Modern Algebra I HW 6 Solutions

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Problem 1.

We will use the identity $(a_1 \dots a_k) = \prod_{i=1}^{k-1}(a_i a_{i+1})$, which immediately gives us the following.

(a) $(15326) = (15)(53)(32)(26)$

(b) $(142)(356)(78) = (14)(42)(35)(56)(78)$

(c) $(1536)(79428) = (15)(53)(36)(79)(94)(42)(28)$

We see that the permutation in item a is even whereas those in b, c are odd because $4, 5, 7$ transpositions respectively appear in the above factorizations.

Problem 2.

Recall that the order of a product of disjoint cycles in S_n is the lcm of the orders of the individual cycles. In what follows, products of cycles are assumed to be disjoint.

(a) The possible cycle types of elements in S_4 are: identity, 2-cycle, 3-cycle, 4-cycle, a product of two 2-cycles. These have orders $1, 2, 3, 4, 2$ respectively, so the possible orders of elements in S_4 are $1, 2, 3, 4$.

(b) Of the above orders of elements in S_4, only 4 is not the order of an element in A_4. So, the possible orders of elements in A_4 are $1, 2, 3$.

(c) The possible cycle types of elements in S_5 are: identity, 2-cycle, 3-cycle, 4-cycle, 5-cycle, product of two 2-cycles, a product of a 2-cycle with a 3-cycle. These have respective orders $1, 2, 3, 4, 5, 2, 6$, so the possible orders of elements in S_5 are $1, 2, 3, 4, 5, 6$.

(d) Examining the above list, we see that only the orders $1, 2, 3, 5$ are orders of elements in A_5.

Problem 3.

The element $(1, 2, 3)(4, 5, 6, 7, 8)$ is in A_{10} because it is the product of a 3-cycle and a 5-cycle, both of which are even. It has order $\text{lcm}(3, 5) = 15$.

Problem 4.

Let $n = 2k + 1$ be odd (k a nonnegative integer) and let σ be an n-cycle, say $\sigma = (a_1 \dots a_n)$. The result is trivial if $n = 1$, Otherwise, we see $(\sigma^2)^i(a_1) = a_{2i+1}$ for $i \leq k$ and in particular $(\sigma^2)^k(a_1) = a_n$. Since $\sigma^2(a_n) = a_2$, we may argue similarly for even indices and find

$$\sigma^2 = (a_1, a_3, \dots, a_n, a_2, \dots, a_{n-1}).$$

Problem 5.

(a) Since $a + \langle 5 \rangle = b + \langle 5 \rangle$ if and only if $a \equiv b \mod 5$, a complete list of the distinct cosets of $\langle 5 \rangle$ is given by $\{i + \langle 5 \rangle : i \in \{0, 1, 2, 3, 4\}\}$. Explicitly,

$$\begin{aligned}
\langle 5 \rangle &= \{0, 5, 10\} \\
1 + \langle 5 \rangle &= \{1, 6, 11\} \\
2 + \langle 5 \rangle &= \{2, 7, 12\} \\
3 + \langle 5 \rangle &= \{3, 8, 13\} \\
4 + \langle 5 \rangle &= \{4, 9, 14\}.
\end{aligned}$$

Since $\mathbb{Z}/15$ is abelian, the left and right cosets agree. Since there are 5 cosets of $\langle 5 \rangle$ in $\mathbb{Z}/15$, its index is 5.

(b) There are a total of $|S_3|/|H| = 6/2 = 3$ left (right) cosets, so once we find 3 distinct left (right) cosets, we'll know that we have them all.

Left cosets:

$$\begin{aligned}
H &= \{1, (23)\} \\
(12)H &= \{(12), (123)\} \\
(13)H &= \{(13), (132)\}
\end{aligned}$$

Right cosets:

$$\begin{aligned}
H &= \{1, (23)\} \\
H(12) &= \{(12), (132)\} \\
H(13) &= \{(13), (123)
\end{aligned}$$

We see that neither of the nontrivial left cosets are right cosets.

Problem 6. Let G be a cyclic group of order n with generator g, so $G = \{1, g, \dots, g^{n-1}\}$. An element of G is a generator if and only if its order is n, and we know that the order of g^i is $n/\gcd(n, i)$ for each i. So g^i is a generator if and only if $n/\gcd(n, i) = n$, which holds if and only if $\gcd(n, i) = 1$. But there are exactly $\phi(n)$ such elements $i \in \{0, 1, \dots, n-1\}$ by definition.

Problem 7.

(a) Since $|S_3| = 6$, the possible orders of subgroups are $1, 2, 3, 6$.

Order 1: Only the trivial subgroup $\{1\}$ has order 1.

Order 2: The order 2 elements of S_3 are 2-cycles, so the order 2 subgroups of S_3 are:

$\{1, (12)\}$

$\{1, (13)\}$

$\{1, (23)\}$

Order 3: These are generated by 3-cycles, which are $(123), (132)$, so the only subgroup of order 3 is $\{1, (123), (132)\}$.

Order 6: The subgroup of order 6 is the full group S_3.

(b) Since $|A_4| = 12$, the possible orders of subgroups are $1, 2, 3, 4, 6, 12$.

Order 1: Only the trivial subgroup $\{1\}$

Order 2: The elements of order 2 in A_4 are products of two disjoint 2-cycles. These give the subgroups

$\{1, (12)(34)\}$

$\{1, (13)(24)\}$

$\{1, (14)(23)\}$

Order 3: These must be generated by three cycles, which are $(123), (132), (134), (143), (124), (142), (234), (243)$. These give the subgroups

$\{1, (123), (132)\}$

$\{1, (134), (143)\}$

$\{1, (124), (142)\}$

$\{1, (234), (243)\}$.

Order 4. If H is a subgroup of order 4, its elements must have order in $\{1, 2, 4\}$. But A_4 doesn't contain any elements of order 4, so H must contain only elements of order $1, 2$. The only elements of order 2 are $(12)(34), (13)(24)$, and $(14)(23)$ and we see that $\{1, (12)(34), (13)(24), (14)(23)\}$ is indeed a subgroup (of order 4).

Order 6: As noted in the problem statement, A_4 has no subgroups of order 6.

Order 12: The whole group A_4.

(c) Since $|\mathbb{Z}/3 \times \mathbb{Z}/3| = 9$, the possible orders of subgroups are $1, 3, 9$.

Order 1: Trivial subgroup $\{(0, 0)\}$.

Order 3: These are cyclic of order 3, and every nontrivial element of $\mathbb{Z}/3 \times \mathbb{Z}/3$ generates such a subgroup. So, the subgroups of order 3 are:

$\{(0, 0), (1, 1), (2, 2)\}$

$\{(0, 0), (1, 0), (2, 0)\}$

$\{(0,0),(0,1),(0,2)\}$

$\{(0,0),(1,2),(2,1)\}$

Order 9: The full group $\mathbb{Z}/3 \times \mathbb{Z}/3$.