\[ \vec{v}_1 = 0 \text{, redundant (zero vector is always redundant)} \]
\[ \vec{v}_3 = 3\vec{v}_2 \text{, } \vec{v}_3 \text{ is redundant} \]
\[ \vec{v}_5 = 4\vec{v}_2 + 5\vec{v}_4 \text{, } \vec{v}_5 \text{ is redundant} \]
\[ \vec{v}_6 = 6\vec{v}_2 + 7\vec{v}_4 \text{, } \vec{v}_6 \text{ is redundant} \]
\[ \vec{v}_2, \vec{v}_4, \vec{v}_7 \text{ are clearly not redundant} \]

Redundant vectors: \( \vec{v}_1, \vec{v}_3, \vec{v}_5, \vec{v}_6 \).
a) If the vectors $\tilde{v}_1, \ldots, \tilde{v}_m$ do not span $V$, choose a vector $\tilde{v}_{m+1} \in V$ but not in the span of $\tilde{v}_1, \ldots, \tilde{v}_m$.

Then $\tilde{v}_1, \ldots, \tilde{v}_m, \tilde{v}_{m+1}$ are linearly independent.

This contradicts the assumption that $m$ is the largest number of linearly independent vectors we can find in $V$. Therefore, $\tilde{v}_1, \ldots, \tilde{v}_m$ is a basis of $V$.

b) By part (a) subspace $V$ admits a basis $\tilde{v}_1, \ldots, \tilde{v}_m$.

Each vector $\tilde{v}_j$ can be represented via its column of coordinates

$$\tilde{v}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix}$$

Put these columns together into a matrix

$$A = (a_{ij}) = \begin{pmatrix} 1 & 1 & 1 \\ \tilde{v}_1 & \tilde{v}_2 & \tilde{v}_m \end{pmatrix}$$

Necessarily, $V = \text{im} (A)$. 
Matrix $A$ is in RREF form. In an RREF matrix, redundant columns are exactly non-pivot columns. In this example, columns 2 and 4 are redundant.

You can also see it directly, denoting columns $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

$\vec{v}_2 = -2 \vec{v}_1$

$\vec{v}_4 = -\vec{v}_1 + 5 \vec{v}_3$
\[ A = \begin{pmatrix} 2 & 4 & 8 \\ 4 & 5 & 1 \\ 7 & 9 & 3 \end{pmatrix} \times \frac{1}{2} \rightarrow \begin{pmatrix} 1 & 2 & 4 \\ 4 & 5 & 1 \\ 7 & 9 & 3 \end{pmatrix}^{-4I} \rightarrow \]

\[ \rightarrow \begin{pmatrix} 1 & 2 & 4 \\ 0 & -3 & -15 \\ 0 & -5 & -25 \end{pmatrix} \times (-\frac{1}{5}) \rightarrow \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix}^{-2(II)} \rightarrow \begin{pmatrix} 1 & 0 & -6 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix} = \text{rref}(A). \]

\text{ker } A = \ker(\text{rref}(A)). \quad \text{dim}(\ker(A)) \text{ is the # of free variables.}

There is one free variable \( x_3 \), and for \( \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \) in \( \ker A \) we have \( x_1 = 6x_3, x_2 = -5x_3 \). Set \( x_3 = 1 \) \( \Rightarrow \) \( x_1 = 6, x_2 = -5 \) and the vector \( \begin{pmatrix} 6 \\ -5 \\ 1 \end{pmatrix} \) is a basis of \( \ker A \).

The basis of image of \( A \) is given by columns of \( A \) that correspond to leading columns of \( \text{rref}(A) \). In our example, these are columns 1 and 2.

Hence, basis of image of \( A \) is given by columns 1 and 2 of \( A \),

\[ \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix}. \]
\( C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \). By inspection, we see that columns 2 and 3 are equal, and that's the only relation on columns of \( C \). This means \( \text{ker} \, C \) is one-dimensional, with basis vector \( \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \).

In matrices \( H, T, X, Y \) last two columns are not equal. This means \( \begin{pmatrix} 0 \\ -1 \end{pmatrix} \) is not in kernel subspace of these matrices. Excluding \( H, T, X, Y \), we are left with \( L \).

In \( L \), last 2 columns are equal, and that's the only relation on columns of \( L \). Therefore \( \text{ker} \, L = \text{ker} \, C \), with basis vector \( \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \).