Homework 5 Solutions

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Exercise 1 (Judson, Chapter 5, Exercise 1)**.** Write the following permutations in cycle notation.

(a)
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \end{pmatrix}$$

(b)
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 1 & 3 \end{pmatrix}$$

(d)
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 2 & 5 \end{pmatrix}$$

Solution. There's not much to say here except that if you forget the meaning of cycle notation, you may look in Chapter 5 of Judson where it is explained. Answers:

(a)
$$(12453)$$

(b)
$$(14)(35)$$

(d)
$$(24)$$

Note you do not need to include trivial cycles of the form (n).

Exercise 2. Simplify the following permutations and write them as products of disjoint cycles:
(a) $(12)(1234)$, (b) $(13)(24)(12)$, (c) $(132)(12)(123)$, (d) $(142)(35)(23)(152)$,
(e) $(1423)(34)(56)(1324)$, (f) $(1542)(365)(2314)$.

Solution. Answers:
(a) (234), (b) (1423), (c) (23), (d) $(13)(24)$, (e) $(12)(56)$, (f) $(1264)(35)$.

Exercise 3. Same question as in exercise 2 for the following permutations:
(a) $(123)^2$, (b) $(13)^7$, (c) $(16352)^3$, (d) $(1342)^{10}$, (e) $(15324)^{-1}$, (f) $(132)(142)^{-1}$,
(g) $((154)(32))^{13}$.

Solution. Let's do (b) first. (13) has length 2, so its square is trivial. Thus $(13)^7 = (13)^1 = (13)$.

To compute $(123)^2$ in (a), simply run through the numbers in the cycle (123) starting at 1, and pick out every other number and put them in a new cycle. The result is just (132). Similarly for $(16352)^3$ in (c), pick out every third number to make the cycle (15623). For (d), note first $(1342)^{10} = (1342)^2$, which is true since the fourth power of (1342) is trivial. Now we can't square exactly according to the method described in (a) and (c). Instead, $(1342)^2$ breaks into two 2-cycles: $(1342)^2 = (14)(23)$, as is easily checked.

There is an algorithm that will raise any cycle to any power, and return the result in cycle notation. Can you conjecture what it is?

To invert a cycle, simply reverse all but the first number in the cycle. For example, in (e), $(15324)^{-1} = (14235)$.

For (f), we note $(132)^4 = (132)$ and $(142)^{-1} = (124)$. Thus, $(132)^4(142)^{-1} = (132)(124) = (243)$.

Finally, for (g), we note that (154) and (32) are disjoint, so they commute. Thus $((154)(32))^{13} = (154)^{13}(32)^{13} = (154)(32)$ which, if we are being pedantic, is equal to (154)(23).

Exercise 4. What are the orders of each of the following permutations
(a) (1), (b) (1523), (c) (153)(24), (d) (165)(243), (e) (1743)(25)(68)?

Solution. These are all products of disjoint permutations. Therefore the orders of the products are just the least common multiples of the orders of each of the terms:
(a) 1, (b) 4, (c) 6, (d) 3, (e) 4.