Modern Algebra II, fall 2020, Instructor M.Khovanov

Homework 10, due Wednesday November 18.

1. (10 points) Which of the following extensions are finite? Which are normal? Which are Galois?
$\mathbb{F}_{64}/\mathbb{F}_2, \quad \mathbb{F}_{64}/\mathbb{F}_4, \quad \mathbb{C}/\mathbb{R}, \quad \mathbb{Q}(\sqrt[5]{7})/\mathbb{Q}, \quad \mathbb{Q}(\sqrt{2},\sqrt{5})/\mathbb{Q}, \quad \mathbb{R}/\mathbb{Q},$

$\mathbb{F}_p(x)/\mathbb{F}_p$, where x is a formal variable.

2. (15 points) Take two prime numbers p_1, p_2 and explain the structure of the field extension $E/\mathbb{Q}$ with $E = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2})$, by analogy with the case $p_1 = 2, p_2 = 3$ done in class. Write down this extension as a splitting field of a suitable polynomial. What is the Galois group $G = \mathrm{Gal}(E/\mathbb{Q})$ and how does it act on the roots of the polynomial? Find all intermediate fields $\mathbb{Q} \subset K \subset E$ and their subgroups $H = \mathrm{Gal}(E/K) \subset G$. Which of the extensions $K/\mathbb{Q}$ are splitting fields?

3. (20 points) (a) Write down a definition of a normal extension E/F. Explain in your own words what is an obstacle to an extension to be normal.
(b) Let E/F be a degree two extension. Prove that E is normal. Hint: pick an element $\alpha \in E \setminus F$. Write down its irreducible polynomial $f(x)$. Can you show that E is a splitting field of $f(x)$?
(c) Look through class notes and find an example of degree 3 extension of $\mathbb{Q}$ which is not normal. Generalize that example and describe a degree n extension of $\mathbb{Q}$ which is not normal, for any $n \geq 3$.
(d) Take a field F of characteristic 2 with an element $a \in F$ which does not have a square root in F (field F is necessarily infinite). Consider the splitting field E of $x^2 - a$. Explain why E is a normal extension which is not Galois.

4. (10 points) Let K/F be a finite extension. Prove that there is an extension E/K so that E/F is a splitting field of some polynomial $f(x) \in F[x]$. (Hint: K/F is finite, hence algebraic, and $K = F(\alpha_1, \dots, \alpha_n)$ for some $\alpha_1, \dots, \alpha_n$. Take E to be the splitting field of $p_1(x) \dots p_n(x)$ where $p_i(x)$ is the irreducible polynomial of α_i over F.

5. (15 points) Consider the splitting field E of the polynomial $f(x) = x^3 - 5$ over $\mathbb{Q}$. Repeat the arguments we used to understand the splitting field of $x^3 - 2$ in this case. What is the Galois group $\mathrm{Gal}(E/\mathbb{Q})$ and how does it act on the roots of $f(x)$? Draw a diagram of all intermediate fields K and list the corresponding subgroups of the Galois group. Which of the extensions K/Q are Galois?

6. (optional) (a) We finished the lecture by discussing the Galois group G of the extension $E = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3})$ over $\mathbb{Q}$, where p_1, p_2, p_3 are distinct prime numbers. Carefully write down arguments sketched in class and describe the Galois group $G = \mathrm{Gal}(E/\mathbb{Q})$ and how to understand and classify all intermediate subfields $\mathbb{Q} \subset K \subset E$. You don't have to list them all, but give some examples, explain how to describe subgroups H of G and which subfields E^H they generate.

(b) Show that E as defined in (a) does not contain $\sqrt[3]{p}$, for any prime p. Try to generalize this result to roots with exponents other than 3.

7. (optional) Complete the arguments given at the beginning of the lecture to show that $E = \mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_n})$ is a degree 2^n Galois extension of $\mathbb{Q}$. Identify the Galois group $G = \mathrm{Gal}(E/\mathbb{Q})$. Can you classify all intermediate extensions $\mathbb{Q} \subset K \subset E$ with $[K : \mathbb{Q}] = 2$?