Homework 7 Solutions

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Exercise 1. A homomorphism $f : G \to H$ of groups is a map such that $f(ab) = f(a)f(b)$ for any $a, b \in G$. Prove that f takes the unit element of G to the unit element of H and that $f(a^{-1}) = f(a)^{-1}$.

Solution. Let e_G, e_H be the unit elements of G and H, respectively. Then

$$e_H f(e_G) = f(e_G) = f(e_G \cdot e_G) = f(e_G)f(e_G).$$

Cancelling, we obtain

$$e_H = f(e_G).$$

As for inverses, we have

$$f(a)f(a^{-1}) = f(a \cdot a^{-1}) = f(e_G) = e_H.$$

Thus, multiplying both sides on the left by $f(a)^{-1}$ gives

$$f(a)^{-1}f(a)f(a^{-1}) = f(a)^{-1}e_H.$$

The left hand side is $e_H f(a^{-1}) = f(a^{-1})$ and the right hand side is $f(a)^{-1}$. So we are done.

Exercise 2. Collect the following groups into isomorphism classes:
(a) C_4, (b) $\mathbb{Z}_6$, (c) $\mathbb{Z}_4$, (d) $\mathbb{Z}_3 \times \mathbb{Z}_2$, (e) $\mathbb{Z}_2 \times \mathbb{Z}_2$, (f) $\mathbb{Z}_5^*$, (g) $\mathbb{Z}_8^*$.

Solution. C_4, $\mathbb{Z}_4$ and $\mathbb{Z}_5^*$ are cyclic of order 4 (the classes of 2 and 3 in $\mathbb{Z}_5^*$ generate that group). Thus they are isomorphic. Also, $\mathbb{Z}_8^* \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. You can check that an isomorphism is given by $f : \mathbb{Z}_8^* \to \mathbb{Z}_2 \times \mathbb{Z}_2$ where f is defined by $f(1) = (0,0)$, $f(3) = (1,0)$, $f(5) = (0,1)$, an $f(7) = (1,1)$. We also note that $\mathbb{Z}_3 \times \mathbb{Z}_2 \cong \mathbb{Z}_6$ because both are cyclic of order 6 ($(1,1)$ generates $\mathbb{Z}_3 \times \mathbb{Z}_2$).

Finally we check that there are no more isomorphisms besides the ones listed. We note that $\mathbb{Z}_4 \ncong \mathbb{Z}_6$ because these groups have different orders, and $\mathbb{Z}_2 \times \mathbb{Z}_2 \ncong \mathbb{Z}_6$ for the same reason. Also, $\mathbb{Z}_4 \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$ because the latter group is not cyclic.

Exercise 3. Show that the groups $\mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ are pairwise non-isomorphic.

Solution. You can check that the maximal order of an element in $\mathbb{Z}_8$ is 8, the maximal order of an element in $\mathbb{Z}_4 \times \mathbb{Z}_2$ is 4, and the maximal order of an element in $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is 2. So it suffices to prove the following lemma (which is maybe pretty obvious, but the proof is not completely trivial to write down. And you should still see a proof of a lemma like this at least once in your life).

Lemma. *Let G and H be groups, and assume $G \cong H$. Then the maximal order of an element in G is equal to the maximal order of an element in H.*

Proof. Let n be the maximal order of an element of G and let m be the maximal order of an element of H. Let $f : G \to H$ be an isomorphism. Let $g \in G$ be an element of maximal order n. Then $f(g)^n = f(g^n) = f(e_G) = e_H$ by Exercise 1. Also, if $0 < k < n$, then $g^k \neq e_G$, and since f is a bijection, $f(g)^k = f(g^k) \neq f(e_G) = e_H$. This shows that $f(g)$ also has order n. Thus H also has an element of order n, and so the maximal order of an element in G is at least the maximal order of an element in H, i.e., $n \leq m$. But this situation is symmetric: The same reasoning in the other direction shows that $m \leq n$. Thus $n = m$, which is what we wanted to show. $\square$

Exercise 6. If a group G has exactly one subgroup H of order k, prove that H is normal in G.

Solution. We need to show that for any $g \in G$, $gHg^{-1} = H$. But if $g \in G$, you can check that gHg^{-1} is a subgroup of G. Furthermore, $h \mapsto ghg^{-1}$ defines a bijection (in fact, an isomorphism) between H and gHg^{-1}. So therefore gHg^{-1} also has order k, and is thus equal to H by hypothesis. This completes the proof.

Exercise 7. (a) Let $H = \{id, (12)(34), (13)(24), (14)(23)\}$. Check that H is a subgroup of S_4. Prove that H is normal in S_4 using that two permutations are conjugate in S_n iff they have the same cycle type (proved in class).
(b) Show that the subgroup H generated by the 4-cycle (1234) is not normal in S_4.

Solution. (a) It is not hard to multiply each element of H by each other element in H and see that the result is still in H. (You can even simplify this computation by noting that multiplication by the identity leaves each element stable, so it suffices to multiply each pair of non-identity elements). I will omit the computation, but I will note that it shows that each element in H squares to the identity, and hence each element is its own inverse. So H is closed under multiplication, inversion, and contains the identity, and is thus a subgroup.

There are no other products of disjoint 2-cycles other than those in H. So by the fact stated in the exercise, for any $\sigma \in S_4$, and for any $h \in H$ which is a product of two disjoint 2-cycles, $\sigma h \sigma^{-1}$ is still a product of two disjoint 2-cycles, hence is in H. Since obviously $\sigma(id)\sigma^{-1} = id$, this proves that H is normal in S_4.

(b) Now H is the subgroup $\{id, (1234), (13)(24), (1432)\}$. Let us conjugate (1234) by (12). We get

$$(12)(1234)(12)^{-1} = (12)(1234)(12) = (12)(134) = (1342).$$

This element is not in H, so H is not normal.