Modern Algebra II, fall 2020, Instructor M.Khovanov

Homework 2, due Wednesday September 23. All rings are assumed commutative unless specified otherwise.

1. (10 points) For each of the following statements, either briefly explain why it is true or give a counterexample.

(a) Element $a+b$ of a ring R is invertible if and only if both a and b are invertible.

(b) Any subring of a field is an integral domain.

(c) Ring $\mathbb{Z}/121$ is an integral domain.

(d) Direct products $R_1 \times R_2$ and $R_2 \times R_1$ are isomorphic rings.

(e) If $e_1, e_2 \in R$ are idempotents then $e_1 e_2$ is an idempotent.

2. (10 points) (a) Show that the direct product $R \times S$ of rings R and S is not an integral domain (unless one of these rings is the zero ring $\mathbf{0} = \{0\}$). Hint: look for zero divisors in $R \times S$.

3. (10 points) Find all zero divisors in the following rings:
(a) $\mathbb{Z}$, (b) $\mathbb{Z}/10$, (c) $\mathbb{Z}/13$, (d) $\mathbb{Z}/2 \times \mathbb{Z}/2$, (e) $\mathbb{Q}$.

4. (20 points) (a) (10 points) An element x of a ring R is called *nilpotent* if $x^n = 0$ for some $n > 0$. Note that $0 \in R$ is always nilpotent. Show that if x, y are nilpotent then $x+y$ is nilpotent (assume that $x^n = 0, y^m = 0$, use that the ring is commutative and apply the binomial theorem from lecture 1 to some large power of $x+y$). Also show that if x is nilpotent then ax is nilpotent for any $a \in R$.

(b) (5 points) Show that 0 is the only nilpotent element of an integral domain R.

(c) (5 points) Find all nilpotent elements in the following rings:

$$\mathbb{Z}, \quad \mathbb{Q}, \quad \mathbb{Z}/8, \quad \mathbb{Z}/12, \quad \mathbb{R}[x].$$

(d) (optional) Suppose now that the ring R is noncommutative. Can you give an example of such R and two nilpotent elements x, y such that $x+y$ is not nilpotent?

5. (10 points) (a) Suppose that R is an integral domain and $S \subset R$ is a subring. Show that S is an integral domain.

(b) Suppose that R is an integral domain. Show that any invertible element of $R[x]$ has degree 0 (thus, it is a constant polynomial). Conclude that $(R[x])^* = R^*$ and explain why $R[x]$ is not a field.

6. (10 points) Prove that a ring R is a field if and only if $R \setminus \{0\}$ is an abelian group under multiplication.

7. (10 points) A subfield K of a field F is a subring that is also a field.
(a) Prove that the intersection of any collection of subfields of a field F is a subfield of F.

(b) Prove that the only subfield of $\mathbb{Q}$ is $\mathbb{Q}$ itself (we can say that $\mathbb{Q}$ has no *proper* subfields). Hint: suppose F is a subfield of $\mathbb{Q}$. Then it should contain 1. Keep going to show that F contains all elements of $\mathbb{Q}$.

8. (10 points) Read through the incomplete proof of the Theorem in lecture 3 (pages 5-6) that Frac(R) is a field for an integral domain R.

(a) Solve the exercise on page 6 of the lecture to show that addition is a well-defined operation in Frac(R), that is, does not depend on a choice of coset representatives.

(b) Solve the next exercise on that page: prove that Frac(R) under addition is an abelian group.

9. (optional) Can you give an example of a noncommutative ring with finitely many elements? (Hint: try matrices $M_n(R)$. What are possible options you have for R?)