Modern algebra II, Midterm exam 1, October 5, 2020.

The exam is a closed book, no internet use exam. Make sure your workspace is visible to me via Zoom. Please write your name on each page of your solutions. You can solve problems in any order. At the end of the exam photograph your solutions, turn your photos into a pdf file and deposit it via Courseworks.

All rings are assumed commutative unless otherwise specified.

1. (20 points) (a) State the definition of an ideal in a ring.
(b) Prove that the sum $I + J = \{i + j \; : \; i \in I, j \in J\}$ of ideals I, J of R is an ideal of R.
(c) Which of the following are ideals I in a ring R? (Provide brief explanations.)
(1) $R = \mathbb{Z}$, set I consists of all even integers.
(2) $R = \mathbb{Z}[x]$, set I consists of polynomials $f(x)$ with the constant and the linear coefficients zero, so that $f(x)$ has the form $a_2x^2 + a_3x^3 + \cdots + a_nx^n$ for some n and $a_2, \ldots, a_n \in R$. The zero polynomial is included in the set.
(3) Set $I = \{1, 2, 3\}$ in the ring $R = \mathbb{Z}/4$ of residues modulo 4.
(4) Set $I = \mathbb{Z}/2$ of constant polynomials in the ring $\mathbb{Z}/2[x]$.

2. (20 points) (a) State the definition of the kernel $\ker(\alpha)$ of a ring homomorphism $\alpha : R \longrightarrow S$ and prove that $\ker(\alpha)$ is an ideal in R.
(b) Determine whether the map $\alpha : R[x] \longrightarrow R[x]$ taking $f(x)$ to $f(2x)$ is a ring homomorphism for all R.

3. (20 points) (a) Compute the greatest common divisor of polynomials $f(x) = 2x^3 - x^2 + 3$ and $g(x) = x^2 + 4$ over the field $\mathbb{F}_5 = \mathbb{Z}/5$ of residues modulo 5. Is the greatest common divisor an irreducible polynomial?
(b) Which of the following polynomials over $\mathbb{F}_2$ are irreducible? Briefly explain.

$$x - 1, \quad x^2 + 1, \quad x^2 - x + 1, \quad x^3 + 1, \quad 2x^2 + 2x + 1. \tag{1}$$

4. (20 points) (a) Prove that any subring S of an integral domain R is an integral domain.
(b) Explain why the direct product $R \times S$ of two integral domains is not an integral domain.
(c) Which of the following rings are integral domains?

$$\mathbb{Z}, \quad \mathbb{Z}/10, \quad \mathbb{Z}/19, \quad \mathbb{Q}[x]/(x+1), \quad \mathbb{Z}[x]/(x^2), \quad \mathbb{R}[x]/(x^2+1).$$

5. (10 points) (a) State the definition of maximal ideal.
(b) Give an example of a prime ideal that is not maximal.

Extra credit: Give an example of a ring R and ideals I, J such that $R/(I+J)$ is a field, but R/I is not an integral domain.