

Modern algebra II, Review problems for the final exam, part 1.

First few problems are related to the material covered in this week's lectures. When preparing for the final, go over the material in Rotman except for the few sections we did not cover: Independence of characters (p.76-79), Galois Theorem (we followed exposition in Friedman, so did not phrase the subgroups-subfields correspondence via lattices as in Rotman), skip pages 90-92, Galois groups of quartics (pages 103-105, we mentioned them only briefly). Please work through Friedman's notes as well.

1. (a) Take the formula $\Delta = -4p^3 - 27q^2$ for the discriminant of $f(x) = x^3 + px + q$. Set $q = 0$ and check that the coefficient of p^3 is indeed -4 by explicitly writing down all roots of $f(x)$ in \mathbb{C} and computing Δ via the product of differences of roots.

(b) Suppose $f(x) \in \mathbb{Q}[x]$ is an irreducible polynomial of degree 3 which has only one real root. Show that the Galois group $Gal(E/\mathbb{Q})$ of its splitting field over \mathbb{Q} is S_3 .

(c) Work through the proof of Theorem 104 (ii) in Rotman (page 101) for the structure of the Galois group when f has three real roots. See Example 38 (Rotman, p.102) for Galois groups of several degree three polynomials.

2. Exercises 100, 101, 102 in Rotman (page 100). Note how complicated the discriminant becomes (exercise 102 ii) if the coefficient a at x^2 is nonzero. Check that all terms in the expression for D are homogeneous (in terms of degrees, as we've discussed). Note that Rotman's D is Friedman's Δ .

3. Determine Galois groups of splitting fields over \mathbb{Q} of the following degree 3 polynomials

$$x^3 - 3x^2 - x - 1, \quad x^3 - x^2 + x - 1, \quad x^3 + 27x - 4, \quad x^3 - 21x + 7$$

4. Determine the Galois group over \mathbb{Q} of the polynomial $x^5 - 12x^2 + 2$.

5. Let $f(x), g(x) \in \mathbb{Q}[x]$ be two irreducible polynomials, each of degree two. Form the splitting field E of the product polynomial $f(x)g(x)$. Describe possible Galois groups $Gal(E/\mathbb{Q})$. Try to generalize your result to the case of a product of n quadratic polynomials and the corresponding splitting field.

6. Show that the Galois group $Gal(\mathbb{R}/\mathbb{Q})$ is trivial (or go through the proof, for instance, in <https://math.stackexchange.com/questions/555661/how-to-get-the-galois-gr>

7. (a) Explain why the angle 25° cannot be constructed using a ruler and compass.

(b) Construct a regular pentagon using a ruler and compass. What field extensions of \mathbb{Q} are involved in its construction?

(c) Given a regular n -gon and a regular m -gon such that n and m are coprime, show that you can create a regular nm -gon using only a ruler and compass.

8. (a) Let G be the Galois group of a degree n polynomial in $F[x]$. Explain why $|G|$ divides $n!$.

(b) What are possible Galois groups of a degree three polynomial $f(x) \in F[x]$? Include the case when $f(x)$ is reducible in your answer.

9. (a) Give an example of a normal inseparable field extension.

(b) Give an example of a finite degree extension E/F which contains infinitely many intermediate subfields (necessarily an inseparable extension).

10. Write down a presentation for the field \mathbb{F}_{27} (similar to what we did for \mathbb{F}_8 and \mathbb{F}_{16} .) What are the subfields of \mathbb{F}_{27} ? Count the orbits of the Galois group action $Gal(\mathbb{F}_{27}/\mathbb{F}_3)$ (you do not need to write them down explicitly).