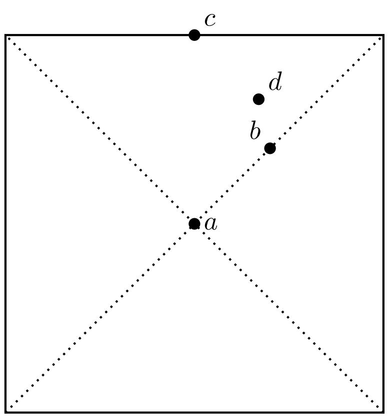
Ryan Abbott
UNI: rwa2110
Modern Algebra 1
Homework 11

1. Since the identity in S_X is the permutation which sends each element of X to itself, we have $\mathrm{id}_{S_X}(x) = x$ for all $x \in X$. Therefore to find the kernel of π we must find the elements of g such that $gx = x$ for all $x \in X$. The statement that $gx = x$ is equivalent to $g \in G_x$, so we have $g \in G_x$ for all $x \in X$, or equivalently $g \in \cap_{x \in X} G_x$. Putting this together we have $g \in \ker \pi \Leftrightarrow g \in \bigcap_{x \in X} G_x$, or

$$\ker \pi = \bigcap_{x \in X} G_x$$

2.



Point	Size of Orbit	Stabilizer	Size of Stabilizer
a	1	D_4	8
b	4	$\langle sr \rangle$	2
c	4	$\langle s \rangle$	2
d	8	$\{1\}$	1

The rotation subgroup C_4 cannot be realized as the stabilizer of a point on the square, since an individual rotation r fixes only one point (labeled a in the diagram above), but the stabilizer of that point is all of D_4.

3. (a) In S_4, conjugate elements are permutations with the same cycle structure. Therefore we may split up according to cycle structure:

   i. e
   ii. $(12), (13), (14), (23), (24), (34)$
   iii. $(123), (132), (124), (142), (134), (143), (234), (342)$
   iv. $(12)(34), (13)(24), (14)(23)$
   v. $(1234), (1243), (1324), (1342), (1423), (1432)$

   Counting the number of elements in each conjugacy class we have the class equation:

$$|S_4| = 24 = 1 + 6 + 8 + 3 + 6$$

   (b) For a generic element of D_5 we may take $s^a r^b$, where $a \in \mathbb{Z}_2$ and $b \in \mathbb{Z}_5$.
   In the case that $a = 1$ we may conjugate by r^2:

$$r^2(sr^b)r^{-2} = r^2 sr^{b-2} = sr^{-2}r^{b-2} = sr^{b-4} = sr^{b+1}$$

Where in the last step we have used $-4 = 1 \mod (5)$ to simplify. This results shows that all element of the type sr^b are of the same conjugacy class, as they may all be reached by repeatedly conjugating by r^2.

In the case that $a = 0$ we may conjugate by a generic $s^\alpha r^\beta$:

$$s^\alpha r^\beta (r^b) r^{-\beta} s^{-\alpha} = s^\alpha r^{\beta+b-\beta} s^{-\alpha} = s^\alpha r^b s^{-\alpha}$$

$$s^\alpha r^b s^{-\alpha} = \begin{cases} r^b & \alpha = 0 \\ r^{-b} & \alpha = 1 \end{cases}$$

Therefore two elements $r^{b_1}, r^{b_2} \in D_5$ are conjugate if and only if $b_1 = \pm b_2 \mod (5)$. This gives three conjugacy classes: r and r^4, r^2 and r^3, and 1.

Putting all this together we have 4 conjugacy classes:

i. 1
ii. r, r^4
iii. $r^2, 4^3$
iv. s, sr, sr^2, sr^3, sr^4

Counting the number of elements in each conjugacy class we have the class equation:

$$|D_5| = 10 = 1 + 2 + 2 + 5$$

(c) Since $\mathbb{Z}_6$ is abelian, every element is its own conjugacy class:

i. 0
ii. 1
iii. 2
iv. 3
v. 4
vi. 5

This gives the class equation:

$$|\mathbb{Z}_6| = 6 = 1 + 1 + 1 + 1 + 1 + 1$$

(d) From earlier work we see that $ji(-j) = ki(-k) = -i$, so $\{i, -i\}$ is a conjugacy class. Similarly $\{j, -j\}$ and $\{k, -k\}$ are conjugacy classes as well. Meanwhile 1 and -1 are in the center of Q_8, so they are within their own conjugacy classes. Putting this together we have:

i. 1
ii. -1
iii. i, -i
iv. j, -j
v. k, -k

And for the class equation we have

$$|Q_8| = 8 = 2 + 2 + 2 + 1 + 1$$

4. (a) The orbits of the action may be uniquely identified by the element r of the orbit is in $[0, 1)$. The orbit associated with a particular r is $r + \mathbb{Z} = \{r + n \mid n \in \mathbb{Z}\}$.

For every $x \in \mathbb{R}$, the stabilizer $\mathbb{Z}_x$ is the set of elements n such that $x + n = x$, or $n = 0$. Therefore we have $\mathbb{Z}_x = \{0\}$ for all $x \in \mathbb{R}$.

(b) The orbit of a given element $z \in \mathbb{C}$ is given by the circle in the complex plane centered about 0 with radius $|z|$. Any other element w of equal magnitude separated by angle θ from z can be obtained by multiplying by $e^{i\theta'} \in \mathbb{T}$, where $\theta' = \theta \mod (2\pi)$. Therefore all orbits consist of concentric circles about the origin in the complex plane.

The stabilizer of 0 is all of $\mathbb{T}$, since $0e^{i\psi} = 0$ for all ψ.

For an arbitrary $z \in \mathbb{C}$, $z \neq 0$, we may describe the stabilizer of z as the set of elements $e^{i\psi}$ such that $ze^{i\psi} = z$, or equivalently, $e^{i\psi} = 1$. This occurs when $\psi = 2\pi n$, $n \in \mathbb{N}$, which corresponds only to the element $1 \in \mathbb{T}$. Therefore the stabilizer for any element other that 0 in $\mathbb{C}$ is $\{1\}$.

5. For any orbit, we know the order of the orbit must divide the order of the group. Therefore the number of elements in the orbit must divide $|C_p| = p$. Since p is prime, this means there must either be 1 or p elements in the orbit.

6. (a) To check that this operation is a group action, we must check that $e(i,j) = (i,j)$ for all i,j, and associativity, i.e. that $(\sigma\tau)(i,j) = \sigma(\tau(i,j))$.

   For identity we have
$$e(i,j) = (e(i), e(j)) = (i,j)$$
   For associativity we have
$$(\sigma\tau)(i,j) = ((\sigma\tau)(i), (\sigma\tau)(j)) = (\sigma(\tau(i)), \sigma(\tau(j))) = \sigma(\tau(i,j))$$
   where we have used the associativity of function composition.

   (b) The two orbits are $O_0 = \{(i,j) \in J \times J \mid i = j\}$ and $O_1 = \{(i,j) \in J \times J \mid i \neq j\}$. To see that these are disjoint note that by the injectivity of $\sigma \in S_n$ we have $\sigma(i) = \sigma(j) \Leftrightarrow i = j$, so no application of the group action can move between the sets.

   To see that these are complete orbits (i.e. they aren't composed of smaller orbits), we must take two arbitrary elements in each orbit and map one to the other.

   For O_0 we take $(i,i) \in J \times J$, which we must map to $(j,j) \in J \times J$. We see that the permutation $\sigma = (ij)$ does the trick, so (i,i) and (j,j) are in the same orbit.

   For O_1, by symmetry of components we need only show that we may arbitrarily set one of the components. Therefore we take $(i,j) \in O_1$ and $(i,k) \in O_1$ and must find σ which sends (i,j) to (j,k). Taking the permutation $\sigma = (jk)$, we see that since $i \neq j$ and $i \neq k$ there will be no effect on i. Meanwhile j and k will swap (or do nothing if $j = k$), so σ will exactly send (i,j) to (j,k) and vice versa.

   (c) For O_0, we select the element $p = (1,1)$. The stabilizer for p is the set of all permutations in S_n which fix 1, which isomorphic to S_{n-1}.

   For O_1, we select the element $p = (1,2)$. The stabilizer for p is the set of all permutations which fix 1 and 2, which is isomorphic to S_{n-2}.

7. To check that this is an action we must check the action of the identity and associativity.

   For identity we have $1A = \{1a \mid a \in A\} = \{a \mid a \in A\} = A$, so the identity checks out.

   For associativity we may apply associativity of the underlying group action:
$$(gh)A = \{(gh)a \mid a \in A\} = \{g(ha) \mid a \in A\} = g\{ha \mid a \in A\} = g(hA)$$
   The stabilizer of the empty set is the whole group, since acting on the empty set will always produce the empty set.

   The stabilizer of an singleton $\{x\}$ is the same as the stabilizer of x under the original group action.