Modern Algebra II, fall 2020, Instructor M.Khovanov

Homework 3, due Wednesday September 30. All rings are assumed commutative unless specified otherwise.

1. (10 points) Which of the following are ideals?
(a) Subset $R \times 0$ of the direct product $R \times S$ of two rings R, S.
(b) The set of diagonal elements $\Delta = \{(x, x) \ : \ x \in R\}$ in the direct product $R \times R$ of a ring R with itself.
(c) The set $Ra + Rb$, where a, b are elements of a ring R.
(d) The set R^* of invertible elements of a ring R.
(e) Polynomials in $\mathbb{Z}[x]$ with all coefficients even.

2. (15 points) Which of the folowing are homomorphisms? Provide very brief explanations.
(a) $\psi : \mathbb{Z} \longrightarrow \mathbb{Z}$, $\psi(n) = -n$.
(b) $\psi : R \times R \longrightarrow R$, $\psi((a, b)) = a + b$, where R is a ring.
(c) $\psi : R \longrightarrow R \times R$, $\psi(a) = (a, 0)$ for $a \in R$.
(d) $\psi : R \longrightarrow R \times R$, $\psi(a) = (a, a)$ for $a \in R$.
(e) $\psi : \mathbb{Q}[\sqrt{2}] \longrightarrow \mathbb{Q}[\sqrt{2}]$, $\psi(a + b\sqrt{2}) = a - b\sqrt{2}$ for $a, b \in \mathbb{Q}$.
(f) $\psi : F[x] \longrightarrow F[y]$, $\psi(f(x)) = f(y^2)$, for $f(x) \in F[x]$ a polynomial with coefficients in the field F.

3. (5 points) Compute the greatest common divisor of polynomials $x^3 - x^2 - 1$ and $x^2 - x + 1$ in the field $\mathbb{F}_3$. (Hint: first divide one polynomial by the other with the remainder. Keep repeating this procedure until the remainder is 0.)

4. (10 points) Show that the ideal $(2, x)$ in $\mathbb{Z}[x]$ is not principal. (Hint: assume it is principal, generated by a polynomial $f(x)$. Look at what the degree of $f(x)$ can be.) Notice the difference with the corresponding ideal $(2, x)$ of $\mathbb{Q}[x]$. The latter is a principal ideal. Can you find its generator?

5. (15 points) In Lecture 5 page 2 (see online notes) we sketched a proof that an inclusion $\alpha : R \longrightarrow F$ of an integral domain R into a field F extends to an inclusion β of its field of fractions $\mathrm{Frac}(R)$ (another notation is $Q(R)$) into F.

(a) Check that β is indeed well-defined on cosets and gives us a map from the field of fractions $Q(R)$ to F.

(b) Check that β is a ring homomorphism. This completes the proof of the proposition on page 2 of the notes.
Remark: since $Q(R)$ is a field, a homomorphism from $Q(R)$ to a non-zero ring is necessarily injective.

6. (10 points) Consider the evalution homomorphism $\mathrm{ev}_a : R[x] \longrightarrow R$ taking $f(x)$ to $f(a)$, for a fixed $a \in R$. Show that the kernel of ev_a is the principal ideal $(x-a)$ of R.

7. (15 points) (a) Prove that the sum $I + J = \{i + j | i \in I,\ j \in J\}$ of two ideals of R is an ideal of R. Recall our discussion in lecture 5 of sum of ideals and its relation to gcd of polynomials or integers.

(b) Compute the sums and intersections of the following ideals of $\mathbb{Z}$:

$$(2)+(3),\quad (4)+(4),\quad (20)+(15),\quad (3)+(0),\quad (5)\cap(3),\quad (12)\cap(15).$$

(c) Compute the following sums and intersections of ideals in $\mathbb{Q}[x]$. Note that all the ideals of $\mathbb{Q}[x]$ are principal, so for each ideal list the monic polynomial which generates the ideal.

$$(x)+(x+2),\quad (x^2)+(2x),\quad (3x^2+2x)+(4x^2+x),$$
$$(3x^2+x+5)+(0),\quad (x)\cap(2x+1),\quad (2x)\cap(3x^2).$$

8. (optional) Let $e \in R$ be an idempotent.
(a) Check that Re and $R(1-e)$ are ideals of R and that their intersection $Re \cap R(1-e) = 0$. Show that any element $a \in R$ has a unique presentation as a sum of an element in Re and an element in $R(1-e)$.
(b) Prove that Re is a ring, with identity e and addition and multiplication inherited from R. Likewise for $R(1-e)$. Note that Re is not a subring of R, according to our definition; the identity e of Re is not the identity 1 of R. Some textbooks allow such generalized subrings, though.
(c) Show that the map from $Re \times R(1-e)$ to R that takes (a,b) to $a+b$ is an isomorphism of rings.
This exercise tells you that an idempotent in a (commutative) ring allows you to decompose the ring as a direct product.