Modern Algebra II, fall 2020, Instructor M.Khovanov

Homework 6, due Wednesday October 21.

1. (20 points) Which of the following extensions have finite degree? Which are algebraic?
(a) $\mathbb{R} \subset \mathbb{C}$.
(b) $\mathbb{F}_p \subset F$, where F is a finite field.
(c) $\mathbb{C} \subset \mathbb{C}(t)$, where $\mathbb{C}(t)$ is the field of rational functions in a variable t with complex coefficients.
(d) $\mathbb{Q} \subset \mathbb{R}$.
(e) $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}, \sqrt[3]{3}, \dots, \sqrt[p]{p}, ...)$, union over all primes p.
(f) $\mathbb{Q}(\sqrt[3]{3})/\mathbb{Q}$.
(f) $F(t^2) \subset F(t)$, where t is a formal variable and $F(t^2)$ is the subfield of $F(t)$ that consists of rational functions in t^2 (explain why it is a subfield).

2. (10 points) Suppose we are given field extensions E/B and B/F and the degree $[E : F]$ is a prime number. Show that either $B = E$ or $B = F$.

3. (15 points) Suppose p and q are distinct prime numbers. Let $\alpha = \sqrt{p} + \sqrt{q}$. Generalize arguments done in class for $p = 2$ and $q = 3$ to show that $\mathbb{Q}(\alpha)/\mathbb{Q}$ is a degree 4 extension that contains $\sqrt{p}$ and $\sqrt{q}$. Write down the irreducible polynomial for α over $\mathbb{Q}$. (You can use that $\sqrt{q} \notin \mathbb{Q}[\sqrt{p}]$ and vice versa.)

4. (15 points) Determine the irreducible polynomial $\mathrm{irr}(\alpha, F)$ for $\alpha = \sqrt{2} + \sqrt{5}$ and the following fields F:
(a) $\mathbb{Q}$, (b) $\mathbb{Q}[\sqrt{2}]$, (c) $\mathbb{Q}[\sqrt{5}]$, (d) $\mathbb{Q}[\sqrt{2}, \sqrt{5}]$, (e) $\mathbb{R}$.

5. (10 points) Prove that an n-th root $\sqrt[n]{\alpha}$ of an algebraic number $\alpha \in \mathbb{C}$ is algebraic. (By an n-th root of a complex number z we mean any complex number w such that $w^n = z$. If $z \neq 0$, there are n such numbers. A complex number is *algebraic* if it's a root of a polynomial $x^m + a_{m-1}x^{m-1} + \cdots + a_0$ with rational coefficients.)

6. (15 points) Consider the field $\mathbb{F}_8 = \mathbb{F}_2[\theta]/(\theta^3 + \theta + 1)$. Check that the polynomial $x^3 + x + 1$ splits in this field as

$$x^3 + x + 1 = (x + \theta)(x + \theta^2)(x + \theta^2 + \theta).$$

Now take the other degree three irreducible polynomial $x^3 + x^2 + 1$ over $\mathbb{F}_2$. Show that it also splits in $\mathbb{F}_8$ and find its roots in $\mathbb{F}_8$.

7. (15 points) (a) Prove that the sum of an algebraic and a transcendental complex number is always transcendental.
(b) Give an example of two transcendental complex numbers α, β such that $\alpha\beta$ is algebraic. You can assume that π is transcendental.

8. (10 points) (a) Prove the product rule $D(fg) = D(f)g + fD(g)$ for the formal derivative $D : F[x] \longrightarrow F[x]$ stated in class.

(b) Derive the power rule for $D(f^n)$ by induction on n.