Modern Algebra I: Homework 13

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1. (a) Show that a non-abelian simple group cannot be solvable.

   Proof. A simple group has two normal subgroups: the trivial group and the entire group itself. If we have a simple group G, we can create only the subnormal series $G = G_0 \supset G_1 = \{1\}$. But we have that the factor group $G_0/G_1 \cong G$ is not abelian by assumption. G is not solvable. ■

   (b) We proved in class that A_5 is simple. Use this and other results (from Gallagher §12) to show that groups S_5, S_6 are not solvable.

   Proof. From (a) we know that A_5 is not solvable because it is non-abelian and simple. THM E (Gallagher §12) derives that any subgroup H of a solvable group G is solvable. Assume toward contradiction that S_5 and S_6 are solvable. Then subgroup $A_5 \subset S_5, S_6$ must be solvable. But since A_5 is not solvable, we know S_5 and S_6 cannot be solvable. ■

2. Show that the direct product $G \times H$ of two solvable groups is solvable. (Hint: How can you construct the required chain of subgroups of $G \times H$ from those for G and H?)

   Proof. Let $G = G_0 \supset G_1 \supset ... \supset G_n = \{1\}$ be the subnormal series for solvable group G, and let $H = H_0 \supset H_1 \supset ... \supset H_m = \{1\}$ be the subnormal series for solvable group H. We can write the subnormal series $G \times H \supset G \times H_1 \supset ... \supset G \times H_{m-1} \supset G \times \{1\} \supset G_1 \times \{1\} \supset ... \supset G_{n-1} \times \{1\} \supset \{1\} \times \{1\}$.

   We must verify that $K_i \triangleleft K_{i-1}$ for each group in the series. We can see this since if $M \triangleleft N$, then $L \times M$ is a normal subgroup of $L \times N$ for any groups L, M, N. This follows trivially from the definition of the direct product. The factor groups K_{i-1}/K_i are isomorphic to factor groups of the subnormal series of G or H. For sub-series $G \times H \supset G \times H_1 \supset ... \supset G \times H_{m-1} \supset G \times \{1\}$, each factor group is isomorphic to a factor group of H, while for sub-series $G \times \{1\} \supset G_1 \times \{1\} \supset ... \supset G_{n-1} \times \{1\} \supset \{1\} \times \{1\}$, each factor group is isomorphic to a factor group of G. Hence all factor groups must be abelian by assumption, and $G \times H$ is solvable. ■

3. Prove that there are no simple groups of order:

   (a) $70 = 2 \times 5 \times 7$

   Proof. Consider G with $|G| = 2 \times 5 \times 7$. From the third Sylow theorem, we have $S_5(G) \equiv 1 \pmod 5$ and $S_5(G) \mid 14$. This implies that $S_5(G) = 1$. It follows that the Sylow 5-subgroup is normal (Theorem 15.10, Judson), and G is not simple. ■

(b) $64 = 2^6$

Proof. From Gallagher §18, a group G of order $n = p^a$ with p prime and $a \geq 2$ has a nontrivial center. Hence $Z(G) = G$ or $Z(G) = H \subset G$ with $H \neq \{1\}$. Since the center is a normal subgroup, the second case implies that G contains a proper normal subgroup. In the first case, we know that G is abelian since $Z(G) = G$ and thus must have every subgroup be normal. Since every group of nonprime order has a proper subgroup of prime order generated by an element of prime order, then G must have a proper normal subgroup of prime order. In either case, G is not simple. ■

(c) $100 = 2^2 \times 5^2$

Proof. We have G of the form $p^a m$ with p prime and $p > m$ $(p = 5, m = 4)$. From Exercise 1 of Gallagher §18 (equivalently, Theorem 15.10 of Judson), a Sylow p-subgroup of G is a proper normal subgroup since there is no divisor $d > p$ of m with $d \equiv 1 \pmod{p}$. This also follows from an application of the third Sylow theorem, since we can see that $S_p(G) \equiv 1 \pmod{p}$ and $S_p(G) \mid m$ implies $S_p(G) = 1$, from which it follows that the Sylow p-subgroup is normal. Hence G is not simple. ■

(d) $65 = 5 \times 13$

Proof. We have G of the form $p^a m$ with p prime and $p > m$ $(p = 13, m = 5)$. From Exercise 1 of Gallagher §18 (equivalently, Theorem 15.10 of Judson), a Sylow p-subgroup of G is a proper normal subgroup since there is no divisor $d > p$ of m with $d \equiv 1 \pmod{p}$. This also follows from an application of the third Sylow theorem, since we can see that $S_p(G) \equiv 1 \pmod{p}$ and $S_p(G) \mid m$ implies $S_p(G) = 1$, from which it follows that the Sylow p-subgroup is normal. Hence G is not simple. ■

(e) $96 = 2^5 \times 3$

Proof. For group G, each subgroup $H \subset G$ with index m contains a normal subgroup K of G with index in G dividing $m!$. We can see this since we have a homomorphism $\phi : G \to S_{G/H}$ (with $S_{G/H}$ having order $m!$) because G acts on G/H by left translation. From the first homomorphism theorem, $G/K \subset S_{G/H}$, so $|G/K| \mid m!$. A corollary of this is that if G has a subgroup H of index m with $m! < |G|$, then G has a proper normal subgroup.

Consider G with $|G| = 2^5 \times 3$. In the corollary, let H be the Sylow 2-subgroup with index 3. Since 3! is less than the order of G, G has a proper normal subgroup by the corollary and is thus not simple. ■

(f) $80 = 2^4 \times 5$

Proof. Consider G with $|G| = 2^4 \times 5$. From the third Sylow theorem, $S_5(G) \equiv 1 \pmod{5}$ and $S_5(G) \mid 16$. Then $S_5(G) = 1$ or $S_5(G) = 16$. In the first case, the Sylow 5-subgroup is normal. In the second case, there are $4 \times 16 = 64$ elements of order 5. But a Sylow 2-subgroup has order 16 since $16 + 64 = 80$. In this case there can be only one Sylow 2-subgroup, which is then normal. In either case, G cannot be simple. ■

4. Determine the number of ways to color vertices of a regular pentagon using 4 colors, up to the symmetries of the pentagon (the symmetry group is D_5). First derive the formula when the number of colors is n, and then specialize to $n = 4$.

   Let X be the set of vertices. Then D_5 acts on X, and the number of orbits of D_5 is the number of colorings up to symmetries. Let k be the number of colorings up to symmetries. Using Burnside's Theorem, we have:

   $$k = \frac{1}{|D_5|} \sum_{g \in D_5} |X_g|$$

   We create a table to count the number of colorings fixed by g, in which we are using n colors:

g	1	r	r^2	r^3	r^4	s	sr	sr^2	sr^3	sr^4
number of colorings	n^5	n	n	n	n	n^3	n^3	n^3	n^3	n^3

   The identity fixes each vertex, so there are n choices of color for all 5 vertices, giving n^5 possible colorings.

   For any rotation, all vertices must have the same color. Hence there are n colorings corresponding to the n different possible colors. This applies to each element of the form $r^i, i \geq 1$.

   For any reflection, we must have the two pairs of opposite vertices be the same color and we allow the fifth vertex along the line of symmetry to be any color. There are n choices for each pair and the singleton vertex, giving n^3 possible colorings. Hence there are n^3 colorings for each element of the form $sr^i, i \geq 0$.

   This gives a general formula in terms of the number of colors n:

   $$\boxed{k(n) = \frac{1}{10}(n^5 + 4n + 5n^3)}$$

   For $n = 4$ colors, we find $k(4) = \boxed{136}$.

5. (a) How many necklaces can you arrange out of ten red and two green beads, up to dihedral symmetries?

   We have the symmetry group D_{12} of order 24. Let X be the set of vertices (beads). Then D_{12} acts on X, and the number of orbits of D_{12} is the number of arrangements up to symmetries. Again we use Burnside's Theorem:

   $$k = \frac{1}{|D_{12}|} \sum_{g \in D_{12}} |X_g|$$

The identity fixes each vertex, so there are $\binom{12}{2} = 66$ arrangements corresponding to selecting 2 vertices out of 12 to be green.

Only the rotation $r^6 \in D_{12}$ preserves the coloring of the two green beads if they are positioned opposite each other. This gives 6 possible arrangements corresponding to the 6 pairs we can choose from the 12 beads.

Consider the two possible reflections:

i. sr^i with i odd: In this case, the axis of symmetry lies in between two pairs of beads. We can choose any of the opposite pairs across the axis of symmetry to be the two green beads. This gives 6 possible arrangements.

ii. sr^i with i even: In this case, the axis of symmetry lies along a pair of beads. We may have the two green beads lie on the axis of symmetry, giving 1 arrangement. We may also have the green beads be any of the 5 opposite pairs across the axis of symmetry. This gives 6 possible arrangements.

There are 12 reflections in D_{12}. In total, we have $6 \times 12 = 72$ arrangements arising from the reflections. Hence, we have $k = \frac{1}{24}(66+6+72) = \boxed{6}$ total arrangements.

(b) Same question, but out of six red, three green, and three blue beads.

We proceed as above.

The identity fixes each vertex. There are $\binom{12}{6}\binom{6}{3} = 18480$ ways to first choose the 6 red beads and then the 3 green beads from the remaining 6 beads.

Consider the rotations. Only $r^4, r^8 \in D_{12}$ give rise to valid arrangements. In this case, we construct the necklace as follows: $GRBRGRBR....$ We must have the rotation be a multiple of 4 so that the triples of green and blue beads remain the same color. Since we must preserve this structure, there are 12 choices for which to place the first bead in the ordering, giving 24 arrangements for the two elements in D_{12}.

Consider the two possible reflections:

i. sr^i with i odd: In this case, the axis of symmetry lies in between two pairs of beads. We cannot find any valid arrangement since if we try to find pairs across the axis of symmetry to have the same color, at least one pair will hold two different color beads.

ii. sr^i with i even: In this case, the axis of symmetry lies along a pair of beads. We must put one green and one blue bead on the axis of symmetry to give rise to a valid arrangement. There are two choices for which color to put on which end of the axis of symmetry. Once this is determined, we must put the two pairs of green and blue beads among the five pairs of opposite beads. There are 5×4 ways to do this. This gives rise to $2 \times 5 \times 4 = 40$ arrangements.

There are 6 reflections of the form sr^i with i even, hence there are $40 \times 6 = 240$ arrangements arising from the reflections. Using Burnside's Theorem, this gives $\frac{1}{24}(18480 + 24 + 240) = \boxed{781}$ total arrangements.