Modern Algebra I, Spring 2017
Homework 1, due Wednesday January 25 before class. Please deposit the homework in the box on the 4th floor labelled "Modern Algebra I".

Read Chapter 1 of the textbook (Judson, Abstract Algebra). Additionally, I recommend reading Chapter 4 in P. Gallagher's notes (Algebra of sets, Algebra of maps). Write solutions to the following problems.

1. List all subsets of the 3-element set $A = \{1, 2, 3\}$. How many subsets does a set with n elements have? How many of these subsets have at most two elements?

2. Simplify descriptions of the following sets. These sets depend on subsets A, B of a universal set X, so that $A' = X \setminus A$, and so on.

(a) $A' \cup A$, $A' \cap A$, $(A' \cup A')' \cup (A' \cap A)$.

(b) $(A \cap B) \cup (A \cup B)$, $(A \cup B') \cap (A' \cap B)$,
$(A \cap B) \setminus B$, $(A \cup B) \setminus B$, $(A \cap B) \cup (A \setminus B)$.

3. Prove $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

4. (a) Consider sets $A = \{a, b\}$ and $B = \{1, 2, 3\}$. How many injective maps are there from A to B? Give an example of such a map. How many injective maps are there from B to A?

(b) Suppose $f : A \longrightarrow A$ is injective and A is a finite set. Prove that f is bijective. Give an example of an infinite set B and an injective map $f : B \longrightarrow B$ which is not surjective.

5. (a) Show that, for any set A, there is exactly one map f from the empty set $\emptyset$ to A. When is f injective? Surjective?

(b) Describe all maps from a set A to the empty set $\emptyset$ (the answer will depend on A).

6. We saw in class that, for sets A and B,

$$|A \cup B| = |A| + |B| - |A \cap B| \tag{1}$$

Prove that, for sets A, B, C,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

(Hint: For instance, try using equation (1) with sets $A, B \cup C$ instead of A, B.)

7. Given maps $f : A \longrightarrow B$ and $g : B \longrightarrow C$ such that gf is surjective, prove that g is surjective. Give an example with surjective gf but not surjective f.