Problem 1. Prove that for each $c \in \mathbb{N}$ and each prime p, $N_a(p^c)$ is equal to the number of (finite) sequences of positive integers $b_1, \ldots, b_r$ for which

$$b_1 \geq \ldots, \geq b_r$$

and $c = b_1 + \ldots + b_r$

Let S_c be the set of sequences (b_i) as above and let $A(p^c)$ be the set of isomorphism classes of abelian groups of order p^c. Define the map $\Phi : S_c \to A(p^c)$ by taking $(b_1, \ldots, b_r)$ to the isomorphism class of $\prod_{i=1}^r C_{p^{b_i}}$. It's well-defined since the order of this product is $\prod_{i=1}^r p^{b_i} = p^{\sum_{i=1}^r b_i} = p^c$. We check that this map is a bijection.

Surjectivity: Let G be an arbitrary abelian group of order p^c. By the classification of finite abelian groups, we have that G is isomorphic to a product of cyclic groups of prime power order, say $G \cong \prod_{i=1}^n C_{p^{m_i}}$. Comparing orders on both sides, we find $c = \sum_{i=1}^m m_i$. Rearrange the m_i to get a decreasing sequence $(b_i)_{i=1}^n$ which is an element of S_c. We see that $\Phi(b_1 \ldots, b_n)$ is the isomorphism class of G since

$$G \cong \prod_{i=1}^n C_{p^{m_i}} \cong \prod_{i=1}^n C_{p^{b_i}}$$

(the order of the factors is permuted.)

Injectivity: Suppose $\Phi(b_1, \ldots, b_r) = \Phi(b'_1, \ldots, b'_s)$ for two sequences $b_1, \ldots, b_r$ and $b'_1, \ldots, b'_s$ in S_c. This means

$$\prod_{i=1}^r C_{p^{b_i}} \cong \prod_{i=1}^s C_{p^{b'_s}}.$$

Theorem C in Gallagher's notes (section 15) tells us that in such a situation, the groups $C_{p^{b_i}}$ and $C_{p^{b'_i}}$ are the same up to reordering since the number of each C_{p^a} is determined by G and p^c. So, the sequences $b_1, \ldots, b_r$ and $b'_1, \ldots, b'_s$ are the same up to reordering. But since both sequences are decreasing, this tells us that they must be equal, as required.

Since Φ is a bijection, its domain and codomain have the same number of elements. Thus, the number of such sequences (b_i) is equal to $|A(p^c)| = N_a(p^c)$.

Problem 2. True or false:

1. Any two groups of order 4 are isomorphic.
2. Any two groups of order 5 are isomorphic.
3. Any two abelian groups of order 6 are isomorphic.
4. Any two groups of order 6 are isomorphic.

1. False: C_4 and $C_2 \times C_2$ both have order 4 but are non-isomorphic since C_4 has elements of order 4 but $C_2 \times C_2$ doesn't.

2. True: let G be a group of order 5 and $g \neq 1$ an element of G. By Lagrange's theorem, the cyclic subgroup generated by g must have order 5 and hence g is a generator; in particular G is cyclic. There is only one cyclic group of order 5 up to isomorphism. (The same exact argument of course works if 5 is replaced by any prime p)

3. True: by the classification theorem, an abelian group of order 6 is isomorphic to a product of prime power-order cyclic groups of order dividing 6. The only factorization of 6 into prime powers is $6 = 2 \cdot 3$ so the only abelian group of order 6 is $C_2 \times C_3$.

4. False: C_6 and S_3 are not isomorphic since C_6 is abelian but S_3 isn't.

Problem 3.

For each n, we will find all factorizations of n into prime powers, up to equivalence (re-ordering of factors). The isomorphism classes of abelian groups of order n will be in natural bijection with these factorizations by the second problem.

(a) Order 16: The factorizations $16 = 2^4 = 2 \cdot 2^3 = 2^2 \cdot 2 \cdot 2 = 2^2 \cdot 2^2$ give the following groups of order 16:

C_{16}

$C_2 \times C_2 \times C_2 \times C_2$

$C_2 \times C_8$

$C_4 \times C_2 \times C_2$

$C_4 \times C_4$

(b) Order 27: The factorizations $27 = 3^3 = 3^2 \cdot 3$ give the groups:

C_{27}

$C_9 \times C_3$

$C_3 \times C_3 \times C_3$.

(c) Order 10: The only factorization into primes is $10 = 2 \cdot 5$, so the only abelian group of order 10 (up to isomorphism) is

$C_2 \times C_5$

(d) Order 40: The factorizations $5 \cdot 2^3 = 5 \cdot 2 \cdot 2^2 = 5 \cdot 2 \cdot 2 \cdot 2$ give the groups:

$C_5 \times C_8$

$C_5 \times C_2 \times C_4$

$C_5 \times C_2 \times C_2 \times C_2$

(e) Order 100: The factorizations $2^2 \cdot 5^2 = 2 \cdot 2 \cdot 5^2 = 2^2 \cdot 5 \cdot 5 = 2 \cdot 2 \cdot 5 \cdot$ give the groups:

$$C_4 \times C_{25}$$
$$C_2 \times C_2 \times C_{25}$$
$$C_4 \times C_5 \times C_5$$
$$C_2 \times C_2 \times C_5 \times C_5.$$

(f) The factorization $49 = 7^2$ gives the groups

$$C_{49}$$
$$C_7 \times C_7.$$

Problem 4. Find a natural $n > 30$ such that there is only isomorphism class of abelian groups of order n and also for $n + 1$.

We claim $n = 33$ works. The only factorizations into prime powers (up to order) of 33 and $n + 1 = 34$ are $33 = 11 \cdot 3$ and $34 = 17 \cdot 2$ so the only groups of these orders up to isomorphism are $C_{11} \times C_3$ and $C_{17} \times C_2$. In general, n will work if and only if both n and $n + 1$ are square-free (so that they have only one factorization into prime powers up to order).