Modern algebra II, Midterm exam 1, October 5, 2020.

The exam is a closed book, no internet use exam. Make sure your workspace is visible to me via Zoom. Please write your name on each page of your solutions. You can solve problems in any order. At the end of the exam photograph your solutions, turn your photos into a pdf file and deposit it via Courseworks.

All rings are assumed commutative unless otherwise specified.

1. (20 points) (a) State the definition of an ideal in a ring.

(b) Prove that the intersection $I \cap J = \{i \ : \ i \in I, \text{ and } i \in J\}$ of ideals I, J of R is an ideal of R.

(c) Which of the following are ideals I in a ring R? (Provide brief explanations.)

(1) $R = \mathbb{Z}$, set I consists of all integers divisible by 20.

(2) Set $I = \{0, 2, 4\}$ in the ring $R = \mathbb{Z}/8$ of residues modulo 8.

(3) Set $I = \{a + bx | a, b \in \mathbb{Q}\}$ of constant and linear polynomials in the ring $\mathbb{Q}[x]$.

2. (20 points) (a) State the definition of the image $\mathrm{im}(\alpha)$ of a ring homomorphism $\alpha : R \longrightarrow S$ and prove that $\mathrm{im}(\alpha)$ is a subring in S. Can any subring in S be realized as the image of some homomorphism?

(b) Determine whether the map $\alpha : R[x] \longrightarrow R[x]$ taking $f(x)$ to $2f(x)$ is a ring homomorphism.

3. (20 points) (a) Compute the greatest common divisor of polynomials $f(x) = x^3 - 2x^2 + x + 1$ and $g(x) = x^2 + 2$ over the field $\mathbb{F}_3 = \mathbb{Z}/3$ of residues modulo 3. Is the greatest common divisor an irreducible polynomial? What are the irreducible factors of $f(x)$ and $g(x)$?

(b) Which of the following polynomials over $\mathbb{F}_5$ are irreducible? Briefly explain.

$$x + 2, \quad x^2 + 2x + 2, \quad x^3 + 2x, \quad x^2 + x + 1.$$

4. (20 points) (a) Prove that a homomorphism $\alpha : F \longrightarrow R$ from a field F into a nonzero ring R is injective. Use the theorem that the kernel of a homomorphism is an ideal.

(c) Which of the following rings are fields? Which are integral domains?

$$\mathbb{Z}[x], \ \ \mathbb{F}_2[x], \ \ \mathbb{Z}/23, \ \ \mathbb{Q}[x]/(x-1), \ \ \mathbb{Q}[x]/(x^3), \ \ \mathbb{R}[x]/(x^2+1), \ \ \mathbb{C}[x]/(x^2+1).$$

5. (10 points) (a) State the definition of a prime ideal.

(b) Give an example of two prime ideals I, J in a ring R such that their intersection $I \cap J$ is not prime. Then compute the sum $I + J$ of your prime ideals.

Extra credit: Give a construction of a 9-element field similar to the construction of an 8-element field done in class. Hint: start with field $\mathbb{F}_3$ and quotient the polynomial ring $\mathbb{F}_3[x]$ by a suitable principal ideal.