s''} = I^{N'+N''+t'+t''-s'-s''} \rightarrow X' \times X'' \rightarrow X' \otimes X''.$$

The last mapping sends the cube $I^{N'+N''+t'+t''-s'-s''}$ into $X'(s')\otimes X''(s'')\subset X(s'+s'')$ and its boundary into

$$X'(s'+r)\otimes X''(s'')\cup X'(s')\otimes X''(s''+r)\subset X(s'+s''+r).$$

It is therefore a relative spheroid of the pair $(X(s'+s''), X(s'+s''+r))$ and so defines an element of $\pi_{N'+N''+t'+t''-s'-s''}(X(s'+s''), X(s'+s''+r))$ (cf. the picture) whose image in $\pi_{N'+N''+t'+t''-s'-s''}(X(s'+s''+1-r), X(s'+s''+1))$ is the very element of $E_r^{s'+s'',t'+t''}(X'\otimes X'')$ that is by definition the product of $\alpha'\in E_r^{s',t'}(X')$ and $\alpha''\in E_r^{s'',t''}(X'')$.



The obtained multiplication commutes with the differentials and coincides on E_2 with the multiplication considered before:

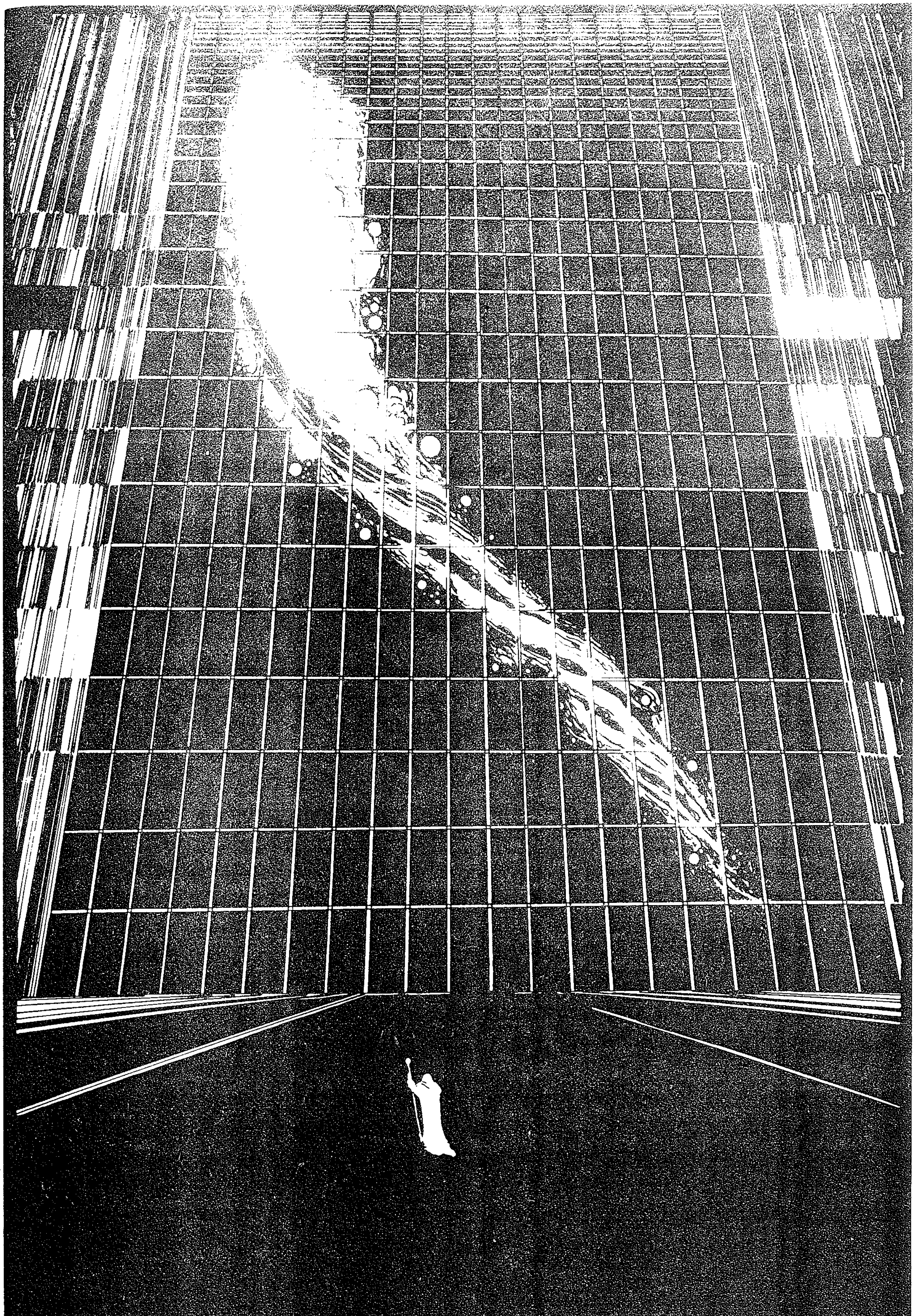
$$\operatorname{Ext}_A^{**}(\tilde H^*(X'), \mathbf{Z}_2)\otimes \operatorname{Ext}_A^{**}(\tilde H^*(X''), \mathbf{Z}_2)\to$$
$$\to \operatorname{Ext}_A^{**}(\tilde H^*(X')\otimes \tilde H^*(X''), \mathbf{Z}_2)$$

further, in limit it yields the multiplication $E_\infty^{**}(X')\otimes E_\infty^{**}(X'')\to E_\infty^{**}(X'\otimes X'')$ which is adjoint with the multiplication

$$\pi_k^S(X')\otimes \pi_l^S(X'')\to\pi_{k+l}^S(X'\otimes X'')$$

as it may easily be verified by the reader.

In the case $X' = X'' = S^0$ the result implies the theorem to be proved. Q.e.d.



Now let $X' = S^0$ and $X'' = X$ be an arbitrary space. By the above construction $E_r^{**}(X)$ is equipped with an $E_r^{**}(S^0)$-module structure that for $r=2$ it coincides with the $H^{**}(A)$-module structure of $\mathrm{Ext}_A^{**}(\tilde{H}^*(X), \mathbf{Z}_2)$ and for $r=\infty$ it is adjoint with the $\pi_*^S(S^0)$-module structure which exists on $\pi_*^S(X)$, for any X (in the sense that elements of the stable homotopy groups of spheres may be naturally considered as "stable homotopy operations" acting on the stable homotopies by composition: the operation $\alpha \in \pi_{N+k-l}(S^{N+k})$ will assign to $\xi \in \pi_{N+k}(\Sigma^N X)$ the composite

$$S^{N+k+l} \xrightarrow{\bar{\alpha}} S^{N+k} \xrightarrow{\bar{\xi}} \Sigma^N X).$$

§34. APPLICATIONS OF THE ADAMS SPECTRAL SEQUENCE

We are going to investigate the problem of the stable homotopy groups of spheres. We begin with computing the E_2 term, i. e. the homology mod 2 of the Steenrod algebra, including the additive and multiplicative structure.

Let us write out a resolution of the A-module $\tilde{H}^*(S^0; \mathbf{Z}_2) = \mathbf{Z}_2$. Clearly A itself may be chosen as the first free module of the resolution. The epimorphism $A \to \mathbf{Z}_2$ sends its unity element into the unique nontrivial element $\mathbf{Z}_2$ while all other elements are sent to zero. The kernel of this epimorphism is the ideal $\tilde{A}$ consisting of the elements of positive degrees. In the next step a system of A-generators is chosen in the ideal. We recall that the system of generators is minimal, which implies that we start selecting the generators in the component of minimal dimension (in dimension one, in the present case). The vector space A_1 has dimension one and is generated by $a_1 = Sq^1$. Next we consider A_2, which is one-dimensional, too. Because $Sq^1 a_1 = 0$, its generator Sq^2 cannot be expressed by a_1 so it must be selected as the next generator. Let it be denoted by a_2. Observe that $\tilde{A}$ is not free as an A-module as we already have found a relation. In the sequel it is useful to know all relations in the A-module $\tilde{A}$. In the dimension 2 we only have $Sq^1 a_1 = 0$. In the dimension 3, by the Serre theorem, the Steenrod algebra has two additive generators Sq^3 and $Sq^2 Sq^1$ that may be expressed by the earlier generators $Sq^3 = Sq^1 a_2$, $Sq^2 Sq^1 = Sq^2 a_1$. There exist no relations in this dimension. In dimension 4 there are two additive generators Sq^4 and $Sq^3 Sq^1$, where $Sq^3 Sq^1 = Sq^3 a_1$ while Sq^4 cannot be written as an expression of Sq^1 and Sq^2, thus it will be introduced as a new generator: $a_3 = Sq^4$. In this dimension there must be two relations (more exactly, a two-dimensional space of relations) because by applying the elements of A to the generators a_1 and a_2 we obtain $Sq^2 a_2$, $Sq^3 a_1$ and $Sq^2 Sq^1 a_1$. Moreover there is the new generator a_3, thus in the absence of relations A_4 should be four-dimensional while its actual dimension is 2. As a basis of the two-dimensional space of relations we may choose the relations that express $Sq^2 a_2$ and $Sq^2 Sq^1 a_1$ by $Sq^3 a_1$. Here we recall the Adem formulas: $Sq^2 a_2 = Sq^3 a_1$, $Sq^2 Sq^1 a_1 = 0$.

Further computation of the homology structure of the Steenrod algebra may be carried out in an algorithmic way. We have heard about many attempts at performing computations on computer. We have no computer at our disposal, nevertheless we did carry out the selecting process by listing the relations as far as dimension 12. The list of generators and relations is to be seen below.

First row

N	*generators*	*relations*
1	a_1	—
2	a_2	$Sq^1a_1 = 0$
3	$Sq^1a_2,\ Sq^2a_1$	—
4	$a_3 \cdot Sq^3a_1$	$Sq^2a_2 = Sq^3a_1,\ Sq^2Sq^1a_1 = 0$
5	$Sq^1a_3,\ Sq^4a_1$	$Sq^3Sq^1a_1 = 0,\ Sq^3a_2 = 0,\ Sq^2Sq^1a_2 = Sq^1a_3 + Sq^4a_1$
6	$Sq^5a_1,\ Sq^4a_2,\ Sq^2a_3$	$Sq^4Sq^1a_1 = 0,\ Sq^5a_1 = Sq^3Sq^1a_2$
7	$Sq^6a_1,\ Sq^5a_2,\ Sq^4Sq^2a_1,\ Sq^3a_3$	$Sq^5Sq^1a_1 = 0,\ Sq^6a_1 = Sq^2Sq^1a_3,\ Sq^4Sq^1a_2 = Sq^5a_2$
8	$a_4,\ Sq^7a_1,\ Sq^6a_2,\ Sq^5Sq^2a_1$	$Sq^6Sq^1a_1 = 0,\ Sq^4Sq^2Sq^1a_1 = o,\ Sq^5Sq^2a_1 = Sq^4Sq^2a_2,$ $Sq^7a_1 = Sq^3Sq^1a_3,\ Sq^5Sq^1a_2 = 0,\ Sq^4a_3 = Sq^7a_1 + Sq^6a_2$
9	$Sq^1a_4,\ Sq^8a_1,\ Sq^7a_2,\ Sq^6Sq^1a_2$ $Sq^6Sq^2a_1$	$Sq^7Sq^1a_1 = 0,\ Sq^5Sq^2Sq^1a_1 = 0,\ Sq^5Sq^2a_2 = 0,\ Sq^5a_3 =$ $= Sq^7a_2,\ Sq^4Sq^2Sq^1a_2 + Sq^4Sq^1a_3 + Sq^6Sq^2a_1 = 0,$ $Sq^4Sq^1a_3 = Sq^1a_4 + Sq^8a_1 + Sq^7a_2$
10	$Sq^9a_1,\ Sq^8a_2,\ Sq^7Sq^1a_2,$ $Sq^7Sq^2a_1,\ Sq^6Sq^3a_1,\ Sq^4Sq^2a_3$	$Sq^8Sq^1a_1 = 0,\ Sq^6Sq^2Sq^1a_1 = 0,\ Sq^6Sq^2a_2 = Sq^6Sq^3a_1,$ $Sq^5Sq^1a_3 = Sq^9a_1,\ Sq^5Sq^2Sq^1a_2 = Sq^7Sq^2a_1 + Sq^9a_1,$ $Sq^6a_3 = Sq^7Sq^1a_2,\ Sq^2a_4 + Sq^4Sq^2a_3 + Sq^8a_2 + Sq^7Sq^2a_1 = 0$
11	$Sq^{10}a_1,\ Sq^9a_2,\ Sq^8Sq^1a_2,$ $Sq^8Sq^2a_1,\ Sq^7Sq^3a_1,\ Sq^3a_4$	$Sq^9Sq^1a_1 = 0,\ Sq^7Sq^2Sq^1a_1 = 0,\ Sq^6Sq^3Sq^1a_1 = 0,\ Sq^6Sq^3a_2$ $= 0,\ Sq^7Sq^2a_2 = Sq^7Sq^3a_1,\ Sq^6Sq^2Sq^1a_2 + Sq^9a_2 + Sq^8Sq^1a_2 +$ $+ Sq^7Sq^3a_1 = 0,\ Sq^9a_2 + Sq^8Sq^1a_2 + Sq^6Sq^1a_3 = 0,\ Sq^5Sq^2a_3 =$ $= Sq^9a_2 + Sq^3a_4,\ Sq^2Sq^1a_4 = Sq^{10}a_1,\ Sq^4Sq^2Sq^1a_3 + Sq^{10}a_1 +$ $+ Sq^8Sq^2a_1 = 0,\ Sq^7a_3 = 0$
12	$Sq^{11}a_1,\ Sq^9Sq^2a_1,\ Sq^8Sq^3a_1,$ $Sq^{10}a_2,\ Sq^9Sq^1a_2,\ Sq^8a_3,\ Sq^4a_4$	$Sq^{10}Sq^1a_1 = 0,\ Sq^8Sq^2Sq^1a_1 = 0,\ Sq^7Sq^3Sq^1a_1 = 0,$ $Sq^7Sq^3a_2 = 0,\ Sq^8Sq^2a_2 = Sq^8Sq^3a_1,\ Sq^7Sq^2Sq^1a_2 =$ $= Sq^9Sq^1a_2,\ Sq^6Sq^3Sq^1a_2 = Sq^9Sq^2a_1 + Sq^8Sq^3a_1,\ Sq^7Sq^1a_3 =$ $= Sq^9Sq^1a_2,\ Sq^6Sq^2a_3 = Sq^{11}a_1 + Sq^9Sq^2a_1 + Sq^{10}a_2 +$ $+ Sq^8Sq^3a_1 + Sq^9Sq^1a_2,\ Sq^5Sq^2Sq^1a_3 + Sq^{11}a_1 + Sq^9Sq^1a_2 = 0,$ $Sq^3Sq^1a_4 = Sq^{11}a_1$

$$\left.\begin{array}{lll} a_1 & Sq^1 & 1 \\ a_2 & Sq^2 & 2 \\ a_3 & Sq^4 & 4 \\ a_4 & Sq^8 & 8 \end{array}\right\} \text{degree}$$

As a matter of fact, the generators may be selected without calculations since the element Sq^{2^s} clearly form such a system. Thus the meaning of this work is just the enumeration of the relations.

The free A-module B_2 has as many generators as has $\tilde{A} = \operatorname{Ker}(B_1 \to \tilde{H}^*(S^0; \mathbf{Z}_2))$, i.e. it is spanned on the generators $\alpha_1, \alpha_2, \alpha_3, \ldots$ etc., of dimensions 1, 2, 4, 8, ... The mapping $B_2 \to B_1 = A$ sends α_k into $Sq^{2^{k-1}}u$ where u is a generator of B_1. The kernel of this epimorphism is isomorphic to the A-module of the relations between the generators $a_1, a_2, a_3, \ldots$ of the A-module $\tilde{A}$. (The additive generators of $\tilde{A}$ are exhibited in the first column of table.)

Thus the next problem we face is to choose a minimal generating system in this module of relations.

The first non-trivial element of the module has dimension 2: it is the relation $Sq^1 a_1 = 0$. Let us denote by b_1 this relation (or this element of $\operatorname{Ker}(B_1 \to \tilde{H}^*(S^0; \mathbf{Z}_2))$. In the dimension 3 there are no relations. What is $Sq^1 b_1$ equal to? By applying the operation Sq^1 to the relation $Sq^1 a_1 = 0$ we get the identity $0=0$ (the element $Sq^1 Sq^1$ is equal to zero in A). Now the relation $0=0$ is the null element in the module of relations. Hence $Sq^1 b_1 = 0$. Further, in the dimension 4 we have two relations. One of them is $Sq^2 b_1$ while the other cannot be expressed by b_1. (First, as Sq^2 is the only operation that increases dimensions by 2, second, as the latter does contain a_2 and so it cannot be obtained from b_1 which does not.) In the dimension 4 thus there are $Sq^2 b_1$ and a new generator b_2. In the dimensions ≤ 12 the A-module has six generators.

Again, the free A-module B_3 is spanned on generators which correspond to the generators selected in $\operatorname{Ker}(B_2 \to B_1)$. The dimension of $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6$ is 2, 4, 5, 8, 9, 10, respectively. The homomorphism $B_3 \to B_2$ acts in the following way:

$$\beta_1 \to Sq^1 \alpha_1$$

$$\beta_2 \to Sq^2 \alpha_2 + Sq^3 \alpha_1$$

$$\beta_3 \to Sq^2 Sq^1 \alpha_2 + Sq^1 \alpha_3 + Sq^4 \alpha_1$$

$$\beta_4 \to Sq^4 \alpha_3 + Sq^7 \alpha_1 + Sq^6 \alpha_2$$

$$\beta_5 \to Sq^4 Sq^1 \alpha_3 + Sq^1 \alpha_4 + Sq^8 \alpha_1 + Sq^7 \alpha_2$$

$$\beta_6 \to Sq^2 \alpha_4 + Sq^4 Sq^2 \alpha_3 + Sq^8 \alpha_2 + Sq^7 Sq^2 \alpha_1$$

....

Second row

N	*generators*	*relations*
2	b_1	—
3	—	$Sq^1 b_1 = 0$

4	b_2, Sq^2b_1	—
5	b_3, Sq^3b_1, Sq^1b_2	$Sq^2Sq^1b_1 = 0$
6	Sq^4b_1, Sq^1b_3	$Sq^2b_2 = Sq^4b_1+Sq^1b_3$, $Sq^3Sq^1b_1 = 0$
7	Sq^5b_1, Sq^2b_3, $Sq^2Sq^1b_2$	$Sq^4Sq^1b_1 = 0$, $Sq^3b_2 = Sq^5b_1$
8	b_4, Sq^6b_1, $Sq^4Sq^2b_1$, Sq^4b_2, $Sq^3Sq^1b_2$, Sq^3b_3	$Sq^5Sq^1b_1 = 0$, $Sq^2Sq^1b_3 = Sq^3Sq^1b_2+Sq^6b_1$
9	b_5, Sq^7b_1, $Sq^5Sq^2b_1$, Sq^5b_2, Sq^4b_3, Sq^1b_4	$Sq^6Sq^1b_1 = 0$, $Sq^4Sq^2Sq^1b_1 = 0$, $Sq^5b_2 = Sq^4Sq^1b_2$, $Sq^7b_1 = Sq^3Sq^1b_3$
10	b_6, Sq^8b_1, $Sq^6Sq^2b_1$, Sq^6b_2, Sq^5b_3, Sq^2b_4, $Sq^4Sq^2b_2$	$Sq^7Sq^1b_1 = 0$, $Sq^5Sq^2b_1 = 0$, $Sq^5Sq^1b_2 = 0$, $Sq^1b_5 = Sq^5b_3+Sq^8b_1+Sq^4Sq^1b_3$, $Sq^4Sq^1b_3+Sq^4Sq^2b_2+$ $+Sq^6Sq^2b_1 = 0$
11	Sq^9b_1, $Sq^7Sq^2b_1$, $Sq^6Sq^3b_1$, Sq^7b_2, $Sq^6Sq^1b_2$, $Sq^4Sq^2Sq^1b_2$, $Sq^4Sq^2b_3$, $Sq^2Sq^1b_4$, Sq^2b_5, Sq^3b_4, Sq^1b_6	$Sq^8Sq^1b_1 = 0$, $Sq^6Sq^2Sq^1b_1 = 0$, $Sq^5Sq^2b_2 = Sq^9b_1+$ $+Sq^7Sq^2b_1$, $Sq^6b_3+Sq^4Sq^2Sq^1b_2+Sq^7b_2+Sq^2Sq^1b_4 = 0$, $Sq^5Sq^1b_3 = Sq^9b_1$
12	Sq^8b_2, $Sq^7Sq^1b_2$, Sq^3b_5, $Sq^6Sq^2b_2$, $Sq^5Sq^2Sq^1b_2$, Sq^7b_3, $Sq^5Sq^2b_3$, Sq^4b_4, $Sq^{10}b_1$, $Sq^8Sq^2b_1$, $Sq^7Sq^3b_1$	$Sq^5Sq^2Sq^1b_2 = Sq^4Sq^2Sq^1b_3+Sq^{10}b_1+Sq^8Sq^2b_1$, $Sq^6Sq^1b_3 = Sq^6Sq^2b_2+Sq^7Sq^3b_1$, $Sq^9Sq^1b_1 = 0$, $Sq^3Sq^1b_4 = Sq^7b_3+Sq^5Sq^2Sq^1b_2$, $Sq^2Sq^1b_5 = Sq^{10}b_1$, $Sq^7Sq^2Sq^1b_1 = 0$, $Sq^6Sq^3Sq^1b_1 = 0$, $Sq^7b_3+Sq^5Sq^2Sq^1b_2+Sq^4b_4+Sq^8b_2+Sq^3b_5+Sq^2b_6 = 0$

		degree
b_1	$Sq^1a_1 = 0$	2
b_2	$Sq^2a_2+Sq^3a_1 = 0$	4
b_3	$Sq^3Sq^1a_2+Sq^1a_3+Sq^4a_1 = 0$	5
b_4	$Sq^4a_3+Sq^7a_1+Sq^6a_2 = 0$	8
b_5	$Sq^4Sq^1a_3+Sq^1a_4+Sq^8a_1+Sq^7a_2 = 0$	9
b_6	$Sq^2a_4+Sq^4Sq^2a_3+Sq^8a_2+Sq^7Sq^2a_1 = 0$	10

Here the kernel is an A-module which is isomorphic to the A-module formed by the relations of the A-module of relations of $\tilde{A}$. Its additive generators are shown on the right column of table. Let us select a minimal generating system in it. In the dimensions at most 12 this system will contain five generators in the dimensions 3, 6, 10, 11 and 12. The generator $\gamma_1, \gamma_2, \ldots$ of the free A-module B_4 are in one-to-one correspondence with them. The homomorphism $B_4 \to B_3$ acts according to

$$\gamma_1 \to Sq^1\beta_1$$

$$\gamma_2 \to Sq^2\beta_2+Sq^1\beta_3+Sq^4\beta_1$$

$$\gamma_3 \to Sq^1\beta_5+Sq^5\beta_3+Sq^8\beta_1+Sq^4Sq^1\beta_3$$

$$\gamma_4 \to Sq^6\beta_3+(Sq^4Sq^2Sq^1+Sq^7)\beta_2+Sq^2Sq^1\beta_4$$

$$\gamma_5 \to Sq^2\beta_6+Sq^3\beta_5+Sq^4\beta_4+Sq^7\beta_3+(Sq^8+Sq^5Sq^2Sq^1)\beta_2$$

. . . .

Third row

N	*generators*	*relations*
3	c_1	
4		$Sq^1c_1 = 0$
5	Sq^2c_1	---
6	c_2, Sq^3c_1	$Sq^2Sq^1c_1 = 0$
7	Sq^4c_1, Sq^1c_2	$Sq^3Sq^1c_1 = 0$
8	Sq^5c_1, Sq^2c_2	$Sq^4Sq^1c_1 = 0$
9	Sq^6c_1, $Sq^4Sq^2c_1$, Sq^3c_3, $Sq^2Sq^1c_2$	$Sq^5Sq^1c_1 = 0$
10	c_3, Sq^7c_1, $Sq^5Sq^2c_1$, Sq^4c_3, $Sq^3Sq^1c_2$	$Sq^6Sq^1c_1 = 0$, $Sq^4Sq^2Sq^1c_1 = 0$
11	c_4, Sq^8c_1, $Sq^6Sq^2c_1$, Sq^5c_2, Sq^1c_3	$Sq^7Sq^1c_1 = 0$, $Sq^5Sq^2Sq^1c_1 = 0$, $Sq^4Sq^1c_2+Sq^5c_2+Sq^8c_1+Sq^1c_3 = 0$
12	c_5, Sq^9c_1, $Sq^7Sq^2c_1$, $Sq^6Sq^3c_1$, Sq^6c_2, $Sq^4Sq^2c_2$, Sq^2c_3, Sq^1c_4	$Sq^5Sq^1c_2 = Sq^9c_1$, $Sq^8Sq^1c_1 = 0$, $Sq^6Sq^2Sq^1c_1 = 0$

		degree
c_1	$Sq^1b_1 = 0$	3
c_2	$Sq^2b_2+Sq^1b_3+Sq^4b_1 = 0$	6
c_3	$Sq^1b_5+Sq^5b_3+Sq^8b_1+Sq^4Sq^1b_3 = 0$	10
c_4	$Sq^6b_3+Sq^4Sq^2Sq^1b_2+Sq^7b_2+Sq^2Sq^1b_4 = 0$	11
c_5	$Sq^2b_6+Sq^3b_5+Sq^4b_4+Sq^7b_3+Sq^8b_2+Sq^5Sq^2Sq^1b_2 = 0$	12

It may analogously be shown that the free A-module B_5 has two generators in the dimensions at most 12, in the dimensions 4 and 11. The action of the homomorphism $B_5 \to B_4$ is given by

$$\delta_1 \to Sq^1\gamma_1$$

$$\delta_2 \to (Sq^4Sq^1+Sq^5)\gamma_2+Sq^8\gamma_1+Sq^1\gamma_3$$

where δ_1 and δ_2 are the generators in point.

Fourth row

N	*generators*	*relations*
4	d_1	---
5	---	$Sq^1d_1 = 0$

6	Sq^2d_1	
7	Sq^3d_1	$Sq^2Sq^1d_1 = 0$
8	Sq^4d_1	$Sq^3Sq^1d_1 = 0$
9	Sq^5d_1	$Sq^4Sq^1d_1 = 0$
10	$Sq^6d_1,\ Sq^4Sq^2d_1$	$Sq^5Sq^1d_1 = 0$
11	$d_2,\ Sq^7d_1,\ Sq^5Sq^2d_1$	$Sq^6Sq^1d_1 = 0,\ Sq^4Sq^2Sq^1d_1 = 0$
12	$Sq^1d_2,\ Sq^8d_1,\ Sq^6Sq^2d_1$	$Sq^7Sq^1d_1 = 0,\ Sq^5Sq^2Sq^1d_1 = 0$

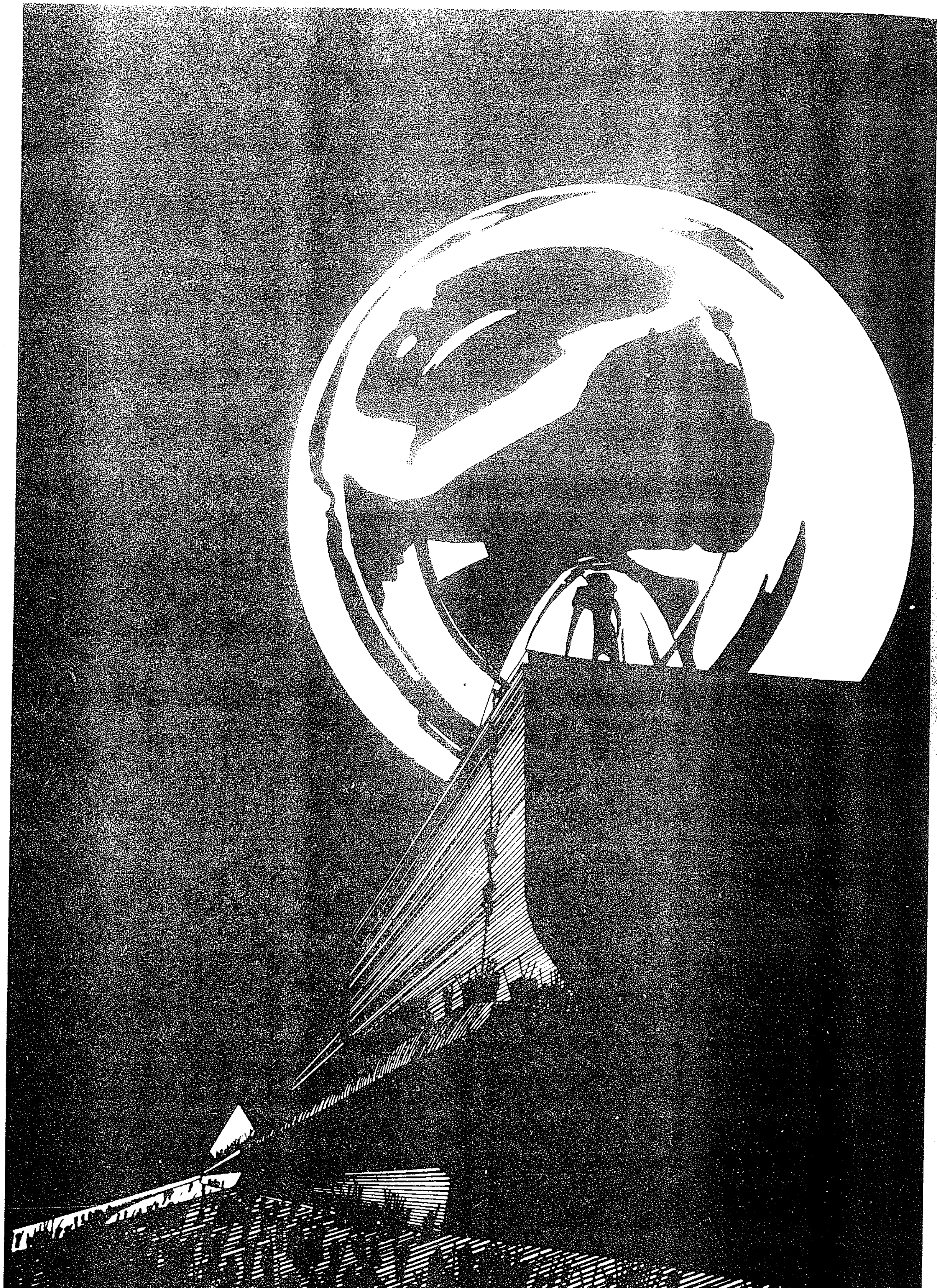
$$\begin{array}{lll} d_1 & Sq^1c_1 = 0 & 4 \\ d_2 & Sq^4Sq^1c_2+Sq^5c_2+Sq^8c_1+Sq^1c_3 = 0 & 11 \end{array}\Bigg\}\ \text{degree}$$

The free A-modules $B_6, B_7, \ldots, B_{13}$ have one generator each, in the dimensions at most 12, whose dimension in 5, 6, ..., 12 respectively. Each homomorphism $B_k \to B_{k-1}$ sends the respective generator into the Sq^1 of the preceding one.

As already shown $\text{Ext}_A^{s,*}(\mathbf{Z}_2, \mathbf{Z}_2) = \text{Hom}_A(B_{s-1}, \mathbf{Z}_2)$ thus the additive generators of the homology $H^{s,t}(A)$ of the Steenrod algebra, in the dimensions at most 12, may be listed completely. That is, we know the additive structure of the second term of the Adams spectral sequence for $t \leq 12$:

					...							
				δ_1							δ_2	
			γ_1			γ_2				γ_3	γ_4	γ_5
		β_1		β_2	β_3			β_4	β_5	β_6		
	α_1	α_2		α_3				α_4				
1												
0	1	2	3	4	5	6	7	8	9	10	11	12

Now it is time to make some general observations. The minimal dimension of a relation in the A-module $\text{Ker}(B_k \to B_{k-1})$ is clearly higher by at least one than the minimal dimension of the generators. Hence the minimal dimension of the generators of B_k is higher than that of B_{k-1}. Actually this observation gives nothing in the present case because it is clear anyway that B_k has a $(k-1)$-dimensional generator and has no generators in lower dimensions. We cannot guarantee therefore that nonzero elements will not appear even on the diagonal $t-s = 1$ in arbitrarily high dimensions. More information can be gained from examining the Adams spectral sequence of the first killing space $S^n|_n$ (in Serre's sense) of the n-dimensional sphere. The A-module of its cohomology is to obtained from the fibration $S^n \xrightarrow{S^n|_n} K(\mathbf{Z}, n)$. It is the part of



dimension $>n$ of the module $\tilde{H}^*(K(\mathbf{Z}, n); \mathbf{Z}_2)$. It is known to be equal (with a shift of dimensions) to the quotient of $\tilde{A}$ by the left ideal generated by the element Sq^1. Thus row zero of the spectral sequence for $S^n|_n$ is the same as the first row for S^0 but the generator α_1 is thrown out and the dimensions are shifted. One may easily show that in higher dimensions, too, the spectral sequences for S^0 and $S^n|_n$ only differ in a shift of dimensions (by 1 upwards and by $n-1$ to the left) and in elements which are on the bisector of the coordinate angle in the former case. In particular, in the dimensions considered, we have the following view of the E_2 term:

	0		$n+1$	$n+2$	$n+3$	$n+4$	$n+5$	$n+6$	$n+7$	$n+8$	$n+9$	$n+10$	$n+11$	$n+12$
												δ_2		
							γ_2				γ_3	γ_4	γ_5	
					β_2	β_3			β_4	β_5	β_6			
0		. . .	α_2		α_3				α_4					

The proof of this theorem is left to the reader. (It is perhaps even more instructuve to verify it by directly computing $\operatorname{Ext}_{\tilde{A}}^{**}(\tilde{A}/\{Sq^1\}, \mathbf{Z}_2)$, which is even simpler than computing $\operatorname{Ext}_{\tilde{A}}^{**}(\mathbf{Z}_2, \mathbf{Z}_2)$.)

Let us now apply the above observation about the minimal dimension of generators of B_k to this spectral sequence. We get that in the Adams spectral sequence for S^0 the dimension of the generator, that is the *second* generator also increases with the increase of k, i. e. the number of empty cells lying on the right side of the bisector in the s-th row does not decrease as s increases. Taking this into account, we obtain that the second term of our spectral sequence is trivial for $t-s\leq 7$, which implies certain consequences concerning the orders of the 2-components of the stable homotopy groups of spheres. Indeed, by consideration of the dimensions there may only be two non-trivial differentials in the domain considered: first, α_2 may possibly be annulled by any differential, and second, β_6 may be mapped onto δ_2 by the differential d_2. The first possibility will not take place because if α_2 disappeared the 2-component of $\pi_{n+1}(S^n)$ would be zero while it is known to be equal to $\mathbf{Z}_2$. The question of $d_2\beta_6$ will be postponed until the multiplicative structure will have been explained. At any case, the orders of the 2-components for $k=1, 2, 3, 4, 5, 6, 7$ are equal to 2, 2, 8, 0, 0, 2, and 8 or 16, respectively.

The multiplicative structure

Clearly the resolution of Ker $(B_k \to B_{k-1})$ may be obtained from the resolution $\mathbf{Z}_2 \leftarrow B_1 \leftarrow B_2 \leftarrow \ldots$ by cutting it off at the term in point. Thus we have

$$\mathrm{Ext}_A^{s,t}(\mathrm{Ker}(B_k \to B_{k-1}), \mathbf{Z}_2) = \mathrm{Ext}_A^{s+k,t}(\mathbf{Z}_2, \mathbf{Z}_2)$$

and the action of $H^{s,t}(A)$ on the two Ext modules is the same.

Suppose that we want to compile a "multiplication table" by a certain element $\alpha_1 \in H^{**}(A)$. Let us choose some $a_1 \in \tilde{A} = \mathrm{Ker}(B_1 \to \mathbf{Z}_2)$ and consider the A-homomorphism $\tilde{A} \to \mathbf{Z}_2$ that maps a_1 onto the generator. It lowers the degrees by one. On the other hand the induced homomorphism of $\mathrm{Ext}_A^{**}(\mathbf{Z}_2, \mathbf{Z}_2)$ into $\mathrm{Ext}_A^{**}(\tilde{A}, \mathbf{Z}_2)$ clearly raises the degrees by one. So we have $\mathrm{Ext}_A^{s,t}(\mathbf{Z}_2, \mathbf{Z}_2) \to \mathrm{Ext}_A^{s,t+1}(\tilde{A}, \mathbf{Z}_2) = \mathrm{Ext}_A^{s+1,t+1}(\mathbf{Z}_2, \mathbf{Z}_2)$. This homomorphism maps the element 1 onto α_1 and any element ξ onto $\xi\alpha_1$.

In order to define the homomorphism between the Ext modules we need the corresponding mapping between the resolution, i. e. a commutative diagram

$$\begin{array}{ccccccccc}
\mathbf{Z}_2 & \xleftarrow{\partial_1} & B_1 & \xleftarrow{\partial_2} & B_2 & \xleftarrow{\partial_3} & B_3 & \xleftarrow{\partial_4} & \ldots \\
\uparrow f_1 & & \uparrow f_2 & & \uparrow f_3 & & \uparrow f_4 & & \\
\tilde{A} & \xleftarrow{\partial_2} & B_2 & \xleftarrow{\partial_3} & B_3 & \xleftarrow{\partial_4} & B_4 & \xleftarrow{\partial_5} & \ldots
\end{array}$$

where the upper row represents the resolution under study of the module $\mathbf{Z}_2$ while the bottom row is the same resolution cut off so that it would be a resolution of $\tilde{A}$, and f_1 is the mapping that sends a_1 into $1 \in \mathbf{Z}_2$. The remaining homomorphisms must be defined so that they lower the dimensions by one and that the diagram is commutative. Because $B_2, B_3, B_4, \ldots$ etc. are free A-modules, the homomorphisms are defined once the image of each generator is given.

Let us begin with B_2. It is a free A-module with generators $\alpha_1, \alpha_2, \alpha_3, \ldots$. We have $\partial_2(\alpha_1) = a_1$ and $f_1(a_1) = 1$. Hence $f_2(\alpha_1)$ is such an element of B_1 for which $\partial_1(f_2(\alpha_1)) = 1$. Now B_1 is the Steenrod algebra and 1 is the image of its unity element u. Hence $f_2(\alpha_1) = u$. The remaining generators a_2, $\alpha_3, \ldots$ are annulled by the composite mapping $B_2 \xrightarrow{\partial_2} \tilde{A} \xrightarrow{f_1} \mathbf{Z}_2$. So we may set $f_2(\alpha_k) = 0$ for $k \geq 2$. We notice that the homomorphism we are constructing is not uniquely determined. We might have chosen for the image α_k any element of the corresponding dimension. We shall construct it as simply as possible, we are only concerned with commutativity of the diagram.

So let $f_2(\alpha_1) = u$ and $f_2(\alpha_k) = 0$ for $k \geq 2$. Next $f_3: B_3 \to B_2$ will be defined. We have $f_2(\partial_3(\beta_1)) = Sq^1 u$. Now we must choose an element of B_2 whose ∂_2-image is $Sq^1 u$; e.g. α_1 will do. We have $\partial_3(f_2(\beta_2)) = Sq^3 u$ and $\partial_2(Sq^1 \alpha_2) = Sq^3 u$. Set $f_3(\beta_2) = Sq^1 \alpha_2$.

Let us carry on this construction. The homomorphisms $f_k\colon B_k \to B_{k-1}$ in the domain considered will act in the following way:

$f_2\colon B_2 \to B_1$	$f_3\colon B_3 \to B_2$	$f_4\colon B_4 \to B_3$	$f_5\colon B_5 \to B_4$
$\alpha_1 \to u$	$\beta_1 \to \alpha_1$	$\gamma_1 \to \beta_1$	$\delta_1 \to \gamma_1$
$\alpha_2 \to 0$	$\beta_2 \to Sq^1\alpha_2$	$\gamma_2 \to \beta_3$	$\delta_2 \to \gamma_3$
...	$\beta_3 \to \alpha_3$	$\gamma_3 \to \beta_5 + Sq^1\beta_4$	...
	$\beta_4 \to Sq^3\alpha_3$	$\gamma_4 \to Sq^2\beta_4 + Sq^1\beta_5$	
	$\beta_5 \to \alpha_4$	...	
	$\beta_6 \to Sq^7\alpha_2$		
	...		

It remains to examine the homomorphisms induced in Hom and Ext. The generators of the vector space $\mathrm{Hom}_A(B_k, \mathbf{Z}_2)$ correspond to the generators of the A-module B_2 (they are even denoted by the same letters). Obviously an element of $\mathrm{Hom}_A(B_{k-1}, \mathbf{Z}_2)$ corresponding with a certain generator of the A-module B_{k-1} is annulled by $\mathrm{Hom}_A(B_{k-1}, \mathbf{Z}_2) \to \mathrm{Hom}_A(B_k, \mathbf{Z}_2)$ whenever the generator is outside the image of the homomorphism $B_k \to B_{k-1}$; if it is the image of a generator of B_k then it is mapped onto the corresponding element of $\mathrm{Hom}_A(B_k, \mathbf{Z}_2)$. So the above generators of $\mathrm{Ext}_A^{**}(\mathbf{Z}_2, \mathbf{Z}_2)$ are mapped by $\mathrm{Ext}_A^{**}(\mathbf{Z}_2, \mathbf{Z}_2) \to \mathrm{Ext}_A^{**}(\mathbf{Z}_2, \mathbf{Z}_2)$ onto the following elements:

$1 \to \alpha_1$	$\alpha_1 \to \beta_1$	$\beta_1 \to \gamma_1$	$\gamma_1 \to \delta_1$
	$\alpha_2 \to 0$	$\beta_2 \to 0$	$\gamma_2 \to 0$
	$\alpha_3 \to \beta_3$	$\beta_3 \to \gamma_2$	$\gamma_3 \to \delta_2$
	$\alpha_4 \to \beta_5$	$\beta_4 \to 0$	$\gamma_4 \to 0$
		$\beta_5 \to \gamma_3$	$\gamma_5 \to 0$
		$\beta_6 \to 0$	

Hence

$$\alpha_1\alpha_1 = \beta_1,\ \alpha_1\alpha_2 = 0,\ \alpha_1\alpha_3 = \beta_3,\ \alpha_1\alpha_4 = \beta_5,\ \alpha_1\beta_1 = \gamma_1,\ \alpha_1\beta_2 = 0,\ \alpha_1\beta_3 = \gamma_2,$$

$$\alpha_1\beta_4 = 0,\ \alpha_1\beta_5 = \gamma_3,\ \alpha_1\beta_6 = 0,\ \alpha_1\gamma_1 = \delta_1,\ \alpha_1\gamma_2 = 0,\ \alpha_1\gamma_3 = \delta_2,\ \alpha_1\gamma_4 = 0,\ \alpha_1\gamma_5 = 0.$$

Analogously we may obtain that $\alpha_2\alpha_2 = \beta_2$, $\alpha_2\alpha_4 = \beta_6$, $\alpha_2\beta_2 = \gamma_2$, $\alpha_2\beta_6 = \gamma_5$, $\alpha_3\alpha_3 = \beta_4$, $\alpha_3\beta_4 = \gamma_5$, and the other products of elements known to us are zero.

The elements of the homology groups of the Steenrod algebra have standard notation. Namely the standard notation for $\alpha_1, \alpha_2, \alpha_3, \ldots$ is $h_0, h_1, h_2, \ldots$. Then the part of the second term of the spectral sequence we are dealing with has the form where a is a new multiplicative generator. By the way, clearly $d_2\beta_6 = d_2(h_1h_3) = 0$, hence the order of the 2-component of $\pi_{n+7}(S^n)$ is 16.

						h_0^6						
					h_0^5							
				h_0^4							$h_0^3h_3$	
			h_0^3			$h_1^3 =$ $= h_0^2h_2$				$h_0^2h_3$	a	$h_2^3 =$ $= h_1^2h_3$
		h_0^2		h_1^2	h_0h_2			h_2^2	h_0h_3	h_1h_3		
	h_0	h_1		h_2				h_3				
1												

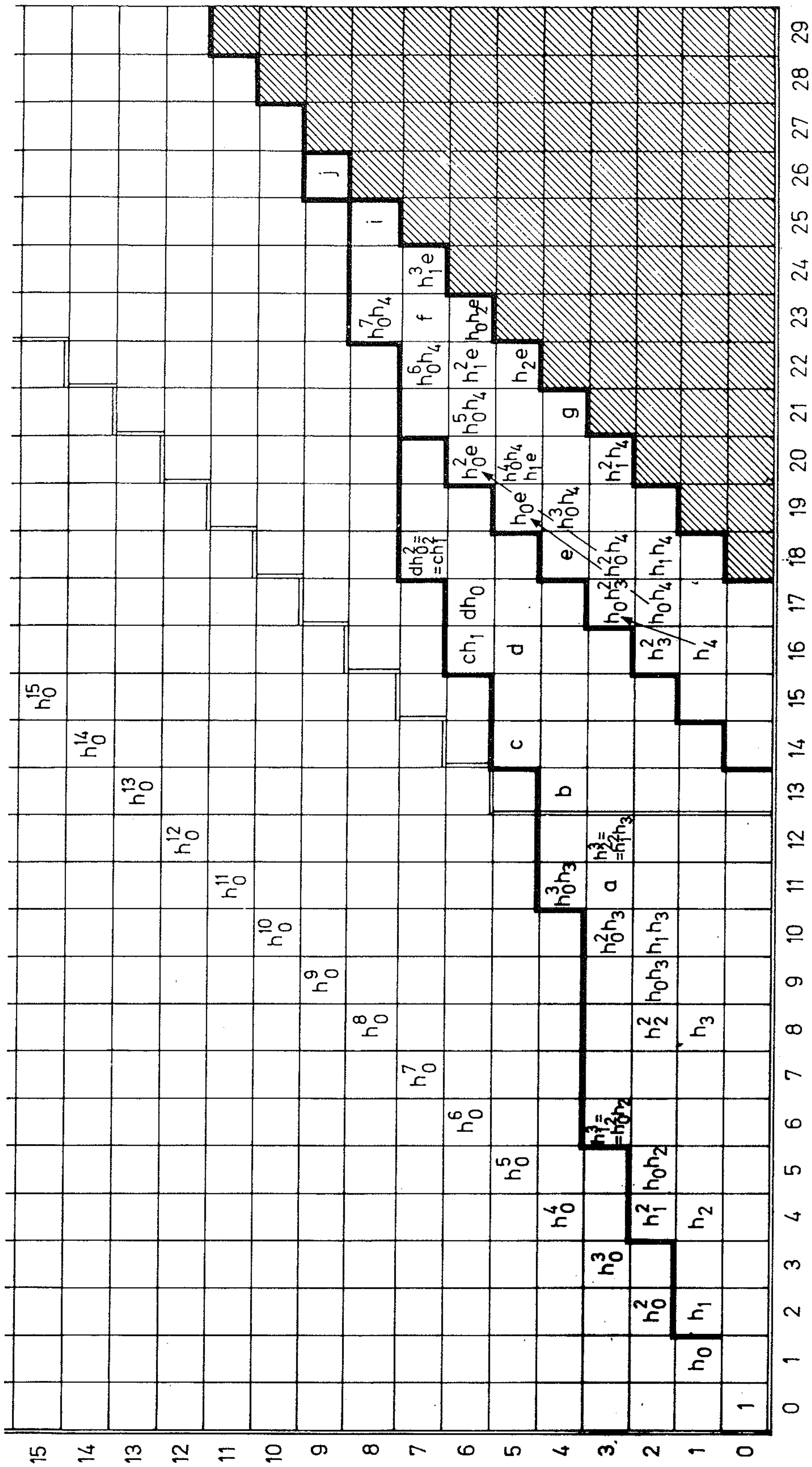
We see that computing the homotopy groups of spheres consists of two steps: computing the homology of the Steenrod algebra and the differentials of the Adams spectral sequence. The first task reduces to a wholly mechanic calculation that may be continued as long as you like. In the book of Adams, "Stable Homotopy Theory", the result of such computation is set forth for $t-s\leq 17$. The diagram of the E_2 term shown on the picture is borrowed from there.

Such a diagram is obtained without any principial difficulty, so it may be regarded as being proved (for compiling it one has to examine the Steenrod algebra up to the dimensions as far as 27 instead of 12 as we did). We obtain then, in particular, that $E_2^{10,t}=0$ for $10<t\leq 27$, consequently E_2 does not contain any new elements for $0<t-s\leq 17$. (See on the next page.)

The differentials of the multiplicative generators $h_0, h_1, h_2, h_3, a, b, c, d, e, f, i, j$ are equal to zero by dimension consideration. Therefore $E_2^{s,t} = E_\infty^{s,t}$ for $t-s\leq 13$. For the homotopy groups of spheres this implies the following information.

The orders of the first thirteen components are 2, 2, 8, 0, 0, 2, 16, 4, 8, 2, 8, 0, 0. The elements 1, h_0, h_0^2, etc. are generators of the group $\mathbf{Z}_2\oplus\mathbf{Z}_2\oplus\mathbf{Z}_2\oplus\ldots$ adjoint to $\pi_0^S(S^0)=\mathbf{Z}$. The filtration in $\mathbf{Z}$ is $\mathbf{Z}\supset 2\mathbf{Z}\supset 4\mathbf{Z}\supset 8\mathbf{Z}\supset\ldots$. Hence h_0^k is a generator of the group ${}^k\mathbf{Z}/2^{k+1}\mathbf{Z}$ and is represented by $2^k\in\mathbf{Z}$ up to elements of higher filtration.

The element $h_2\in E_\infty^{1,4}$ is the generator of the quotient group of $\pi_3^S(S^0)$ by some subgroup. Let us choose a representative $\alpha\in\pi_3^S(S^0)$. Because h_0h_2 and $h_0^2h_2$ are different from zero, the composite of α with 2 and $4\in\pi_0^S(S^0)$ is not trivial, i. e. $4\alpha\not\equiv 0 \pmod 2$, i. e. α has degree ≤ 8. Now the order of the 2-component of $\pi_3^S(S^0)$ is 8, so the group is equal to $\mathbf{Z}_8$. Analogously the 2-components of $\pi_7^S(S^0)$ and $\pi_{11}^S(S^0)$ are $\mathbf{Z}_{16}$ and $\mathbf{Z}_8$, respectively. The group adjoint to $\pi_8^S(S^0)$ has two generators h_1h_3 and a. The product $h_0h_1h_3$ is zero in $E_\infty^{3,11}$. Then the composite of $2\in\pi_8^S(S^0)$ and a representative


1
h0
h1
h0^2
h0^3
h1^2
h2
h0^4
h0h2
h0^5
h1^3 = h0^2h2
h0^6
h0^7
h2^2
h3
h0^8
h0^9
h0h3
h0^10
h0^2h3
h1h3
h0^11
h0^3h3
a
h0^12
h1^3 = h1^2h3
h0^13
b
h0^14
c
h0^15
ch1
d
h3^2
h4
dh0
h0h3^2
h0h4
h1h4
dh0^2 = ch1^2
e
h0^2h4
h0^3h4
h0e
h1^2h4
h0^2e
h0^4h4
h1e
g
h0^5h4
h2e
h1^2e
h0^6h4
h0h2e
f
h0^7h4
h1^3e
i
j
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

of h_1h_3 has filtration >3. Now, as it is obvious from the spectral sequence, there exist no such elements at all in $\pi_8^S(S^0)$. Hence the 2-component of $\pi_9^S(S^0)$ is equal to $\mathbf{Z}_2 \oplus \mathbf{Z}_2$. Finally, $\pi_9^S(S^0)$ has a 2-component of order 8. By applying to this group the argumenting used at $\pi_8^S(S^0)$ we obtain that it cannot be equal to $\mathbf{Z}_8$. So it is either $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ or $\mathbf{Z}_4 \oplus \mathbf{Z}_4$. In the latter case the generator of $\mathbf{Z}_4$ is represented in E_∞ by $h_2^3 = h_1^2h_3$, i. e. it is (up to elements of higher filtration, i. e. of order 2) the composite $\alpha \circ \alpha \circ \alpha$ where α is the generator of $\pi_3^S(S^0)$. Now this composite has the order 2, as has already $\alpha \circ \alpha$ which is an element of $\pi_6^S(S^0)$ whose 2-component is $\mathbf{Z}_2$.

So we have the list of the 2-components of $\pi_n^S(S^0)$ for $n \leq 13$:

$$\mathbf{Z}_2,\ \mathbf{Z}_2,\ \mathbf{Z}_8,\ 0,\ 0,\ \mathbf{Z}_2,\ \mathbf{Z}_{16},\ \mathbf{Z}_2 \oplus \mathbf{Z}_2,\ \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2,\ \mathbf{Z}_2,\ \mathbf{Z}_8,\ 0,\ 0.$$

Let η denote the generator of $\pi_1^S(S^0)$. Then $\pi_2^S(S^0)$ is generated by η^2. Further $\eta^3 = 4\alpha$ where α is the generator of $\pi_3^S(S^0)$. Again $\pi_3^S(S^0)$ is generated by α^2. The elements of $\pi_7^S(S^0)$ do not decompose. Let its generator be denoted by β. The group $\pi_8^S(S^0)$ is generated by $\eta\beta$ and an indecomposable γ; $\pi_9^S(S^0)$ is generated by $\alpha^3 = \eta^2\$\beta$ and two further generators one of which is clearly indecomposable while the second is possibly equal to $\eta\gamma$. Next $\pi_{10}^S(S^0)$ is generated by η multiplied by one of these generators; $\pi_{11}^S(S^0)$ is generated by an indecomposable element.

The whole of this lot is contained in the result we have about spectral sequences.

Finally something about the p-components of these groups for $p>2$.

The only difficulty arises in the case $p=3$ where one must turn to the Adams spectral sequence. It will be recalled that the first non-trivial p-component occurs in $\pi_{2p-3}^S(S^0)$ and the second in $\pi_{4p-5}^S(S^0)$. Both of them are equal to $\mathbf{Z}_p$. Thus the groups $\pi_n^S(S^0)$, $n \leq 13$ have no p-components for $p>3$ except $\mathbf{Z}_5 \subset \pi_7^S(S^0)$ and $\mathbf{Z}_7 \subset \pi_{11}^S(S^0)$.

The 3-components of these groups are

$$0,\ 0,\ \mathbf{Z}_3,\ 0,\ 0,\ 0,\ \mathbf{Z}_3,\ 0,\ 0,\ 0,\ \mathbf{Z}_9,\ 0,\ \mathbf{Z}_3.$$

(the reader may check this with use of the mod 3 Adams spectral sequence: it is much easier than our mod 2 job).

The composition product is trivial in these groups by consideration of the dimensions. Hence

$$\begin{aligned}
\pi_n(S^n) &= \mathbf{Z} && (n \geq 1)\\
\pi_{n+1}(S^n) &= \mathbf{Z}_2 && (n \geq 3)\\
\pi_{n+2}(S^n) &= \mathbf{Z}_2 && (n \geq 4)\\
\pi_{n+3}(S^n) &= \mathbf{Z}_{24} && (n \geq 5)\\
\pi_{n+4}(S^n) &= 0 && (n \geq 6)\\
\pi_{n+5}(S^n) &= 0 && (n \geq 7)\\
\pi_{n+6}(S^n) &= \mathbf{Z}_2 && (n \geq 8)\\
\pi_{n+7}(S^n) &= \mathbf{Z}_{240} && (n \geq 9)\\
\pi_{n+8}(S^n) &= \mathbf{Z}_2 \oplus \mathbf{Z}_2 && (n \geq 10)\\
\pi_{n+9}(S^n) &= \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 && (n \geq 11)
\end{aligned}$$

$\pi_{n+10}(S^n) = \mathbf{Z}_2$ $(n \geq 12)$
$\pi_{n+11}(S^n) = \mathbf{Z}_{504}$ $(n \geq 13)$
$\pi_{n+12}(S^n) = 0$ $(n \geq 14)$
$\pi_{n+13}(S^n) = \mathbf{Z}_3$ $(n \geq 15)$

In computing the 14th and subsequent groups we face some difficulties, because consideration of the dimensions cease to ensure the triviality of the differentials. Indeed a nontrivial differential appears at the first possibility: $d_2(h_4) = h_0h_3^2$, $d_3(h_0h_4) = h_0i$, $d_3(h_0^2h_4) = h_0^2i$.

Adams' theorems on $E_2^{s,t}$

"...it is shown that homological algebra can be applied to stable homotopy theory. In this application, we deal with A-modules, where A is the mod p Steenrod algebra. To obtain a concrete geometrical result by this method usually involves work of two distinct sorts. To illustrate this, we consider the spectral sequence:

$$\mathrm{Ext}_A^{s,t}(H^*(Y; \mathbf{Z}_p), H^*(X; \mathbf{Z}_p) \Rightarrow {}_p\pi_*^S(X, Y).$$

Here each group $\mathrm{Ext}^{s,t}$ which occurs in the E_2 term can be effectively computed; the process is purely algebraic. However, no such effective method is given for computing the differentials d_r in the spectral sequence, or for determining the group extension by which ${}_p\pi_*^S(X, Y)$ is built up from the E_∞ term; these are topological problems.

A mathematical logician might be satisfied with this account: an algorithm is given for computing E_2; to find the maps d_r still requires intelligence. The practical mathematician, however, is forced to admit that the intelligence of mathematicians is an asset at least as reliable as their willingness to do large amounts of tedious mechanical work. In fact, when a chance has arisen to show that such differencial d_r is non-zero, it has been regarded as an interesting problem, and duly solved.

However, the difficulty of computing groups $\mathrm{Ext}_A^{s,t}(L, M)$ has remained the greatest obstacle to the method."

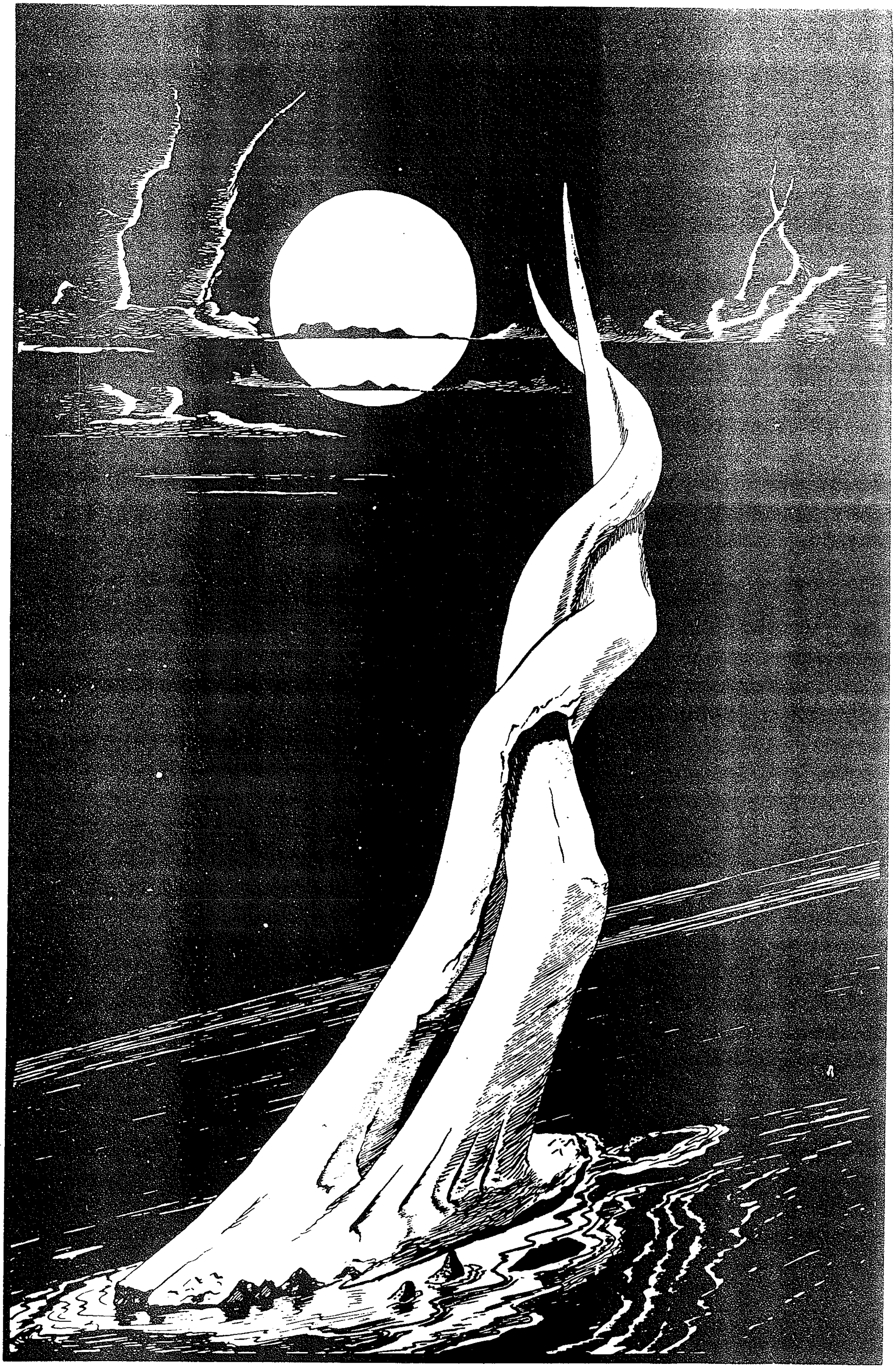
As seen from this text cited from the introduction to Adams' paper "A periodicitiy theorem in homological algebra", its author does not consider that algorithmic computability of the homology of the Steenrod algebra solves the problem of their computation once and for all. Adams devoted a series of papers to the question. Some of the main results are the following theorems.

Theorem on the three bottom rows (Adams J., Ann. Math. 1960, 72, N1, 20–104).

(i) The group $E_2^1 = \bigoplus_t E_2^{1,t}$ is additively generated by linearly independent elements h_i of dimensions 2^i.

(ii) The group $E_2^2 = \bigoplus E_2^{2,t}$ is additively generated by linearly independent elements h_jh_i with $j \geq i \geq 0$, $j \neq i+1$. The products $h_{i+1}h_i$ are equal to zero for all i.

(iii) In the group $E_2^3 = \bigoplus_t E_2^{3,t}$ the following relations hold:



$$h_{i+2}h_i^2 = h_{i+1}^3, \quad h_{i+2}^2 h_i = 0.$$

If the elements $h_{j+1}h_jh_i$, $h_kh_{i+1}h_i$, $h_{i+2}h_i^2$, $h_{i+2}^2h_i$ are omitted, the remaining products $h_kh_jh_i$, $k \geq j \geq i \geq 0$ are linearly independent in E_2^3.

Actually this does not fully describe the third row of E_2 as it may also contain elements that cannot be expressed by h_i, like $a \in E_2^{3,11}$.

Triviality theorem (Adams J., Proc. Cambr. Phil. Soc., 1966, 61).

$E_2^{s,t} = 0$ for $s < t < f(s)$, where

$$\begin{aligned} f(4n) &= 12n-1, && (n>0),\\ f(4n+1) &= 12n+2, && (n \geq 0),\\ f(4n+2) &= 12n+4, && (n \geq 0),\\ f(4n+3) &= 16n+6, && (n \geq 0). \end{aligned}$$

Here $E_2^{s,f(s)}$ is indeed a non-trivial group.

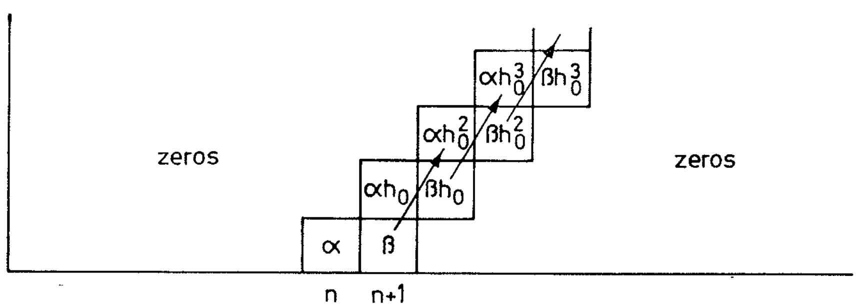
One can see on the diagram the domain where there are only zeros by the theorem

Periodicity theorem (the same article).

For any k there exists a neighbourhood N_k of the line $t = 3s$ where the groups $E_2^{s,t}$ are being repeated periodically, with period 2^{k+2} in s and period $3 \cdot 2^{k+2}$ in t. The union of such neighbourhoods is a domain $s < t < g(s)$, where $g(s)$ is some function with $4s \leq g(s) \leq 6s$.

Finally we show an example of an Adams spectral sequence which clearly contains a nontrivial differential.

Let $X = K(\mathbf{Z}_4, n)$, with n sufficiently large. Up to dimension $\approx 2n$ it has the following non-trivial stable homotopy groups: $\pi_n^S(X) = \mathbf{Z}_4$ and $\pi_i^S(X) = 0$ for $i \neq n$. Thus in these



Clearly $d_2\beta = \alpha h_0^2$, otherwise the order of $\pi_n^S(X)$ could not be equal to four.

dimensions the algebra $\tilde{H}^*(X; \mathbf{Z}_2)$ is isomorphic to

$$\tilde{H}^*(K(\mathbf{Z}, n); \mathbf{Z}_2) \otimes \tilde{H}^*(K(\mathbf{Z}, n+1); \mathbf{Z}_2),$$

i. e. it has two generators $\alpha \in H^n(X; \mathbf{Z}_2)$ and $Sq^1\alpha = Sq^1\beta = 0$. Then the E_2 term contains two diagonals that are filled with by non-trivial groups

§35. PARTIAL OPERATIONS

In this section we do not wish to give new information, it is rather aiming at a deeper comprehension of the former material.

Our account of the Adams spectral sequences did not employ partial operators we did not even mention this notion. Perhaps it should have been proper to introduce it at the introductory section along with the general ideas about Adams spectral sequences. In fact one may say that the notion of partial operation is the fundament underlying to the method of Adams.

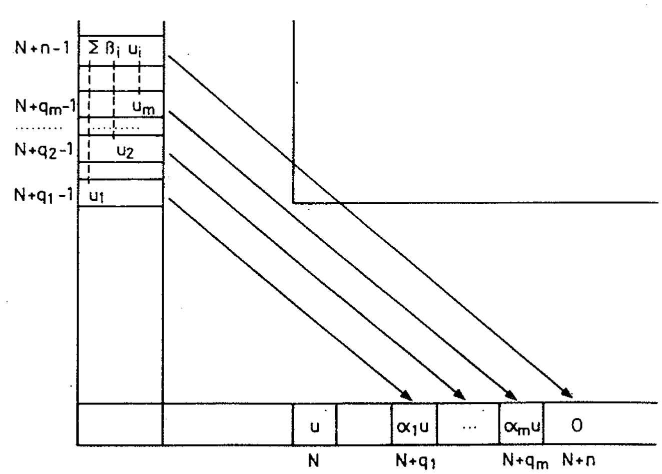
This section will contain almost no proofs. We hope that the reader will take it as a series of interesting exercises.

Construction of partial operations

Let
$$\sum_{i=1}^{m} \beta_i \alpha_i = 0 \qquad (*)$$
be a relation in the Steenrod algebra $A_{(p)}$. (Here α_i, β_i are stable operations of degrees q_i and $n-q_i$, respectively.) For every N each α_i defines a mapping
$$\tilde{\alpha}_i\colon K(\mathbf{Z}_p, N) \to K(\mathbf{Z}_p, N+q_i),$$
all together they define
$$\tilde{\alpha}\colon K(\mathbf{Z}_p, N) \to \Pi_i K(\mathbf{Z}_p, N+q_i).$$

The induced fibration
$$\begin{array}{ccc}
E & \longrightarrow & * \\
\Big\downarrow {\scriptstyle \Pi_i K(\mathbf{Z}_p, N+q_i-1)} & & \Big\downarrow {\scriptstyle \Pi_i K(\mathbf{Z}_p, N+q_i-1)} \\
K(\mathbf{Z}_p, N) & \xrightarrow{\ \tilde{\alpha}\ } & \Pi_i K(\mathbf{Z}, N+q_i)
\end{array}$$

has a spectral sequence of the form



The element $\Sigma\beta_i u_i$ remains in E_∞ to define a coset in the group $H^{N+n-1}(E;\, \mathbf{Z}_p)$ by the subgroup $\operatorname{Im}[H^{N+n-1}(K(\mathbf{Z}_p, N)) \to H^{N+n-1}(E;\, \mathbf{Z}_p)]$.

Let us choose an arbitrary element of the coset and denote it by v. By definition the homomorphism $H^{N+n-1}(E;\, \mathbf{Z}_p) \to H^{N+n-1}(\Pi_i K(\mathbf{Z}_p,\, N+q_i-1);\, \mathbf{Z}_p)$ maps v onto $\Sigma_i \beta_i u_i$. Again $u \in H^N(K(\mathbf{Z}_p, N);\, \mathbf{Z}_p)$ remains in E_∞ to define an element of $H^N(E;\, \mathbf{Z}_p)$ which we are going to denote by u, too. Now for any CW complex X we define a natural homomorphism

$$\bigcap_{i=1}^{m} \operatorname{Ker}[\alpha_i\colon H^N(X;\, \mathbf{Z}_p) \to H^{N+q_i}(X;\, \mathbf{Z}_p)] \to$$

$$\to H^{N+n-1}(X;\, \mathbf{Z}_p) / \bigoplus_{i=1}^{m} \operatorname{Im}[\beta_i\colon H^{N+q_i-1}(X;\, \mathbf{Z}_p) \to H^{N+n-1}(X;\, \mathbf{Z}_p)]$$

which will be called a *secondary operation.*

Let $\xi \in H^N(X;\, \mathbf{Z}_p)$ and $\alpha_i \xi = 0$ for $i = 1, 2, \ldots, m$. The mapping $\tilde{\xi}\colon X \to K(\mathbf{Z}_p, N)$, corresponding to ξ, will be homotopy trivial if composed by $\tilde{\alpha}$. So the fibration over X induced from $* \to \Pi_i K(\mathbf{Z}_p, N+q_i)$ by the mapping $\tilde{\alpha} \circ \tilde{\xi}$.) Thus there exists a section

$$\begin{array}{ccc} E' \approx X \times \Pi_i(K(\mathbf{Z}_p,\, N+q_i-1)) & \to & E \\ \downarrow\uparrow & & \downarrow \\ X & \xrightarrow{\quad \tilde{\xi} \quad} & K(\mathbf{Z}_p,\, N) \end{array}$$

By composing it with the upper row we get a commutative triangle

$$\begin{array}{ccc} & & E \\ & \overset{\tilde{\eta}}{\nearrow} & \downarrow \\ X & \longrightarrow & K(\mathbf{Z}_p,\, N) \end{array}$$

Clearly $\tilde{\eta}^* u = \xi$. Let $\varphi(\xi) \in H^{N+n-1}(X;\, \mathbf{Z}_p)$ be equal to $\tilde{\eta}^* v$. It is not uniquely defined. Its definition depends on the choice of the section, so if the latter is not fixed the mapping is not wholly defined.

To what extent is the section determined? Its existence is a consequence of the homeomorphism between E' and $X \times \Pi_i K(\mathbf{Z}_p,\, N+q_i-1)$, and it has been given by the formula $x \mapsto (x, *)$. Any other section is given by $x \mapsto (x, \zeta(x))$ where $\zeta\colon X \to \Pi_i K(\mathbf{Z}_p,\, N+q_i-1)$ is an arbitrary continuous mapping. The reader will prove that such substitution changes $\varphi(\xi)$ into $\varphi(\xi) + \sum_{i=1}^{n} \beta_i \xi^*(u_i)$ where

$$u_i \in H^{N+q_i-1}(K(\mathbf{Z}_p,\, N+q_i-1);\, \mathbf{Z}_p)$$

are the fundamental classes. Thus $\varphi(\xi)$ is uniquely defined as an element of the corresponding quotient group.

So we have defined, by relation $(*)$, a secondary operation φ. It represents a family of partially-defined multi-valued homomorphisms which are given for every X and N. (Partial because they are defined on the intersection of the kernels of the operations α_i and multi-valued because they are defined up to images of the operations β_i.)

Basic properties of the secondary operations, including *stability* and *naturality*, may be formulated and proved by the reader.

An example of a secondary operation is the "Second Bockstein homomorphism". It is constructed by using the relation $\beta\beta = 0$ and is defined on elements $\xi \in H^N(X; \mathbf{Z}_p)$ for which $\beta\xi = 0$. An alternative direct definition is the following. Let x be a cocycle representing ξ and $\tilde{x}$ be an integral cochain which is projected onto x. In view of $\beta\xi = 0$, there exists an integral cochain y such that $\frac{1}{p}\delta\tilde{x} \equiv \delta y \,(\mathrm{mod}\, p)$, i. e. $\delta(\tilde{x} - py) \equiv 0$ reduced *mod p*. Analogous are the definitions of the "Third Bockstein homomorphisms"

$$\left(\frac{1}{p^3}\,\delta(\tilde{x} - py - p^2 y')\right)$$

and so on.

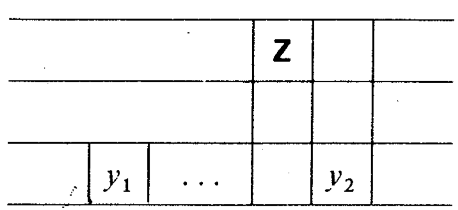
These are examples for tertiary etc. operations. In general a tertiary operation is defined by a relation $\Sigma\beta_i\varphi_i = 0$ where φ_i are secondary and β_i are primary, i. e. ordinary operations. The reader, interested in the topic, may develop the theory of n-ary operations for arbitrary n.

Moreover if we do not confine ourselves to stable operations, and take not only $\mathbf{Z}_p$ for coefficients but arbitrary groups, we shall have such a plenty of operations that the homotopy type of the space will already be completely determined by them.

It is not easy to formulate this theorem exactly, but once it is done, it is already obvious.

Secondary operations and second differential in the Adams spectral sequence

Connection between secondary operations and the differential will not be discussed here in whole extent. Rather we focus our interest on a simple case. Suppose that in the Adams spectral sequence of some space we have elements $y_1 \in E_2^{s-n+1,t}$, $y_2 \in E_2^{s,t}$ and $z \in E_2^{s+2,t-1}$, where y_1 and y_2 are generators of the A-module $\tilde{H}^*(X(s); \mathbf{Z}_p)$. Suppose that $y_1 = \varphi(y_2)$ where φ is a secondary operation defined by the relation $\Sigma\beta_i\alpha_i = 0$.



By definition $\alpha_i y_1 = 0$ for all i. Let the operation β_i be applied to this relation (in the A-module $\tilde{H}^*(X; \mathbf{Z}_2)$) and take the sum of the relations obtained. The result is the relation $0=0$. Hence we have a relation in the module of relations. Assume that it is one of the generators of the A-module of the relations in the module of relations in the module $\tilde{H}^*(X(s); \mathbf{Z}_2)$, and that this generator is the very element z. (This is permitted by the dimensions.)

Then $d_2 y = z$ in the Adams spectral sequence.

The proof is left to the reader.

Partial operations and homotopy groups of spheres

The homology of the Steenrod algebra has obvious connection with the partial operations. The first row contains the primary operations, the second row—the relations in the Steenrod algebra, i. e. the secondary operations, the third—the relations between them, i. e. the ternary operations, etc. Every element of the p-component of the homotopy group of a sphere comes from some element of the E_2 term of the Adams spectral sequence. What is the connection between operations and elements of homotopy groups of a sphere?

An element $\alpha \in \pi_{N+k}(S^N)$ defines a mapping $S^{N+k} \to S^N$.

Let the ball D^{N+k+1} be sewn on S^N along the mapping. We obtain a complex X_α of two cells whose cohomology is nontrivial in the dimensions N and $N+k+1$. As it turns out the partial operation corresponding to the element of E_2 that gives α in the E_2 term is nontrivial in X_α.

This statement may easily be proved by considering the Adams spectral sequences for X_α and S^n and the mapping induced between them by the inclusion $S^n \subset X_\alpha$.

For example, consider the elements of the bottom row: $Sq^2, Sq^4, Sq^8, \ldots \in \tilde{A}$. Those surviving until E_∞ define elements of $\pi_{n+1}(S^n)$, $\pi_{n+3}(S^n)$, $\pi_{n+7}(S^n)$, ... such that the operations Sq^2, Sq^4, $Sq^8, \ldots$ are not trivial in the respective complexes $S^n \cup D^{n+2}$, $S^n \cup D^{n+4}$. $S^n \cup D^{n+8}, \ldots$.

As proved by Adams, not every element $h_4, h_5, \ldots$ does reach E_∞, i. e. some of them have non-trivial differentials. (For example, $d_2 h_4 = h_0 h_3^2 \neq 0$.)

Therefore if there exists any two-cell complex $X = S^n \cup D^{n+q}$ such that the operation Sq^q is not trivial on it, then necessarily $q = 1, 2, 4, 8$. In particular for these q alone may the group $\pi_{2q-1}(S^q)$ have an element with odd Hopf invariant.

APPENDIX 3
POSTNIKOV'S NATURAL SYSTEMS

The natural systems of Postnikov* should be mentioned first, because the term is widely used in the literature and, second, because Postnikov's paper from the year 1949 anticipated the further investigations in restoring homotopy properties of a space by using the algebraic invariants. The language of that paper is of course different from what we have been using. It does not contain either the notion of the Leray spectral sequence or even the term fibration.

As it is well known, the homotopy groups do not fully determine the homotopy type of a space. There are two exceptions: when all homotopy groups are trivial, or all are trivial except one. In both cases the space is determined by its homotopy groups up to weak homotopy equivalence.

In the case of spaces with two nontrivial homotopy groups the situation is already different. Indeed, let π_1 and π_2 be Abelian groups, and n_1 and n_2 be natural numbers such that $1 < n_1 < n_2$. Assume that $\pi_{n_1}(X) = \pi_1$ and $\pi_{n_2}(X) = \pi_2$, and $\pi_n(X) = 0$ for the remaining n. There exists an obvious mapping of X into $K(\pi_1, n_1)$. Consider the spectral sequence of the equivalent fibration (with fibre $K(\pi_2, n_2 - 1)$) with coefficients in π_2. Then $e \in H^{n_2-1}(K(\pi_2, n_2 - 1); \pi_2)$ is mapped by the transgression onto an element of $H^{n_2}(K(\pi_1, n_1); \pi_2)$ which, on one hand, may be any element of the group and, on the other hand, wholly defines the homotopy type of X. Thus the homotopy type of the space with two nontrivial homotopy groups is determined by the groups and an element of $H^{n_2}(K(\pi_1, n_1); \pi_2)$ which is regarded as a cohomology operation and called the Postnikov factor of the space X.

Assume now that X has three nontrivial homotopy groups π_1, π_2 and π_3 in dimensions $n_1 < n_2 < n_3$. By attaching the highest homotopy group we obtain a mapping of X into a space Y with two nontrivial homotopy groups $\pi_{n_1}(Y) = \pi_1, \pi_{n_2}(Y) = \pi_2$. It may be considered as a fibration with fibre $K(\pi_3, n_3 - 1)$. The fundamental class of the fibration is mapped by the transgression onto an element of $H^{n_3}(Y; \pi_3)$. This element may be chosen arbitrarily, and fully determines the homotopy type of X.

It may again be regarded as a secondary cohomology operation defined on the kernel of the primary cohomology operation that is the first Postnikov factor of the space Y. This secondary operation is the so-called second Postnikov factor of X. We conclude that the homotopy type of a space X is determined by the homotopy groups and two cohomology operations: a primary one defined on the n_1 dimensional cohomology with coefficients in π_1 taking its values in the n_2 dimensional cohomology with coefficients in π_2, and a secondary one which is defined on the kernel of the former and takes its values in n_3 dimensional cohomology with π_3 coefficients. By continuing the construction and performing the limit transition which makes no trouble in the case of finite complexes we obtain the following result:

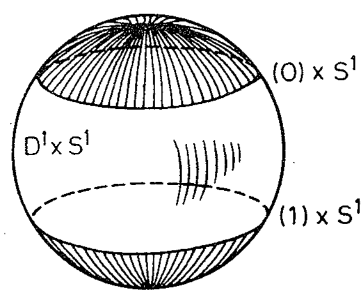
* The modern term is "the Postnikov tower".

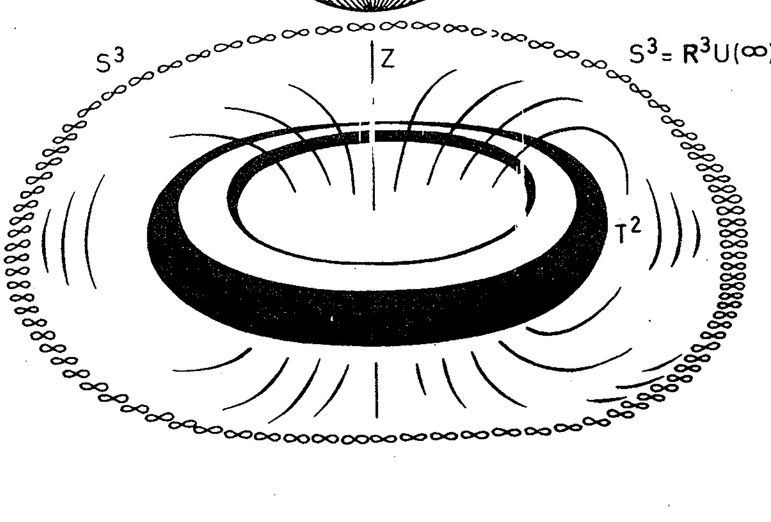
The homotopy type of any simply connected finite CW complex X is determined by its homotopy groups and a sequence of homotopy operations: a primary, a secondary, a ternary one, etc., each successive operation being defined on the kernel of the preceding one. They are called the Postnikov factors of the space X. The whole system of invariants is the natural Postnikov system of the space X.

APPENDIX 4
THE J-HOMOMORPHISM

Next we are going to formulate without proof some theorems of Adams that describe certain subgroups of the stable homotopy groups of spheres in terms of the so-called J-homomorphism. In order to remain within the frame of the present book, we give a purely geometric definition of this homomorphism.

Let us be given a continuous mapping f: $S^m \times S^n \to S^n$. We shall define a mapping of the sphere S^{m+n+1} into the sphere S^{n+1}. The sphere S^{m+n+1} contains the direct product $S^m \times S^n$ (as will be shown later in more detail) as the common boundary of two solid torii $(D^{m+1} \times S^n)$ and $(S^m \times D^{n+1})$ (here D^{m+1} and D^{n+1} are disks of the corresponding dimensions), and the sphere S^{m+n+1} is obtained by sewing them together. In the special cases $m=0$, $n=1$ and $m=1$, $n=1$ the decomposition of the spheres S^2 and S^3 into two solid torii is shown on the following pictures:





The solid tori Π_1 and Π_2 are sewn together to the sphere S^{m+n-1} in the following way:

Let us consider the restriction f' of the obvious homeomorphism

$$\varphi\colon D^{m+1}\times D^{n+1}\to D^{m+n+2}$$

to the boundary

$$\varphi'\colon \partial(D^{m+1}\times D^{n+1})\to\partial D^{m+n+2}.$$

We obtain

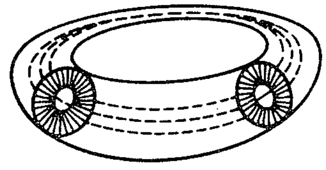
$$\varphi'\colon (S^m\times D^{n+1})\cup(D^{m+1}\times S^n)\to S^{m+n-1},$$

which gives the decomposition of the sphere S^{m+n+1} as the union of two solid torus Π_1 and Π_2: $\Pi_1=S^m\times D^{n+1}$, $\Pi_2=D^{m+1}\times S^n$.

Both solid torus Π_1 and Π_2 are of the form $(S^m\times S^n)\times I/R_i$ where, according to the

$$S^{m+n+1}=\Pi_1\cup_e\Pi_2\to C(S^m\times S^n)\cup_e C(S^m\times S^n)=\Sigma(S^m\times S^n)$$

Both solid tori Π_1 and Π_2 are of the form $(S^m\times S^n)\times I/R_i$ where, according to the equivalence relation $R_i (i=1,2)$, the points $(p,q,1)$ on the upper face of the cylinder are identified with $(p_0,q,1)$ (for $i=1$) and $(p,q_0,1)$ (for $i=2$) respectively. (Here p_0 and q_0 are the base points of S^m and S^n, respectively.)



It follows from the above consideration that there exists a continuous mapping

$$S^{m+n+1}=\Pi_1\cup_e\Pi_2\to C(S^m\times S^n)\cup_e C(S^m\times S^n)=\sum(S^m\times S^n)$$

(where e is the identity mapping and C is a cone). To obtain it we have to contract two spheres into a single point: S^m into Π_1 and S^n into Π_2 (where S^m and S^n are the "axes" of the solid torus Π_1 and Π_2). Since $f\colon S^m\times S^n\to S^n$ is already given, the mapping $H(f)\colon S^{m+n+1}\to S^{n+1}=\Sigma S^n$ is constructed in an obvious way. The mapping $H(f)$ may be given by simple formulas. Any vector x of the Euclidean space $\mathbf{R}^{m+n+2}$ may may be given in the form (p,q), $p\in\mathbf{R}^{n+1}$, $q\in\mathbf{R}^{m+1}$. The vectors (p,q) with $|p|^2+|q|^2=1$ belong to the sphere S^{m+n+1}, with $(p,0)$, $|p|=1$ running through the sphere S^n, and with $(0,q)$, $|q|=1$ running through the sphere S^m. Every point $(p,q)\in S^{n+m+1}$ is uniquely represented as $p'\cos\varphi+q'\sin\varphi$ where $p'\in S^n$ and $q'\in S^m$. Now if f maps $S^n\times S^m$ into S^n then the formulas for $H(f)$ are

$$[H(f)\,(p,q)]_i = \sin 2\varphi \cdot f_i(p',q') \text{ if } |p|\cdot|q| \neq 0 \ (i=1,2,\ldots,n+1);$$

$$H(f)\,(0,q) = H(f)\,(p,0) = 0;$$

$$[H(f)\,(p,q)]_{n+2} = \cos 2\varphi,$$

where $(p,q) = p' \cos\varphi + q' \sin\varphi$.

Here the index i indicates the corresponding coordinate of the vector

$$H(f)\,(p,q) \in S^{n+1} \subset \mathbf{R}^{n+2},$$

and p' and q' are the projections of (p,q) into $\mathbf{R}^{n+1}$ and $\mathbf{R}^{m+1}$, respectively; $\sin 2\varphi = 2|p|\,|q|$, $\cos 2\varphi = |q|^2 - |p|^2$.

In this notation the boundary of the solid torus Π_1 and Π_2, i. e. $S^m \times S^n$ has equation $|p| = |q|$; $(p,q) \in S^{m+n+1}$; while the solid torus Π_1 and Π_2 are given by the inequalities $|p| \geq |q|$ and $|p| \leq |q|$, respectively.

Let us now consider the group $SO(n+1)$. Let $[g] \in \pi_m(SO(n+1))$, and $g(S^m) \subset$ $\subset SO(n+1)$ be a corresponding spheroid.

On the direct product $S^n \times S^m$ the continuous mapping $g^*\colon S^n \times S^m \to S^n$ is given by the formula $g^*(p',q') = [g(q')](p')$ where $p' \in S^n$, $q' \in g(S^m) \subset SO(n+1)$.

So g^* maps $S^n \times S^m$ into S^n. Thus by assigning the mapping $H(g^*)$ to g we finally obtain that $H\colon \mapsto H(g^*)$ maps the set $\pi_m(SO(n+1))$ into the set $\pi_{m+n+1}(S^{n+1})$, since the replacement of g by its homotopic image $\tilde{g}$ results a replacement $H(\tilde{g}^*)$ homotopic to $H(g^*)$. The proof of the following statement is left to the reader.

Theorem. The mapping H is a homomorphism of the group $\pi_m(SO(n+1))$ into the group $\pi_{m+n+1}(S^{n+1})$.

It can be shown that the homomorphism $H_{m,n}\colon \pi_m(SO(n+1)) \to \pi_{m+n+1}(S^{n+1})$ is an isomorphism if $m=1, n>1$, or $m=2, n>1$. It is well known that $\pi_1(SO(n+1)) = \mathbf{Z}_2$ and $\pi_{n+2}(S^{n+1}) = \mathbf{Z}_2$ for $n>1$.

For a large enough n, the homomorphism $H_{m,N}$ is a homomorphism between the stable homotopy groups

$$\pi_m^S(SO) \to \pi_m^S(S^0)$$

and is called J-homomorphism.

An alternative definition of the J-homomorphism is the following.

Any transformation $g \in SO(n)$ can be regarded as a continuous mapping $S^{n-1} \to S^{n-1}$.

We obtain then an imbedding of $SO(n)$ into the set of all continuous mappings of the sphere S^{n-1} into itself: $SO(n) \xrightarrow{i} \Pi(S^{n-1}, S^{n-1})$. Any mapping $\alpha\colon S^{n-1} \to S^{n-1}$ uniquely defines a mapping $\Sigma\alpha\colon \Sigma S^{n-1} \to \Sigma S^{n-1}$ which preserves the base point x_0 (e.g. the north pole) of the sphere $S^n = \Sigma S^{n-1}$: thus we obtain an imbedding

$$\Pi(S^{n-1}, S^{n-1}) \xrightarrow{j} \Pi_{x_0}(S^n, S^n)$$

where on the right-hand side we have the space of continuous, base-point preserving mappings of S^n into itself, i. e. the n-fold loop space $\Omega^n(S^n)$ (by virtue of the natural homeomorphism $\Omega^n(X) \cong \Pi_*(S^n, X)$. Here $\Pi_*(S^n, X)$ denotes the space of those continuous maps from the sphere S^n into the space X which map the base point of S^n into the base point of X, valid for any space X).

Thus there is a chain of imbeddings

$$SO(n) \xrightarrow{i} \Pi(S^{n-1}, S^{n-1}) \xrightarrow{j} \Pi_{x_0}(S^n, S^n) \cong \Omega^n(S^n)$$

which induces a chain of homomorphisms of the homotopy groups

$$\pi_m(SO(n)) \to \pi_m(\Omega^n(S^n)) \cong \pi_{m+n}(S^n) = \pi_{m+n}(\Sigma^n S^0).$$

For a large enough n, we obtain a homomorphism between stable homotopy groups $J: \pi_m(SO) \to \pi_m^S(S^0)$ called a J-homomorphism.

Exercise. Prove that the two definitions are equivalent.

The stable homotopy groups $\pi_m(SO)$ of the full orthogonal group $SO = \lim_{n\to\infty} SO(n)$ are well known. They are periodic with period 8 (orthogonal Bott periodicity) and have the following form

0	1	2	3	4	5	6	7	8	9
$\mathbf{Z}_2$	$\mathbf{Z}_2$	0	$\mathbf{Z}$	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$

The following theorems are due to Adams.

Theorem 1. For $m \equiv 0 \pmod 8$ and $m > 0$ (in this case $\pi_m(SO) = \mathbf{Z}_2$) the J-homomorphism is a monomorphism and its image is a direct summand in the group $\pi_m^S(S^0)$.

The methods developed by Adams for computing the image of the "stable" J-homomorphism makes it possible to show that for $m \equiv 1$ or $\equiv 2 \pmod 8$ and $m > 0$, the group $\pi_m^S(S^0)$ contains an element μ_m of order 2 (for example, $\mu_1 = \eta$, $\mu_2 = \eta^2$ where $\eta \in \pi_1^S(S^0) = \mathbf{Z}_2$ coresponds to the generator of $\pi_{n+1}(S^n)$, $n > 2$, and η^2 is taken in the sense of composition product in the ring $\pi_*^S(S^0)$).

The elements μ_m, $m \geqslant 1$ are characterized by a series of interesting properties that specify their set among the elements of stable homotopy groups of spheres as a particular class.

Theorem 2. For $m \equiv 2 \pmod 8$ and $m > 0$ the group $\pi_m^S(S^0)$ contains $\mathbf{Z}_2$ as a direct summand generated by μ_m.

Theorem 3. For $m \equiv 1 \pmod 8$ and $m > 1$ the J-homomorphism is a monomorphism and the group $\pi_m^S(S^0)$ contains a direct summand of the form $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ such that the first summand in it is generated by μ_m and the second is Im J.

Before passing to the formulation of the further theorems of Adams we define the Bernoulli numbers.

Definition. The rational numbers $B_m (m \geq 1)$ given by the expansion

$$\frac{t}{e^t - 1} = 1 - \frac{t}{2} + \sum_{s=1}^{\infty} (-1)^{s-1} \frac{B_s}{(2s)!} t^{2s}$$

are called the Bernoulli numbers.

Here are the first twelve Bernoulli numbers.

$$B_1 = \frac{1}{6}, \quad B_2 = \frac{1}{30}, \quad B_3 = \frac{1}{42}, \quad B_4 = \frac{1}{30}, \quad B_5 = \frac{5}{66},$$

$$B_6 = \frac{691}{2730}, \quad B_7 = \frac{7}{6}, \quad B_8 = \frac{3617}{510}, \quad B_9 = \frac{43\,876}{798},$$

$$B_{10} = \frac{174\,611}{330}, \quad B_{11} = \frac{854\,513}{138}, \quad B_{12} = \frac{236\,364\,091}{2730}.$$

Theorem 4. For $m = 4s - 1 \equiv 3 \pmod 8$ (in this case $\pi_m^S(SO) = \mathbf{Z}$) the image of the J-homomorphism is a cyclic group of order $\tau(s)$, which is a direct summand in $\pi_m^S(S^0)$. Here $\tau(2s)$ is the denominator of the irreducible fraction form of $B_s/4s$ where B_s is the s-th Bernoulli number.

Theorem 5. For $m = (4s - 1) \equiv 7 \pmod 8$ (in this case $\pi_m^S(SO) = \mathbf{Z}$) the image of the J-homomorphism is a cyclic group of degree either $\tau(s)$ or $2\tau(s)$. Moreover there exists a homomorphism $\omega\colon \pi_m^S(S^0) \to \mathbf{Z}_{\tau(2s)}$ such that $\omega \circ J\colon \pi_m^S(SO) \to \mathbf{Z}_{\tau(s)}$ is an epimorphism. Hence, if the order of $\operatorname{Im} J$ is equal to $\tau(s)$ then the image is a direct summand in $\pi_m^S(S^0)$. Such cases are, for example, $m = 7$ and $m = 15$.

As it is shown by Mahowald this is the case for all $r = 2^s - 1$.

Homotopy groups of spheres
$\pi_{n+k}(S^n)$

$k \backslash h$	2	3	4	5	6	7	8	9	10	$>k+1$
1	$\mathbf{Z}$	$\mathbf{Z}_2$	...	...	...	...	...	...	...	$\mathbf{Z}_2$
1	$\mathbf{Z}$	$\mathbf{Z}_2$	...	...	...	...	...	...	...	$\mathbf{Z}_2$
2	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	...	...	...	...	...	...	$\mathbf{Z}_2$
3	$\mathbf{Z}_2$	$\mathbf{Z}_{12}$	$\mathbf{Z} \oplus \mathbf{Z}_{12}$	$\mathbf{Z}_{24}$	...	...	...	...	...	$\mathbf{Z}_{24}$
4	$\mathbf{Z}_{12}$	$\mathbf{Z}_2$	$(\mathbf{Z}_2)^2$	$\mathbf{Z}_2$	0	...	...	...	...	0
5	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$(\mathbf{Z}_2)^2$	$\mathbf{Z}_2$	$\mathbf{Z}$	0	...	...	...	0
6	$\mathbf{Z}_2$	$\mathbf{Z}_3$	$\mathbf{Z}_{24} \oplus \mathbf{Z}_3$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	...	...	$\mathbf{Z}_2$
7	$\mathbf{Z}_3$	$\mathbf{Z}_{15}$	$\mathbf{Z}_{15}$	$\mathbf{Z}_{30}$	$\mathbf{Z}_{60}$	$\mathbf{Z}_{120}$	$\mathbf{Z} \oplus \mathbf{Z}_{120}$	$\mathbf{Z}_{240}$	...	$\mathbf{Z}_{240}$
8	$\mathbf{Z}_{15}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{24} \oplus \mathbf{Z}_2$	$(\mathbf{Z}_2)^3$	$(\mathbf{Z}_2)^4$	$(\mathbf{Z}_2)^3$	$(\mathbf{Z}_2)^2$	$(\mathbf{Z}_2)^2$
9	$\mathbf{Z}_2$	$(\mathbf{Z}_2)^2$	$(\mathbf{Z}_2)^3$	$(\mathbf{Z}_2)^3$	$(\mathbf{Z}_2)^3$	$(\mathbf{Z}_2)^4$	$(\mathbf{Z}_2)^5$	$(\mathbf{Z}_2)^4$	$\mathbf{Z} \oplus (\mathbf{Z}_2)^3$	$(\mathbf{Z}_2)^3$

Stable homotopy groups of spheres

k	10	11	12	13	14	15	16	17	18
$\pi_k^S(S^0)$	$\mathbf{Z}_6$	$\mathbf{Z}_{504}$	0	$\mathbf{Z}_3$	$(\mathbf{Z}_2)^2$	$\mathbf{Z}_{480} \oplus \mathbf{Z}_2$	$(\mathbf{Z}_2)^2$	$(\mathbf{Z}_2)^4$	$\mathbf{Z}_8 \oplus \mathbf{Z}_2$

k	19	20	21	22
$\pi_k^S(S^0)$	$\mathbf{Z}_{264} \oplus \mathbf{Z}_2$	$\mathbf{Z}_{24}$	$(\mathbf{Z}_2)^2$	$(\mathbf{Z}_2)^2$

k	23	24	25	26	27	28	29	30
$\pi_k^S(S^0)$	$\mathbf{Z}_{65520} \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2$	$(\mathbf{Z}_2)^2$	$(\mathbf{Z}_2)^2$	$\mathbf{Z}_6 \oplus \mathbf{Z}_2$	$\mathbf{Z}_{24}$	$\mathbf{Z}_2$	0	$\mathbf{Z}_2$

k	31	32	33	35	36	37
$\pi_k^S(S^0)$	$\mathbf{Z}_{16320} \oplus \mathbf{Z}_2$	$(\mathbf{Z}_2)^2$	$\mathbf{Z}_4 \oplus (\mathbf{Z}_2)^4$	$\mathbf{Z}_{28728} \oplus (\mathbf{Z}_2)^2$	$\mathbf{Z}_2$	$\mathbf{Z}_6 \oplus (\mathbf{Z}_2)^2$

ABOUT THE ILLUSTRATIONS

The pictures scattered everywhere in the text are sometimes based on particular topological constructions, sometimes they are graphical realizations of certain theorems and sometimes reflect the atmosphere and colours of a specific group of ideas underlying the text of the book. Beside that, some of the pictures contain further information not connected with spectral sequences or those parts of general homotopy or homology theory that is expounded in the first two chapters. Picture 1 represents the "sphere with horns" of Alexander, i. e. the example of an imbedding $S^2 \subset \mathbf{R}^3$ which is not locally flat at a point and which divides $\mathbf{R}^3$ to domains that are not simply connected. The first step in constructing this sphere is shown on Picture 2. The basic elements of Pictures 3 and 4 are discussed in the text of Chapter I. Picture 5 shows a locally-compact Hausdorff space which is not locally homologically connected (in the sense of Czech) in the dimension 1 (because each open neighbourhood of the endpoint has non-trivial homology group). The decomposition of the same space on its elementary particles is shown on Picture 8 which also contains a solenoid (the first step of the construction, the necklace of Antoin (the first step of the construction), and an example of J. F. Adams. The necklace, resulted as the intersection of the sequence of rus T_2 is a completely disconnected compact perfect metric space, and as such, homeomorphic to the Cantor set. Its complement is not simply connected. The example of Adams can be obtained by taking the sum of the Möbius band with the "triple Möbius band" given in cylindric coordinates by the following formulas: $r = 1 + \varepsilon t \cos u$, $\Theta = 3u$, $z = \varepsilon t \sin u$, ε=constant, $0 \leq t \leq 1$, $0 \leq u \leq 2\pi$. The result is a complex X which is not a manifold, because at $t = 0$ we have a singular line where the three sheets intersect at angles $\frac{2\pi}{3}$. The boundary of the complex is the circle S^1, nevertheless it is a retract of the space X, as it trivially follows from the theorem of Hopf on the extension. That example has a not insignificant role as an illustration for various aspects of contemporary theory of minimal surfaces: S^1 is a boundary in the case when the group of coefficients is $\mathbf{Z}_2$, and is not a boundary in the case of the group $U=S^1$. On the same picture, on the right-hand gallery we see a torus T^2.

Picture 7 can be regarded as an illustration for the covering homotopy property of Serre fibrations. The contents of Picture 9 is the theorem about unlinking of complexes P and Q if dim $P+$dim $Q<n-1$. The scheme of the unlinking procedure is shown also on the Figure on page 70. Picture 10 applies the process of turning the sphere S^2 inside out in $\mathbf{R}^3$ which has been caught at the moment $t=\frac{1}{2}$, after that the series of actions will be repeated in the opposite order.

Picture 11 is based on the division of the sphere S^3 to a pair of tores. The scheme of the same decomposition is shown on page (together with the Hopf fibration $S^3 \to S^2$). The detailed description of the procedure is also given in appendix 4. The action of the fundamental group as a group of left-side operators on the higher homotopy groups motivates Pictures 12 and 13. Picture 14 is devoted to the orthogonal version of the Bott periodicity: the flattened body is the group $SO(16m)$; one can well distinguish the quaternion Grassmann manifolds $G^H_{4m,2m}$ (the white ribs) diverging in both directions with changing n; in the centre we see its component of maximal dimension, $G^H_{4m,2m}$ which contains the other loop spaces that take part in the periodicity at $4 \leq k \leq 8$.

All pictures in the second part of the book admit interpretation in terms of the Adams spectral sequences. The elements coming from the term E_2 are most frequently used. For example Picture 24 illustrates the difference between the annihilating processes of Adams and Serre. The long ledges projections represent the dimensions. The elements in the first dimension are already annihilated. Far away behind we see the cohomology ring already "cleaned".

On page 38 we see the "overwound triangle" which first occured in the literature in a paper of the English genetician L. S. Penrose and his son Roger Penrose, "Impossible object. A special form of optical illusion". British Journal of Psychology, February 1958. The figure in point is the torus, with a trajectory of the type $4a+3b$, i. e. going round three times along the meridian and four times along the parallel. On page 50 we have a figure obtained by putting together two such triangles.