Algebraic topology, Fall 2013

Homework 6, due Wednesday, October 14

1. How would you define a $K(G, 0)$ space, where G is a group?

2. Given two complexes A and B, a homotopy h is a collection of homomorphisms $h_n : A_n \longrightarrow B_{n+1}$ (you can assume that A and B are complexes of abelian groups, although everything works for complexes of modules over any ring R). We say that $f : A \longrightarrow B$ is null-homotopic if $f = dh + hd$ for some h.
(a) Check that f, for any homotopy h, is a homomorphism of complexes.
(b) Any homomorphism of complexes induces a homomorphism of their homology groups. Show that a null-homotopic morphism induces the zero map $H(A) \longrightarrow H(B)$.
(c) Given three complexes and two homomorphisms $A \xrightarrow{f} B \xrightarrow{g} C$ the composition gf is null-homotopic if either f or g is null-homotopic.
(d) The sum and difference of null-homotopic maps is null homotopic.
Conclude that null-homotopic maps constitute an ideal in the category of complexes. The quotient by this ideal is called the homotopy category of complexes.

3. Explain why the complex of abelian groups

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathbb{Z}/4 \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

(with the obvious differential) is acyclic but not contractible

4. Using classification of complexes of vector spaces obtained in class, show that a complex of vector spaces of acyclic if and only if it is contractible.

5. Give an example of a short exact sequence of complexes

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

such that the boundary map $H_n(C) \longrightarrow H_{n-1}(A)$ is nontrivial for some n.

Hatcher exercises 1, 4, 9 in Section 2.1 page 131.

Round table: Understand the proof that the sequence of homology groups associated to short exact sequence of complexes

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is exact (Hatcher Theorem 2.16 Section 2.1 page 117).

Sample computations of homology groups (for instance, exercises 5, 8 in Hatcher on page 131, Section 2.1).