# Modern Algebra I HW 8 Solutions

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Problem 1:

A homomorphism $\phi : \mathbb{Z} \to \mathbb{Z}_4$ is determined by $\phi(1)$ since $\phi(n) = n \cdot \phi(1)$ for every $n \in \mathbb{Z}$. Also, for any $a \in \mathbb{Z}_4$, we can get a homomorphism $\mathbb{Z} \to \mathbb{Z}_4$ taking 1 to $a$ by sending $n$ to the reduction mod 4 of $an$. So, there are four homomorphisms $\phi : \mathbb{Z} \to \mathbb{Z}_4$, one for each value in $\mathbb{Z}_4$.

If $\phi(1) = 0$, we get the zero map. Its kernel is all of $\mathbb{Z}$ and its image is $\{0\}$.

If $\phi(1) = 1$, our map is just reduction mod 4, which is clearly surjective; that is, its image is all of $\mathbb{Z}_4$. We see that an element is sent to zero if and only if it's a multiple of 4, so $\ker(\phi) = 4\mathbb{Z}$.

If $\phi(1) = 2$, our map takes $n$ to the reduction of $2n$ mod 4. The image is generated by 2, which is $\langle 2 \rangle = \{0, 2\}$. The kernel is the set of elements of $n$ such that $2n$ is a multiple of 4. This is the set of even integers, $2\mathbb{Z}$.

If $\phi(1) = 3$, our map takes $n$ to the reduction of $3n$ mod 4. The image is generated by $\phi(1) = 3$ and so is all of $\mathbb{Z}_4$ (so it's surjective). Since $3n$ is a multiple of 4 if and only if $n$ is a multiple of 4, the kernel is $4\mathbb{Z}$.

Problem 3:

(a) Let $\mathrm{Hom}(\mathbb{Z}_n, G)$ be the set of homomorphisms from $\mathbb{Z}_n$ to $\mathbb{Z}_m$. Define $ev_1 : \mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z}_m) \to G$ ("evaluation at 1") by $ev_1(\phi) = \phi(1)$ for all $\phi \in \mathrm{Hom}(\mathbb{Z}_n, G)$. For every such $\phi$, we notice $\phi(1)^n = \phi(n \cdot 1) = \phi(0) = 1$ So, the image of $ev_1$ is contained in the subset $G_n \subset G$ consisting of elements of order dividing $n$, and we can therefore restrict its codomain to get a map $\mathbb{Z}_n \to G_n$. We want to show that this map is now a bijection.

Injective: If $\phi$ and $\psi$ are homomorphisms as above with $\phi(1) = \psi(1)$, then $\phi(k) = \phi(1)^k = \psi(1)^k = \psi(k)$ for all $k \in \mathbb{Z}_n$, which means $\phi = \psi$.

Surjective: Let $g$ be an arbitrary element of $G$ with $g^n = 1$. There is a well-defined homomorphism $\phi : \mathbb{Z}_n \to G$ given by $\phi(i) = g^i$ because if $a \equiv b \mod n$, say $a = b + kn$, then $g^a = g^b g^{kn} = g^b$.

(b) By the previous part, the set of homomorphisms $\mathbb{Z}_n \to \mathbb{Z}_m$ is in bijection with the elements of $\mathbb{Z}_m$ with order dividing $n$. But every element of $\mathbb{Z}_m$

also has order dividing $m$. The only common divisor of $m, n$ is 1 by assumption. The only element of order 1 is 0, and so the only homomorphism $\mathbb{Z}_n \to \mathbb{Z}_m$ is the trivial (zero) one.

Problem 4:

Homomorphisms $\mathbb{Z}_3 \to S_3$: The elements of order dividing 3 in $S_3$ are $1, (123), (132)$. By the previous problem, the homomorphisms $\mathbb{Z}_3 \to S_3$ are given by taking 1 to these elements. Explicitly, these homomorphisms $\phi : \mathbb{Z}_3 \to S_3$ are:

$\phi(n) = 1$ for all $n \in \mathbb{Z}_3$

$\phi(n) = (123)^n$ for $n \in \mathbb{Z}_3$

$\phi(n) = (132)^n$ for $n \in \mathbb{Z}_3$

The elements in $S_3$ with order dividing 4 are just the identity and transpositions. Thus the homomorphisms $\phi : \mathbb{Z}_4 \to S_3$ are defined by:

$\phi(n) = 1$

$\phi(n) = (12)^n$

$\phi(n) = (13)^n$

$\phi(n) = (23)^n$

Problem 5:

(a) Firstly, $6 - 4 = 2 \in H + N$, so $\langle 2 \rangle \subset H + N$. Since $H$ and $K$ are both subsets of $\langle 2 \rangle$, we also have $H + N \subset \langle 2 \rangle$ too. It follows that $H + N = \langle 2 \rangle = \{0, 2, \ldots, 22\}$. By similar arguments, $H \cap N = \langle 12 \rangle = \{0, 12\}$.

(b) The three elements of $(H + N)/N = \langle 2 \rangle / \langle 6 \rangle$ are

$N = \{0, 6, 12, 18\}$

$2 + N = \{2, 8, 14, 20\}$

$4 + N = \{4, 10, 16, 22\}$.

(c) The $[H : H \cap N] = 6/2 = 3$ cosets in $H/(H \cap N)$ are:

$H \cap N = \langle 12 \rangle = \{0, 12\}$

$4 + H \cap N = \{4, 16\}$

$8 + H \cap N = \{8, 20\}$

(d) The isomorphism of the second isomorphism theorem is given by taking $h + (H \cap N)$ to $h + N$. With this, we see:

$H \cap N$ is taken to $N$

$4 + H \cap N$ is taken to $4 + N$.

$8 + H \cap N$ is taken to $8 + N = 2 + N$.

Problem 7:

Let $r, s$ be as in the usual presentation of the dihedral group. We have $srs = r^{-1}$. If $D_n$ were abelian, we would have $srs = s^2r = r$. Since $n \geq 3$, we have $r \neq r^{-1}$ and hence $D_n$ is nonabelian.