Modern Algebra I HW 12 Solutions

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Problem 1.

Fact: If $p < q$ are primes and q is not congruent to 1 modulo p, then the only group of order pq up to isomorphism is C_{qp}. This is 15.10 in Judson.

(a) By the above fact, the only group of order $35 = 5 \cdot 7$ up to isomorphism is C_{35}.

(b) 59 is prime so the only group of order 59 up to isomorphism is C_{59} by Lagrange's theorem.

(c) The fact above shows that the only group of order $77 = 7 \cdot 11$ up to isomorphism is C_{77}.

(d) We factor $26 = 2 \cdot 13$. The only abelian group of order 26 up to isomorphism is C_{26}. Suppose now that our group G is non-abelian. Let P be a 13-Sylow subgroup and $Q = \{1, g\}$ a 2-Sylow subgroup. Since P has prime order, it is cyclic, say with generator x. Note that P is normal in G, since $[G : P] = 2$. Consider the element $gx \in G$; it isn't an element of P since $g \notin P$, and in particular $gx \neq 1$. Also, gx can't have order 13 since P is the only subgroup of G of order 13 (since it's a normal Sylow 13-subgroup), nor can it have order 26 since we assumed G was not abelian (and in particular not cyclic). By process of elimination, we have found $(gx)^2 = 1$. Thus, G is a group of order 26 with generators x, g satisfying the relations $x^{13} = g^2 = gxgx = 1$. These relations should be familiar from our work with the dihedral group; r, s in the usual presentation of D_{13} satisfy analogous relations and this defines D_{13}. These relations tell us that there is a well-defined homomorphism $D_{13} \to G$ taking r to x and s to g, which is necessarily surjective. But a surjective map between finite sets of the same size is a bijection, so we conclude that there is an isomorphism between D_{13} and G.

Remark 1. *For $p < q$ primes and $q \equiv 1 \mod q$, there is exactly one non-abelian group of order pq up to isomorphism. This is most easily seen using the concept of the semi-direct product, to be introduced later in the class.*

(e) Let G be a group of order 325. We factor $325 = 5^2 \cdot 13$. The number of Sylow 5-subgroups is 1 modulo 5 and divides 13. The only natural number satisfying these constraints is 1, so there is exactly one order 25 subgroup of G, say P. Similarly, the number of Sylow 13-subgroups is 1 modulo 13 and divides 25, so it must be 1- call the Sylow 13-subgroup Q. Now, P and Q are normal subgroups of G that intersect trivially (because they have coprime orders) and by comparing orders, we find $G = PQ$. So, G is an internal direct product of P and Q. We have $Q \cong C_{13}$ and P can be isomorphic to either $C_5 \times C_5$ or C_{25}. So, up to isomorphism, the only groups of order 325 are $C_{13} \times C_{25}$ and $C_{13} \times C_5 \times C_5$.

Problem 2. If H is a normal subgroup of a finite group G and $|H| = p^k$, show that H is contained in every Sylow p-subgroup of G

By Sylow's theorems, H is contained in some p-Sylow subgroup P. Let P' be any other p-Sylow subgroup. By Sylow's theorems again, P' is a conjugate of P, say $P' = gPg^{-1}$. But $H \subset P$ implies $gHg^{-1} \subset P'$. Since H is normal, we have $gHg^{-1} = H$ and hence $H \subset P'$, as required.

Problem 3. What are the orders of Sylow p-subgroups of A_4 for $p = 2, 3, 5$? For each of these p, give an example of a Sylow p-subgroup of A_4. Which of your examples are normal subgroups of A_4?

We have $|A_4| = 12 = 2^2 \cdot 3$; the highest powers of $2, 3, 5$ that divide 12 are 4, 3, and 1 respectively, so these are the orders of the Sylow p-subgroups.

- $p = 2$: From an earlier homework, we know that the only order 4 subgroup of A_4 is $H = \{1, (12)(34), (13)(24), (14)(23)\}$. Since any conjugate of H is a subgroup of order 4 and hence equal to H, we see that H is normal.
- $p = 3$: Take the subgroup generated by a 3-cycle, say $K = \{1, (123), (132)\}$. It is not normal because, for instance, $(134)(123)(143) = (243) \notin K$
- $p = 5$ The trivial subgroup $\{1\}$ is the only subgroup of order 1.

Problem 4. What is the order of a Sylow p-subgroup of the symmetric group S_5 for $p = 2, 3, 5$? For each of these p, give an example of a Sylow p-subgroup of S_5.

We factor $|S_5| = 5! = 2^3 \cdot 3 \cdot 5$. It follows that a Sylow p-subgroup has order $8, 3, 5$ for $p = 2, 3, 5$ respectively.

- $p = 2$: We may view D_4 as a subgroup of S_4 and hence of S_5. Explicitly, if we label the vertices of a square by $1, 2, 3, 4$, then the dihedral group of order 8 is generated by the permutations (1234) (rotation) and $(14)(23)$ (reflection). This subgroup is $\langle (1234), (14)(23) \rangle = \{1, (13), (24), (12)(34), (14)(23), (12)(34), (1234), (1432)\}$.

- $p = 3$: As in the previous problem, we may use the subgroup generated by a 3-cycle, say $\{1, (123), (132)\}$
- $p = 5$: We may use the cyclic subgroup generated by a 5-cycle, say $\langle(12345)\rangle = \{1, (12345), (13524), (14253), (15432)\}$

Problem 5. Show that every group of order 45 has a normal subgroup of order 9.

Let G be a group of order 45. Since $45 = 9 \cdot 5$, a Sylow 3-subgroup of G must have order 9. The number n_3 of such subgroups is 1 modulo 3 and divides $45/9 = 5$. The only natural number with this property is 1, so G has only one Sylow 3-subgroup; call it H. For any $g \in G$, we see that gHg^{-1} is also a subgroup of order 9 and hence a Sylow 3-subgroup, but by the above this means $gHg^{-1} = H$. So, H is an order 9 normal subgroup of G

Problem 6. Suppose that G is a finite group of order $p^n k$, where $k < p$ and p is a prime. Show that G must contain a normal subgroup.

As the problem becomes trivial if we allow the normal subgroup to be G itself or the trivial subgroup, we disallow these cases. But with these restrictions, we also require $n > 1$, as otherwise taking G to be the cyclic group of order p would give a counterexample. We separate into the cases $k > 1$ and $k = 1$.

- $k > 1$: In this case, the proof is analogous to the previous problem. A p-Sylow subgroup has order p^n since $p \nmid k$ follows from $k < p$. The number of p-Sylow subgroups is 1 mod p and divides k. Since $k < p$, this means that there is only one p-Sylow subgroup, which is necessarily normal by the reasoning in the previous problem.
- $k = 1$. The above arguments no longer work because a subgroup of order p^n is no longer proper ($|G| = p^n$). To deal with this case, we separate into the cases that G is abelian and G is nonabelian.

  G is abelian: Any subgroup of G is normal in this case. Let $g \in G$ be any element other than 1. If g doesn't generate G, then $\langle g \rangle$ is a nontrivial, proper normal subgroup of G. Otherise, if g is a generator of G (so G is cyclic of order p^n), then g^p generates a subgroup of order p^{n-1} which is normal, proper, and nontrivial (since $p^{n-1} > 1$ comes from $n > 1$).

  G is nonabelian: The center $Z(G)$ is a normal subgroup of G. It is nontrivial since G is a p-group and is not all of G because G is not abelian.