Modern Algebra I, Spring 2017

Homework 9, due Wednesday March 29 before class.

Read Gallagher §14 (Characters of finite abelian groups) and the first two pages of §15.

1. Write down a proof that, for a group G and an abelian group H, the set of all homomorphisms $\mathrm{Hom}(G, H)$ from G to H is an abelian group. (Hint: you'll need to explain why the product of homomorphisms into an abelian group is a homomorphism, associativity, and what the identity homomorphism and the inverse homomorphism are. Most of this was done in class.)

2. (a) Write down the character table of the cyclic group C_3, denoting a generator of C_3 by h, $C_3 = \{1, h, h^2\}$. Label the characters $\psi_0, \psi_1, ...$, with ψ_0 being the trivial character.
(b) Do the same for the cyclic group C_6, denoting a generator by g, so that $C_6 = \{1, g, g^2, \dots, g^5\}$. Label the characters $\chi_0, \chi_1, \dots$, with χ_0 being the trivial character.
(c) Now view C_3 as a subgroup of C_6 by taking $h = g^2$. For each character of C_3 list the characters of C_6 that extend it.

3. Recall that $\widehat{G}$, for a finite abelian group G, denotes the character group of G. We proved that $|\widehat{G}| = |G|$. Show that, if H and K are finite abelian groups, then

$$\widehat{H \times K} \cong \widehat{H} \times \widehat{K}$$

For this, given characters $\psi \in \widehat{H}$ and $\phi \in \widehat{K}$, check that

$$\psi \times \phi \ : \ H \times K \longrightarrow \mathbb{T}$$

given by

$$(\psi \times \phi)(h, k) = \psi(h)\phi(k)$$

is a character, thus $\psi \times \phi \in \widehat{H \times K}$, and that the map sending ψ, ϕ to $\psi \times \phi$ is an isomorphism from $\widehat{H} \times \widehat{K}$ to $\widehat{H \times K}$. (Surjectivity follows from injectivity for this map, why?)

4. Write down the character table of the group $C_2 \times C_2$ (recall that this also the Klein four group V_4). You can either use the previous problem to classify all characters, or just directly find all homomorphisms from this group to the circle group $\mathbb{T}$.

If you'd like additional practice with character tables, write them down for cyclic groups C_4 and C_5. Another problem is to show that the character group $\widehat{Z}$ of $\mathbb{Z}$ is isomorphic to $\mathbb{T}$.For these questions do not turn your solutions in.

We talked about the center $Z(G)$ of a group G in class. Judson delegates group centers to exercises.

5. (a) Write down proofs that the center of the quaternion group Q_8 is $\{1, -1\}$ and the center of the symmetric group S_n is trivial for $n \geq 3$. What is the center of S_2?

(b) Look at the dihedral group D_4 of all symmetries of a square. Can you determine the center of D_4?

6. Look at the groups Q_8 and D_4. These are both nonabelian groups of order 8. Can you show that these groups are not isomorphic?