# Homework 9 Solutions

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**Exercise 1.** Write down a proof that, for a group $G$ and an abelian group $H$, the set of all homomorphisms $\mathrm{Hom}(G, H)$ from $G$ to $H$ is an abelian group.

**Solution.** Let $f, g \in \mathrm{Hom}(G, H)$. Define their product $fg$ by

$$(fg)(x) = f(x)g(x), \quad x \in G.$$

Let us show that $fg$ is a homomorphism. Let $x, y \in G$. Then

$$\begin{aligned}(fg)(xy) &= f(xy)g(xy)\\ &= f(x)f(y)g(x)g(y)\\ &= f(x)g(x)f(y)g(y)\\ &= (fg)(x)(fg)(y),\end{aligned}$$

hence $fg$ is a homomorphism. This shows $\mathrm{Hom}(G, H)$ is closed under this product.

Next we show associativity of this product. Let $h \in \mathrm{Hom}(G, H)$ be another homomorphism. We need to check $f \cdot (gh) = (fg) \cdot h$. We check this on elements. Let $x \in G$. Then

$$\begin{aligned}(f \cdot (gh))(x) = f(x)((gh)(x)) = f(x)(g(x)h(x))&\\ = (f(x)g(x))h(x) = ((fg)(x))h(x) = ((fg) \cdot h)(x).&\end{aligned}$$

Thus our product is associative.

Now define $1 \in \mathrm{Hom}(G, H)$ to be the homomorphism such that $1(x) = e$ for all $x \in G$, where $e$ is the identity of $H$. Then 1 is a homomorphism and for any $f \in \mathrm{Hom}(G, H)$,

$$(1f)(x) = 1(x)f(x) = ef(x) = f(x).$$

Thus 1 is an identity element for our product. (We only need to check it is a one-sided identity because we will show the product is commutative at the end.)

Next, for $f \in \mathrm{Hom}(G, H)$, define $f^{-1} \in \mathrm{Hom}(G, H)$ by $f^{-1}(x) = f(x)^{-1}$. This is a homomorphism because $H$ is abelian. We have

$$(ff^{-1})(x) = f(x)f^{-1}(x) = f(x)f(x)^{-1} = e = 1(x).$$

Thus our product has (right) inverses.

Finally, we show our product is commutative. We have, for $f, g \in \mathrm{Hom}(G, H)$,

$$(fg)(x) = f(x)g(x) = g(x)f(x) = (gf)(x)$$

because $H$ is abelian. Thus we have finished showing $\mathrm{Hom}(G, H)$ is an abelian group.

**Exercise 2** (a)**.** Write down the character table of the cyclic group $C_3$, denoting a generator of $C_3$ by $h$, $C_3 = \{1, h, h^2\}$. Label the characters $\psi_0, \psi_1, \ldots$, with $\psi_0$ being the trivial character.

**Solution.** Let $\omega \in \mathbb{T}$ be a third root of unity. The character table for $C_3$ is

| | $1$ | $h$ | $h^2$ |
|---|---|---|---|
| $\psi_0$ | $1$ | $1$ | $1$ |
| $\psi_1$ | $1$ | $\omega$ | $\omega^2$ |
| $\psi_1$ | $1$ | $\omega^2$ | $\omega$ |

**Exercise 4.** Write down the character table of the group $C_2 \times C_2$.

**Solution.** Write $C_2 = \{1, h\}$. The character table for $C_2 \times C_2$ is

| | $(1,1)$ | $(h,1)$ | $(1,h)$ | $(h,h)$ |
|---|---|---|---|---|
| $\psi_0$ | $1$ | $1$ | $1$ | $1$ |
| $\psi_1$ | $1$ | $1$ | $-1$ | $-1$ |
| $\psi_1$ | $1$ | $-1$ | $1$ | $-1$ |
| $\psi_1$ | $1$ | $-1$ | $-1$ | $1$ |

**Exercise 5** (a)**.** Write down proofs that the center of the quaternion group $Q_8$ is $\{1, -1\}$ and the center of the symmetric group $S_n$ is trivial for $n \geq 3$. What is the center of $S_2$?

**Solution.** We have by definition that $1, -1 \in Z(Q_8)$. We check

$$ij = k, \quad ji = -k,$$

so $i$ and $j$ do not commute. Similarly,

$$k(-i) = -j, \quad (-i)k = j$$

and

$$(-j)(-k) = i, \quad (-k)(-j) = -i,$$

so no other elements are in $Z(Q_8)$ besides $\pm 1$. Thus $Z(Q_8) = \{1, -1\}$.

As for $S_n$, let $n \geq 3$. Let $\sigma \in S_n$ be a nonidentity permutation. Then there is an $i \in \{1, \ldots, n\}$ such that $\sigma(i) \neq i$. Write $j = \sigma(i)$, and let $k \in \{1, \ldots, n\}$ with $k \neq i, j$. This is possible since $n \geq 3$. Consider the 2-cycle $\tau = (jk)$. Then $(\sigma\tau)(i) = \sigma(\tau(i)) = \sigma(i) = j$ while $(\tau\sigma)(i) = \tau(\sigma(i)) = \tau(j) = k$. Since $j \neq k$, we must have $\sigma\tau \neq \tau\sigma$. Since $\sigma \neq \mathrm{id}$ was arbitrary, we see that $Z(S_n)$ is trivial if $n \geq 3$.

If $n = 2$, then one easily checks that $S_2$ is commutative. Thus $Z(S_2) = S_2$.

**Exercise 6.** Look at the groups $Q_8$ and $D_4$. These are both nonabelian groups of order 8. Can you show that these groups are nonisomorphic?

**Solution.** It is easy to see that $-1$ is the only element of order 2 in $Q_8$ (all other nontrivial elements in $Q_8$ square to $-1$.) However, there are two distinct elements in $D_4$ which have order 2, for instance $r^2$ and $s$ (and in fact there are three others.) Thus $Q_8$ and $D_4$ cannot be isomorphic (and, unlike on HW7, I will not check implications like this one in detail here.)