Modern Algebra II, fall 2020, Instructor M.Khovanov

Homework 4, due Wednesday October 7. All rings are assumed commutative unless specified otherwise.

1. (15 points) Which of the following ideals are (A) prime, (B) maximal?

1abcd) Ideals (0), (10), (7) and (1) in $\mathbb{Z}$.

2) Ideal (0) in $\mathbb{Q}$.

3) Ideal (0) in $\mathbb{Z}[x]$.

4ab) Ideals (x) and (x^2) in $F[x]$. (Here and in later homeworks we use F to denote a field.)

2. (10 points) Describe all ideals of $\mathbb{Z}$ that contain

(a) ideal (3),

(a) ideal (4),

(c) ideal (20).

Don't forget special cases (the ideal itself, etc.) Which of these ideals are prime?

To solve this problem (or similar problems for other rings R) you need to recall the criterion for the inclusion of ideals $(a) \subset (b)$, for $a, b \in R$: element a must be a multiple of b. When R is an integral domain, we additionally have a criterion for the equality of ideals $(a) = (b)$: there is an invertible element $r \in R$ such that $b = ar$.

3. (10 points) (a) We proved that a polynomial of degree d in $F[x]$, where F is a field, has at most d roots in F. Prove that a polynomial of degree d in $R[x]$, where R is an integral domain, has at most d roots in R.

(Hint: embed R into its field of fractions $F = Q(R)$ and work in the larger ring $F[x]$.)

(b) Take the ring $R = \mathbb{Z}/4$ and consider the polynomial $f(x) = 2x$. It is linear, so have degree 1. Check that it has two roots in R. Why is this not a contradiction with part (a)?

4. (15 points) Which of the following polynomials in $\mathbb{F}_3$ are irreducible?

$$2, \; x+1, \; 2x^2 - x + 1, \; x^2 + x + 1, \; x^3 - x + 1, \; 2x^3 - x + 1.$$

For polynomials that are reducible, write down their factorizations into irreducible polynomials. Recall that a polynomial of degree 2 or 3 is irreducible over a field F iff it does not have a root in F.

5. (15 points) Which of the following rings are fields? Use the theorem that describes when $F[x]/(f(x))$ is a field.

$$\mathbb{R}[x]/(x^2-2),\ \mathbb{R}[x]/(x^2+2x+2),\ \mathbb{Q}[x]/(x^2+x),$$
$$\mathbb{Q}[x]/(x^2+x-6),\ \mathbb{C}[x]/(x^2+9),\ \mathbb{F}_2[x]/(x^3),\ \mathbb{F}_3[x]/(x-2).$$

6. (15 points) In lecture 6 (see notes, page 2) we looked at the the 8-element field $F = F_2[x]/(x^3+x+1)$. Consider instead a different quotient ring $R = F_2[x]/(x^3+x^2+1)$ (note that the polynomial is different from the one in the lecture).
(a) Check that the polynomial x^3+x^2+1 is irreducible over $\mathbb{F}_2$. Conclude that R is a field. Check that R is finite. How many elements does it have?
(b) Write down all elements of R. Check directly that the group R^* is cyclic, similar to what we showed in the lecture for F. Find the inverse of each element of R^*.
(c) (optional). Show that fields F and R are isomorphic. You'd need to find an element of R that behaves like the generator x of F. It may help to relabel x in R to y for convenience.

7. (10 points) (a) Prove that the zero ideal in a ring R is a prime ideal if and only if R is a domain.
(b) If F is a field and $a \in F$, prove that the kernel of the evaluation map $\mathrm{ev}_a : F[x] \longrightarrow F$ is a maximal ideal in $F[x]$. (Hint: find a generator for this ideal first.)

Additional problems to look at to prepare for the exam: Rotman, problems 10, 12, 15, 16, 18, 21 (pages 16-17), problems 23, 25, 26, 29i, 32, 33 (pages 19-20), problem 38 (page 23), problems 50, 53ii (pages 37-38). That's a long list of problems; don't feel obliged to go through all of them. There'll also be more concrete problems on the exam similar to homework problems: manipulate ideals and polynomials, determine whether a subset of a ring is an ideal or a subring, determine whether a map is a homomorphism, which of the following rings are PIDs or fields, etc.