M.KHOVANOV

NEW GEOMETRICAL CONSTRUCTIONS IN LOW-DIMENSIONAL TOPOLOGY

JULY 1990

MOSCOW STATE UNIVERSITY

(THE SECOND GRADE)

ADDRESS: MOSCOW , 117312 , FERSMAN STREET ,
5 - 2 - 211 ,PHONE 124 - 98 - 81 .

NEW GEOMETRICAL CONSTRUCTIONS IN LOW-DIMENSIONAL TOPOLOGY

1. NEW LINK DIAGRAM

We recall that a link in R^3 is a smooth embedding of finite number of disjoint circles. If we choose some general projection $p: R^3 \to R^2$, then the image of the link is called it's diagram. The fig. 1 transformations of diagrams called Reidemeister moves. It s easy to see that two diagrams correspond to the same link if and only if they are connected

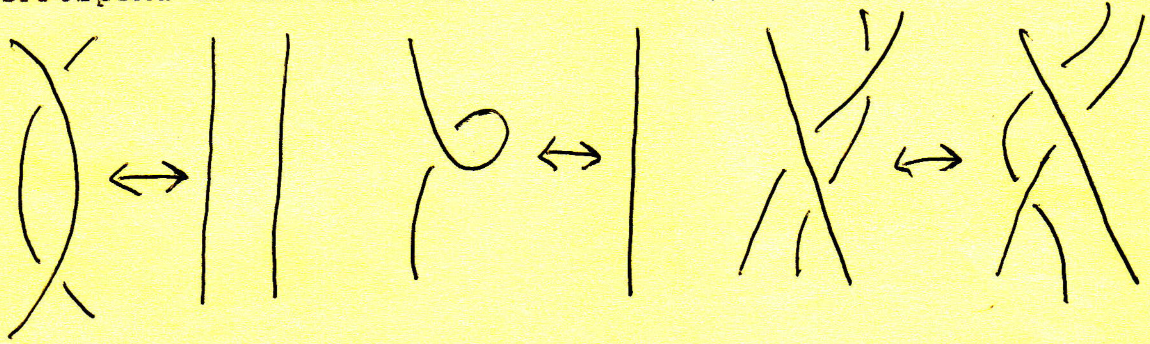


FIG. 1

by a number of Reidemeister moves. Another type of diagram can be obtained by closing the braid. Braids are formed then n points on a horizontal plane are connected by n strings to n points on another horizontal plane directly below the first n points. The strings are not allowed to go back upwards at any point in their travel. The isotopy classes of braids constitute the group $B(n)$ with generations

$$\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$$

and the following relations:

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad |i-j| > 1\ ,\ i,j = 1,2,\dots,n-1$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad i = 1,2,\dots,n-2$$

Closing the braid one can obtain a link (fig. 2).

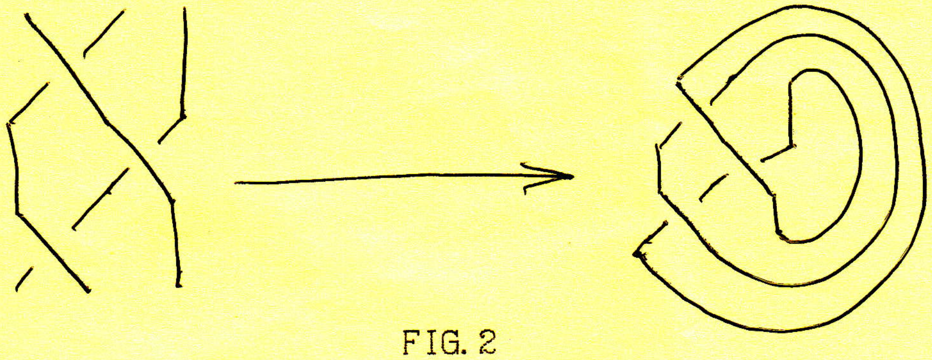


FIG. 2

Here are two classical theorems on this matter

1) Every link can be obtained from some braid .

2) Markov's theorem: if two links are isotopical, then corresponding braids can be obtained from each other by a sequence of elementary transformations (called Markov's moves) (fig. 3)

$$\alpha \longleftrightarrow \alpha \sigma_n^{\pm 1} \quad (\alpha \in B(n))$$

$$\alpha \longleftrightarrow g^{-1} \alpha g \quad (\alpha, g \in B(n))$$

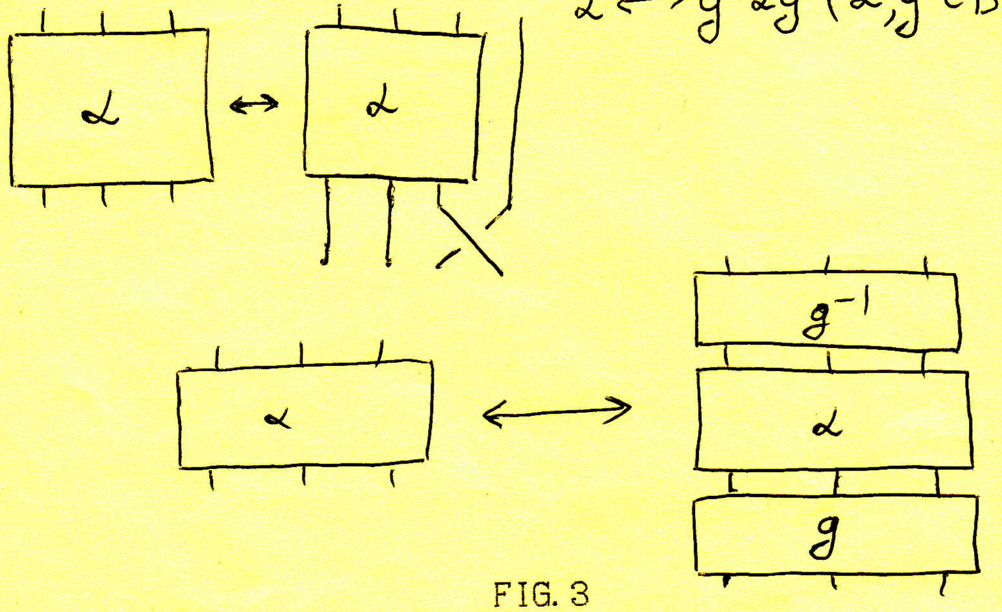


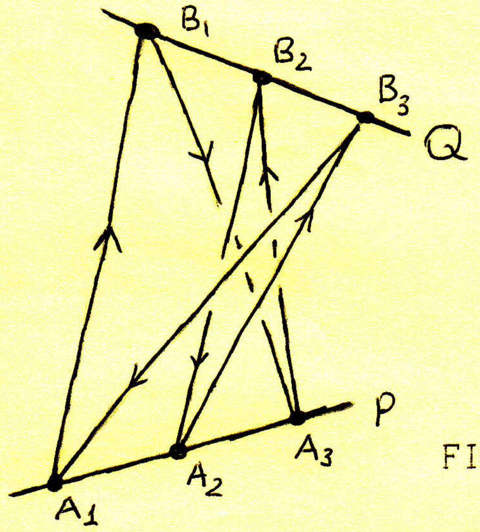
FIG. 3

I propose to consider another type of diagram: a symmetrical giagram of a link. Let $(\alpha, \beta) \in S(n) \times S(n)$. Here $S(n)$ is a symmetrical group (group of transformations). Then we consider two nonintersecting straight lines P and Q in R^3 and choose n ordered points on them

on P: $A_1, A_2, \ldots, A_n$;

on Q: $B_1, B_2, \ldots, B_n$.

Then we join the points A_i and $B_{\alpha(i)}$, B_i and $A_{\beta(i)}$ (fig. 4) by straight segments for every i . As a result we get a link .

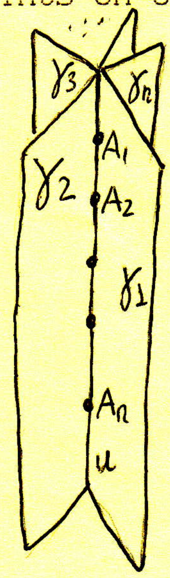


$\alpha : (1)(23)$

$\beta : (13)(2)$

FIG. 4

We will call this diagram the symmetrical diagram of a link . It's convenient to represent this diagram in a slightly different way. Let U be a vertical line, $A_1, A_2, \ldots, A_n$ - points on U, $\gamma_1, \ldots, \gamma_n$ -n halfplanes, having U as a boundary.



$\alpha = (14253)$

$\beta = (1)(2)(3)(4)(5)$

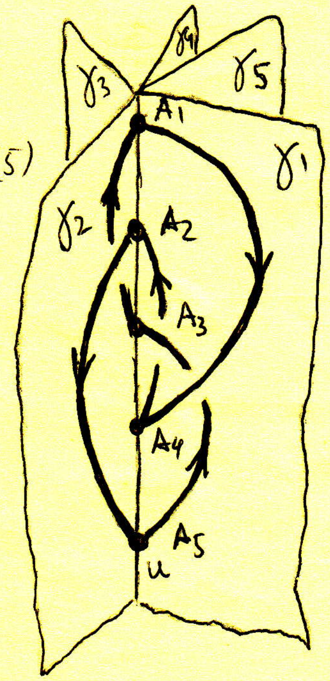


fig. 5

$\langle \alpha, \beta \rangle \longleftrightarrow (\beta\alpha, \alpha)$

Let $<\alpha,\beta>\in S(n)\times S(n)$. Then we join point A_i with $A_{\alpha(i)}$ by a curve, belonging to a halfplane $\gamma_{\beta(i)}$ (fig. 5) for every i.
The following theorem is obvious:
THEOREM 1. Every link possesses a symmetrical diagram.
One can ask a Qestion:
Which diagrams correspond to isotopical link ?
It's obvious that the fig. 6 transformations preserve the isotopical type of a link. The theorem is valid
THEOREM 2. Two symmetrical diagrams define isotopical links if and only if they can be obtained from each other by a sequence of transformations of fig. 6 .

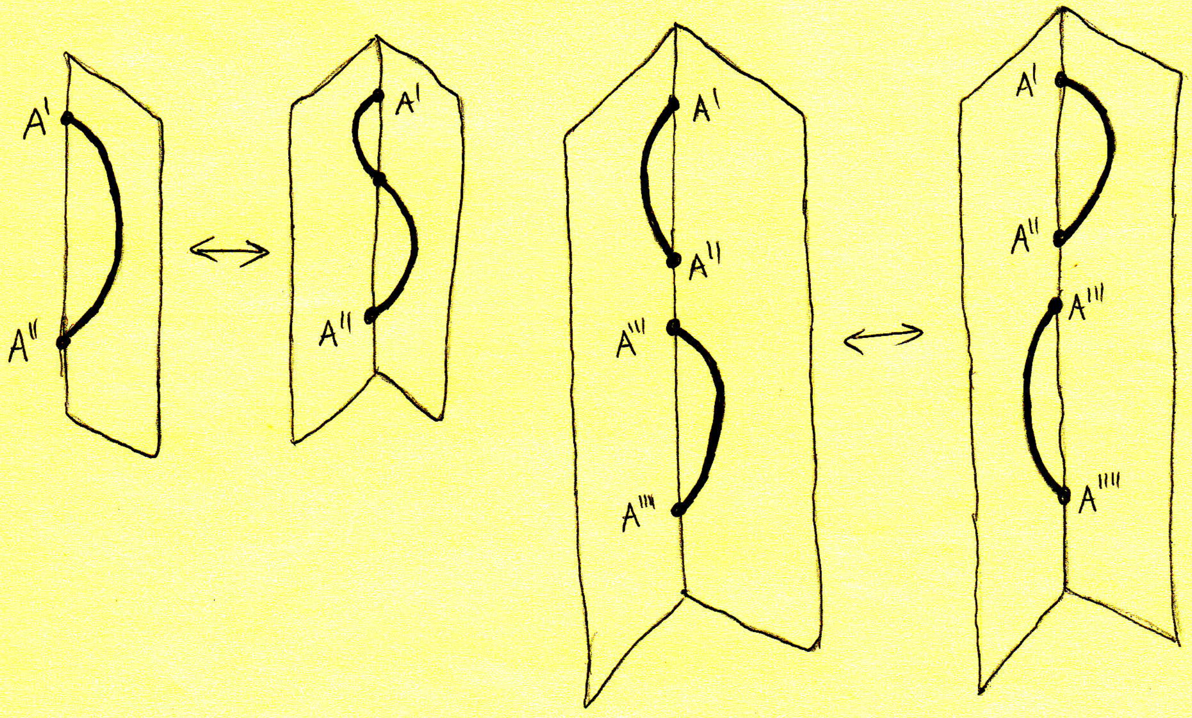


FIG. 6

The Markov's theorem follows easy from Theorem 2 if we assosiate the braid with the symmetrical diagram such as on fig. 7

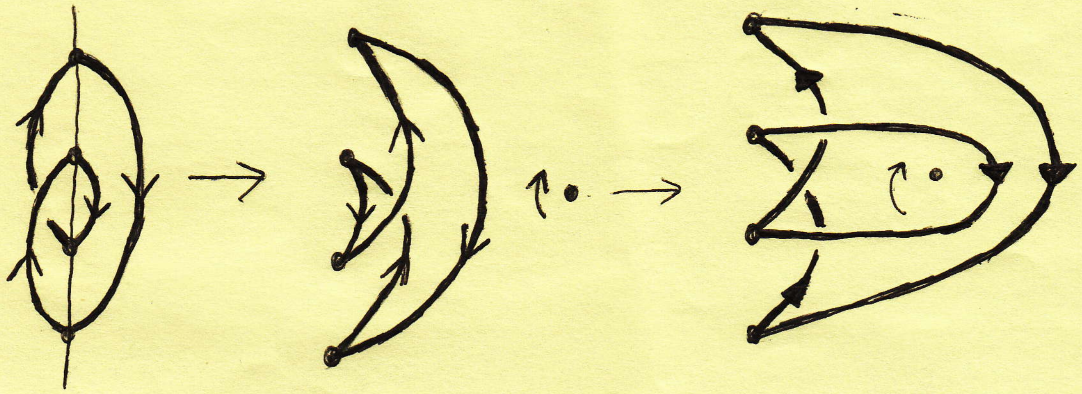


FIG. 7

Also it's easy to verify, that each S-diagram $\langle a, e\rangle$ (where e is the unit of group $S(n)$) gives positive link and each positive link has the diagram $\langle a, e\rangle$ for certain a .

THEOREM 3. Let d be a sublink in f. Then each S-diagram for d can be extended to S-diagram for f.

REMARK We can define a partial order on $S(\infty)=\bigcup_{n=1}^{\infty} S(n)$:

$\alpha, \beta \in S(\infty)$

$$\alpha \leq \beta \overset{def}{\iff} \forall \gamma \in S(\infty)\ \exists \gamma' \in S(\infty)$$

such that $\langle \alpha, \gamma\rangle \overset{isotopy}{\sim} \langle \beta, \gamma'\rangle$

2. TRIPLE IN BRAID THEORY

Fenn and Taylor {1} introduced the conception of doodle. My definition of dooodle differs from theirs. I consider that the doodle is a set of disjoint circles on S^2 without triple points of intersection (fig. 8)

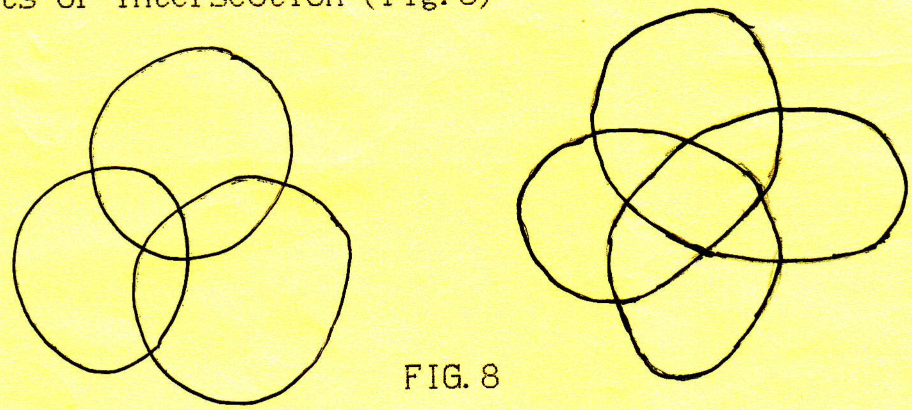


FIG. 8

The isotopy of configuration and fig. 9 moves preserve the doodle. The transformation over double point (fig. 10) is forbiden.

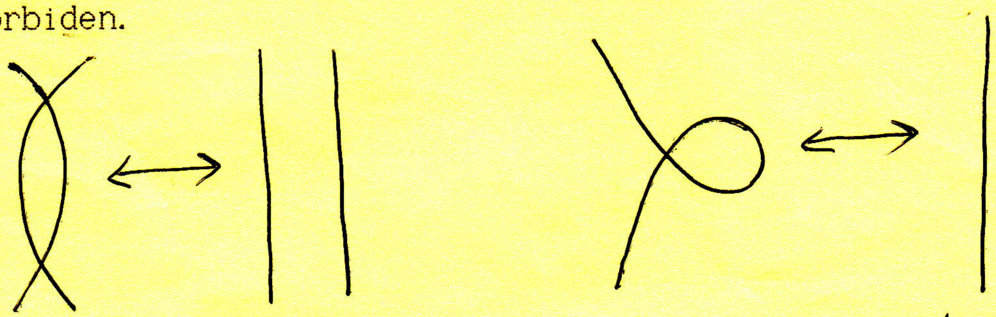


FIG. 9

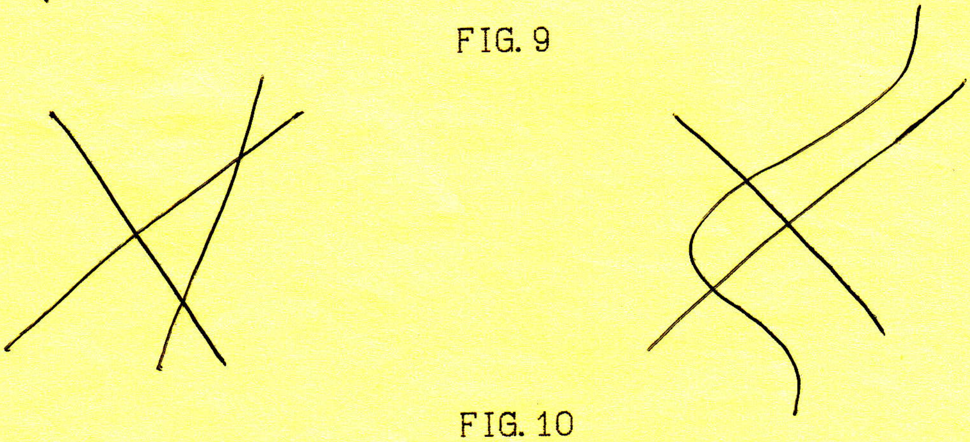
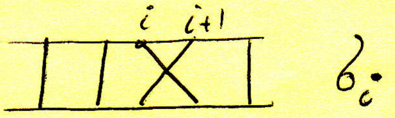


FIG. 10

Now, suppose that n+1 points on horizontal line are connected by n+1 strings with n+1 points on another horizontal line directly below the first n+1 points . The strings can't have the triple points of intersection, but, perhaps, they have double points of intersection (fig. 11)

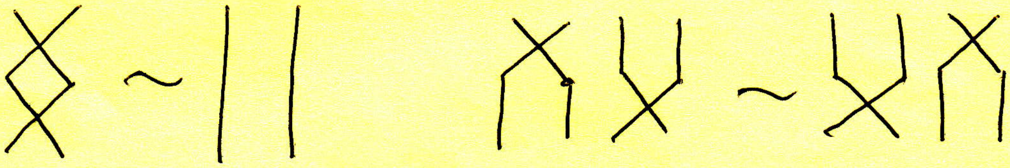
FIG. 11

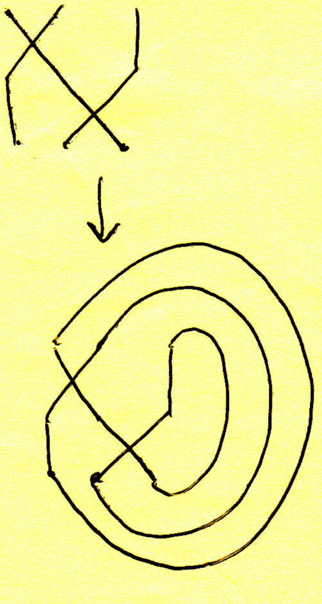
The definition of the multiplication of such configurations is the same as in braids. Then we'll get a group with generations $\sigma_1, \sigma_2, \dots, \sigma_n$ (see fig. 12) and relations



$\sigma_i^2 = 1$ $\qquad$ i= 1,2,...,n

$\sigma_i\sigma_j = \sigma_j\sigma_i$ $\qquad$ $|i-j| > 1$, $\quad$ i,j = 1,2,...,n





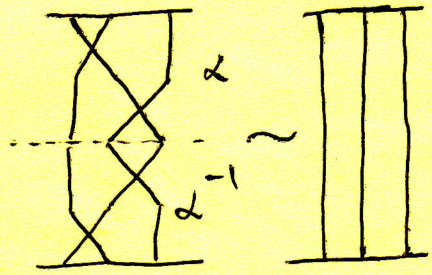


FIG. 12

Let us call this group TWIN-group (TW (n+1)) and each element a twin . The closure of the twin is ~~braid~~ doodle (fig. 12) .

THEOREM 4. Each doodle is a closure of a certain twin .

A question arises: when do two twins give the equal doodles? It's easy to see that operations

$$\alpha \Longleftrightarrow \alpha \sigma_n \sigma_{n-1} \dots \sigma_{k+1} \sigma_k \sigma_{k+1} \dots \sigma_n \qquad k \le n, \ \alpha \in TW(n)$$

$$\alpha \longleftrightarrow \alpha' \sigma'_1 \dots \sigma'_{k-1} \sigma'_k \sigma'_{k-1} \dots \sigma'_1 \qquad \alpha \in TW(n) \quad \sigma_i \to \sigma'_{i+1}, \quad TW(n) \to TW(n+1)$$

$$\alpha \longleftrightarrow g \alpha g^{-1} \qquad \alpha, g \in TW(n)$$

preserve the correspondent doodle. I would try to prove Markov's theorem for twins with the help of right analogue of symmetrical diagrams for link (see theorem 2). It should be noted that the structure of doodles is more simple than the link structure. So, the theorem is valid

THEOREM 5. Each doodle has a unique diagram with the minimal number of intersection's points (fig. 13) . This diagram can be constructed from any other doodle's diagram with the help of finite number of moves which decrease the number of intersection's points on 1 or 2 (fig. 14).

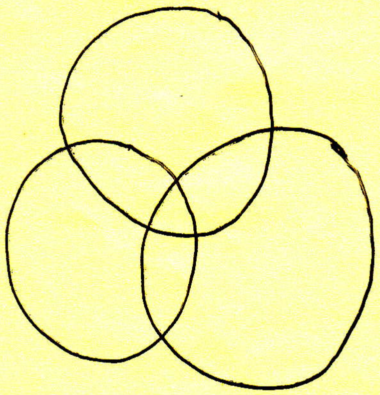


FIG. 13

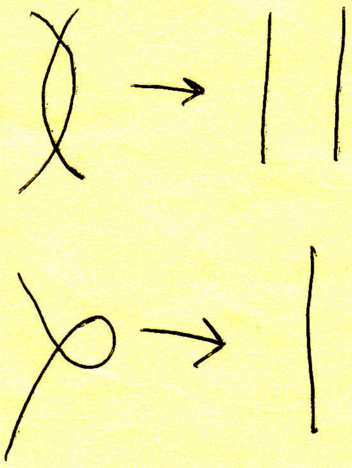


FIG. 14

After the finding of right analogue of Markov's theorem for twins, the hypotetic construction of doodles polynomials would probably used a right quantalization of Yang-Baxter equation:

$$\sigma_{i+1}\sigma_i\sigma_{i+1} = q\,\sigma_i\sigma_{i+1}\sigma_i + (1-q)\,\sigma_{i+1}$$

REMARK There is a subgroup of pure twins $TW_0(n)$. It's evident that

$$TW_0(n) \simeq \pi_1\,(\,R^n \setminus \{\, x_i = x_j = x_k \;,\; i \neq j \;,\; i \neq k \;,\; j \neq k \;,\; i,j,k = 1,\dots,n \;\; x_i \in R^1, i = 1,2,\dots,n \,\}\,) \quad R^n = (x_1,\dots,x_n),\, x_i \in R^1,\, i=1,\dots n$$

Now I recall the famous coset representation of the symmetrical group

$$\sigma_i^2 = 1 \qquad i=1,\dots,n \qquad (1)$$

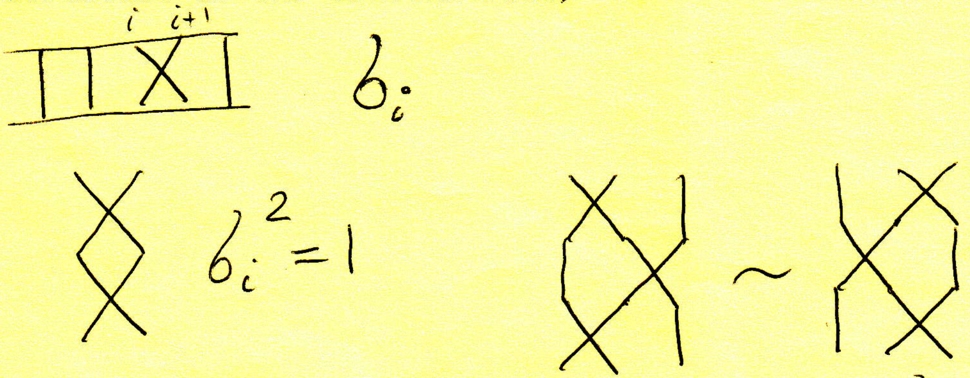
$$\sigma_i\sigma_j = \sigma_j\sigma_i \qquad |i-j|>1 \;, \quad i,j=1,\dots,n \qquad (2)$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \qquad i=1,\dots n-1 \qquad (3)$$

Relations (2) and (3) define braid group .
Relations (1) and (2) define twin group .
And what is the geometrical meaning of the group with the following coset representation:

$$\sigma_i^2 = 1 \qquad i=1,\dots,n$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \qquad i=1,\dots,n-1$$

ANSWER: Suppose n+1 points on horizontal line are connected by n+1 strings to n+1 points on another horizontal line directly below the first n+1 points. The strings may have the double and triple points of intersection, but without the four points of intersection and all intersection's points are situated on different horizontal lines (fig.15) (the structure of the trivial flow)



σ_i

$\sigma_i^2 = 1$

$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

FIG. 15

Obviously, these objects constitute the group. We will call this group -the triplet group. The theory of this group is similar to the theory of twins and braids (we will only complete the trivial flow on R^2 by the flow on S^2 with two singular points) .

Summing up , we come up with an interesting triple: braids, twins, triplets. We may say that the braid group is the space object, but other two groups are the plane objects . Can we define the doodles and triplets in space ? Yes, we can .

Let us examine the example of doodles . The braid consists of n nonintersecting strings . Now suppose that these strings have the double points of intersections and these new objects are equivalent under the fig.16 moves .

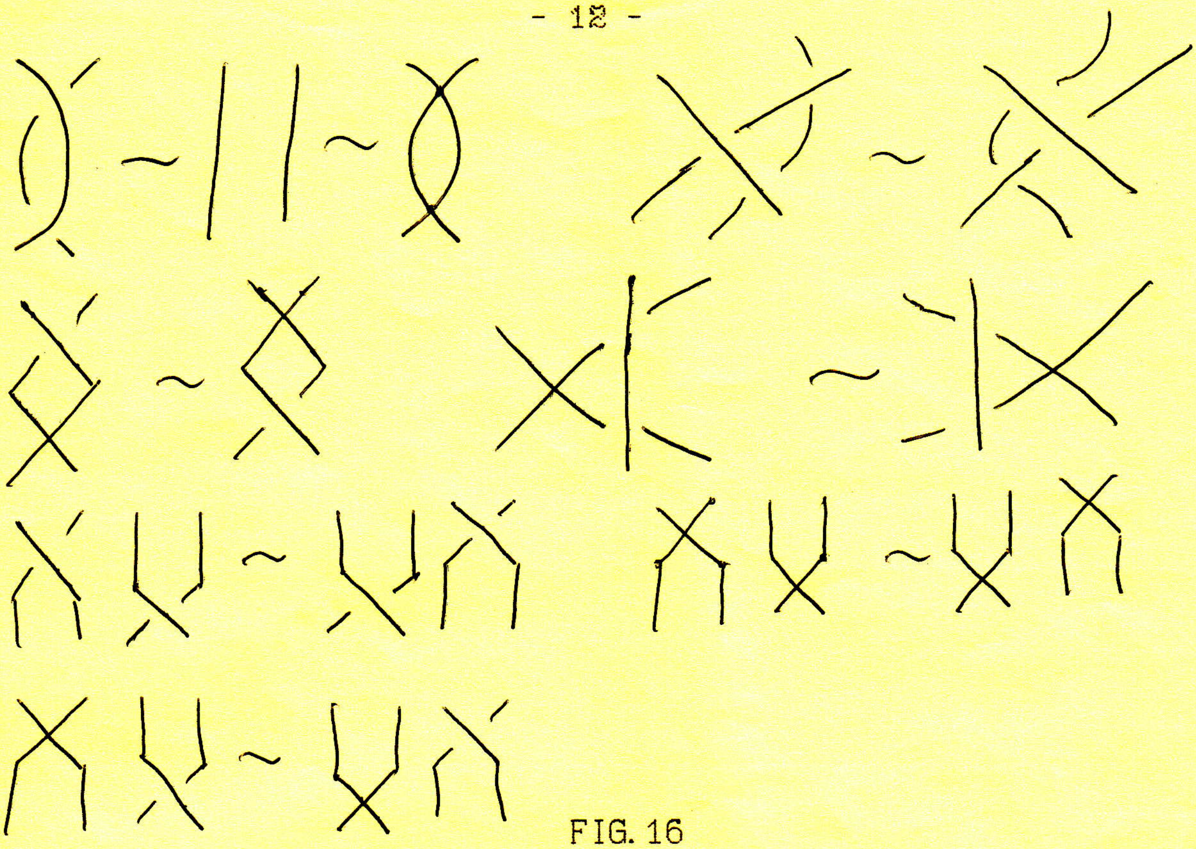


FIG. 16

This twin-braids will form the group with the following coset representation :

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad i=1,\dots,n-2 \qquad p_i p_j = p_j p_i$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad |i-j|>1 \quad i,j=1,\dots,n-1 \qquad |i-j|>1,\ i,j=1,\dots,n-1$$

$$p_i^2 = 1, \quad i=1,\dots,n-1 \qquad \sigma_{i+1} p_i \sigma_{i+1}^{-1} = \sigma_i^{-1} p_{i+1} \sigma_i \quad i=1,\dots,n-2$$

$$p_i \sigma_i = \sigma_i p_i \quad i=1,\dots,n-1 \qquad p_i \sigma_j = \sigma_j p_i \quad |i-j|>1,\ i,j=1,\dots,n-1$$

We also may construct the braid-triplet group and twin-triplet group. In the last group the following relations between twins and triplets are valid (fig. 17)

$$q_{i+1} p_i q_{i+1} = q_i p_{i+1} q_i$$

p -twin
q -triplet

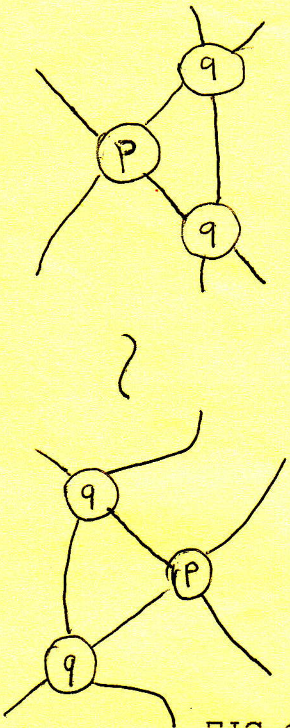


FIG. 17

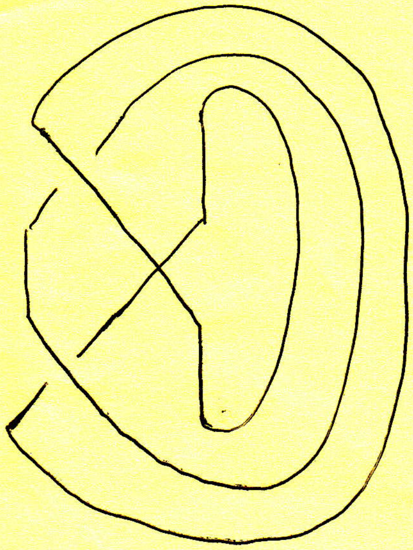


FIG. 18

Surely, there exist a braid-twin-triplet group . The braids are most free from these three types of objects, because braids may pass throught twins and triplets, and triplets are more free than twins (fig. 17) .

REMARK I think that it is interesting to find the connections between the closure braid-twin (fig. 18) and 3-manifolds. We must, probably, cut out the neighbourhood of knot-twin and paste something back .

REMARK The twin and triplet groups are the reflection groups in lobachevskii space H^p :

TW (n+1) $$\begin{cases} \sigma_i^2 = 1 & i=1,\dots,n \\ \sigma_i\sigma_j = \sigma_j\sigma_i & |i-j|>1 . \end{cases} \Leftrightarrow \begin{cases} \sigma_i^2 = 1 \\ (\sigma_i\sigma_j)^2 = 1 & |i-j|>1 \end{cases}$$

TR (n+1) $$\begin{cases} \sigma_i^2 = 1 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \end{cases} \Leftrightarrow \begin{cases} \sigma_i^2 = 1 \\ (\sigma_i\sigma_{i+1})^3 = 1 \end{cases}$$

3. SPIN-BRAIDS

We take braid group and represent each generation as the composition of two configurations (fig. 19)

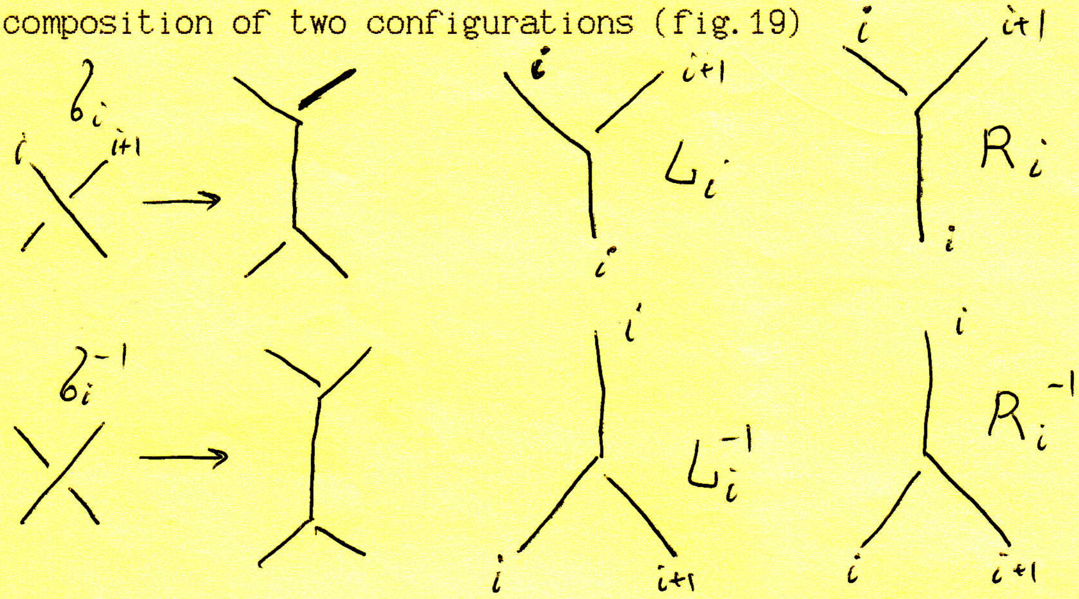


FIG. 19

So, $\sigma_i = L_i R_i^{-1}$, $\sigma_i^{-1} = R_i L_i^{-1}$.

L_i is a merging of two strings with the left string being the main one. R_i is a merging of two strings with the right string being the main one (the other string is secondary). We will call the finite production of $L_i^{\pm 1}, R_i^{\pm 1}$ ($i \in N$) a spin-braid (the word spin were suggested by my adviser prof. Yu. P. Soloviov). I suggest that fig. 20's moves give the equal spin-braids .

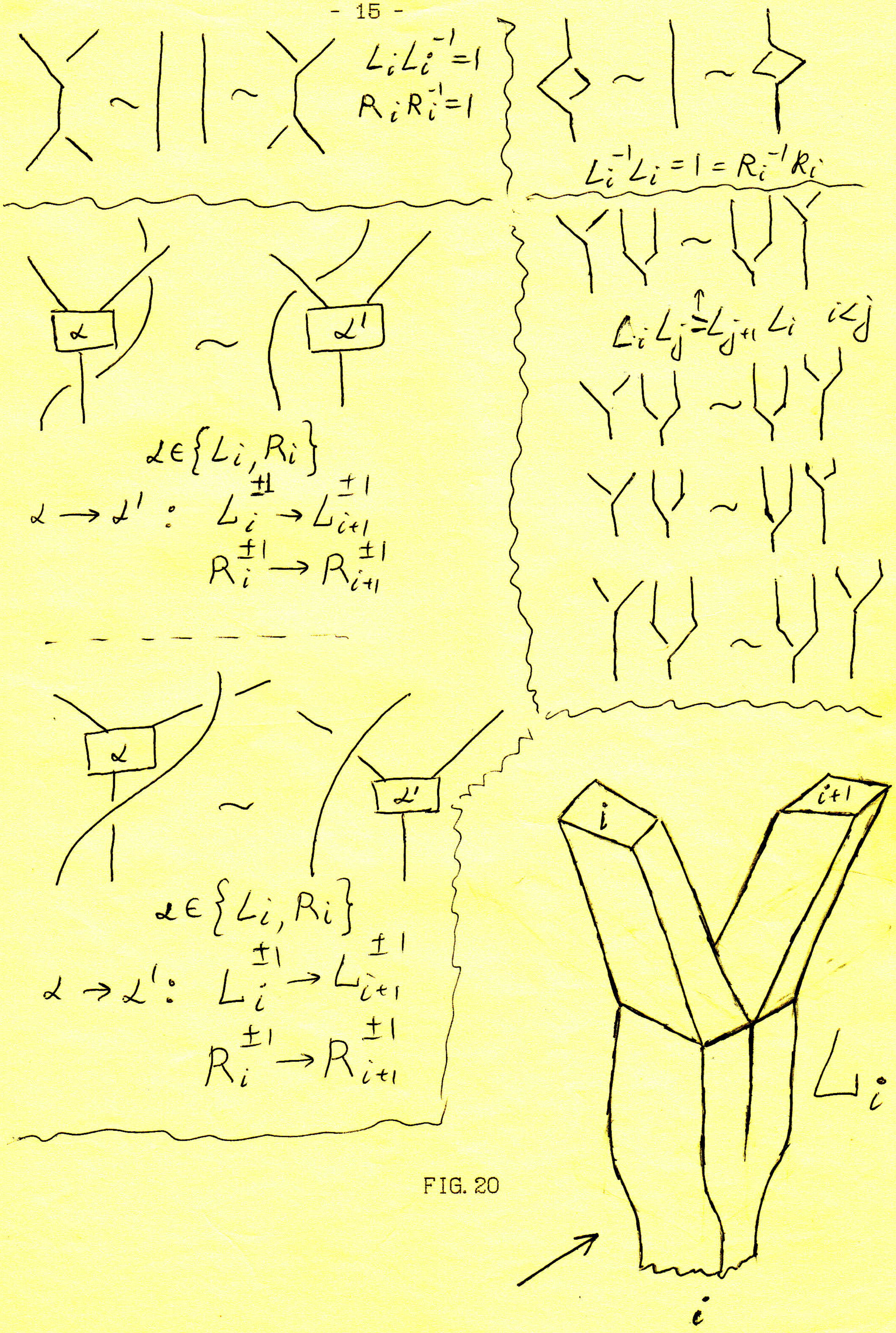

$L_i L_i^{-1} = 1$
$R_i R_i^{-1} = 1$
$L_i^{-1} L_i = 1 = R_i^{-1} R_i$
$\alpha \in \{L_i, R_i\}$
$\alpha \to \alpha'$: $L_i^{\pm 1} \to L_{i+1}^{\pm 1}$
$R_i^{\pm 1} \to R_{i+1}^{\pm 1}$
$L_i L_j \stackrel{\uparrow}{=} L_{j+1} L_i \quad i < j$
$\alpha \in \{L_i, R_i\}$
$\alpha \to \alpha'$: $L_i^{\pm 1} \to L_{i+1}^{\pm 1}$
$R_i^{\pm 1} \to R_{i+1}^{\pm 1}$
i
i+1
L_i
i

FIG. 20

It's obvious that spin-braids form the group . We will call it spin-braid group (SPINB) . By definition, SPINB group has the following generations and relations:

$$L_i L_j = L_{j+1} L_i \quad i,j \in \mathbb{N},\ j>i \qquad R_{i+1} L_{i+1}^{-1} L_i L_i R_i^{-1} = L_i R_i^{-1} L_{i+1}$$

$$L_i R_j = R_{j+1} L_i \quad i,j \in \mathbb{N},\ j>i \qquad R_{i+1} L_{i+1}^{-1} R_i L_i R_i^{-1} = L_i R_i^{-1} R_{i+1}$$

$$R_i L_j = L_{j+1} R_i \quad i,j \in \mathbb{N},\ j>i \qquad i \in \mathbb{N} \qquad (\ast)$$

$$R_i R_j = R_{j+1} R_i \quad i,j \in \mathbb{N},\ j>i \qquad L_{i+1} R_{i+1}^{-1} L_i R_i L_i^{-1} = R_i L_i^{-1} L_{i+1}$$

$$L_{i+1} R_{i+1}^{-1} R_i R_i L_i^{-1} = R_i L_i^{-1} R_{i+1}$$

REMARK There is a natural inclusion: $B(\infty) \longrightarrow$ SPINB

$\forall\ i \in N \qquad b_i \longrightarrow L_i R_i^{-1}$

REMARK There is a pure spin-braid group (SPINB_0)-the subgroup of SPINB group . The geometrical meaning of pure spin-braids is that each substring of each string must return to her original place. For instance, the spin-braid in fig. 21 is not pure spin-braid .

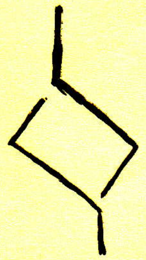


FIG. 21

The pure spin-braid group is, obviously, normal in SPINB . Therefore, we can define spin-symmetrical group (SPINS)-the group $\mathrm{SPINB}/\mathrm{SPINB}_0$.

There is a short exact sequence

$$1 \to \mathrm{SPINB}_0 \to \mathrm{SPINB} \to \mathrm{SPINS} \to 1$$

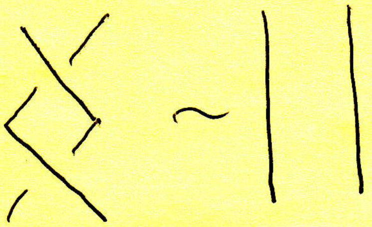
SPINS is a group of "quasiautomorphisms" $f: N \to N$.

That is, each element of N has the image consisting of finite number of elements, and the elements of prototype of each element have a specific order (this is for first n_1 and n_2 elements of N ;for other elements $\forall\ m > n_1\ \ f(m) = \{m + (n_2 - n_1)\}$, $\forall k > n_2\ \ f^{-1}(k) = \{k + (n_1 - n_2)\}$).

The coset representation of SPINS is the coset representation of spin-braid group with the supplementary relation (fig. 22)

$$(R_1 L_1^{-1})^{-2}=1 \;(\Rightarrow (R_i L_i^{-1})^{-2}=1 \quad \forall i \in N)$$

$$\wr$$



$$\leftarrow (L_1 R_1^{-1})^2 = 1$$

FIG. 22

There is a natural inslusion:

$S(\infty) \longrightarrow SPINS \qquad (S(\infty) = \bigcup_{i=1}^{\infty} S(i)$

$\sigma_i \in S(\infty) \qquad \sigma_i \longrightarrow L_i R_i^{-1}$

And there is a map from one short exact sequence to other short exact sequence:

$$\begin{tikzcd}
1 \arrow[r] & B_0(\infty) \arrow[r] \arrow[d] & B(\infty) \arrow[r] \arrow[d] & S(\infty) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & SPINB_0 \arrow[r] & SPINB \arrow[r] & SPINS \arrow[r] & 1
\end{tikzcd}$$

The elements which belong to SPINS of finite order are not only the elements of $S(\infty)$ (fig. 23 for example)

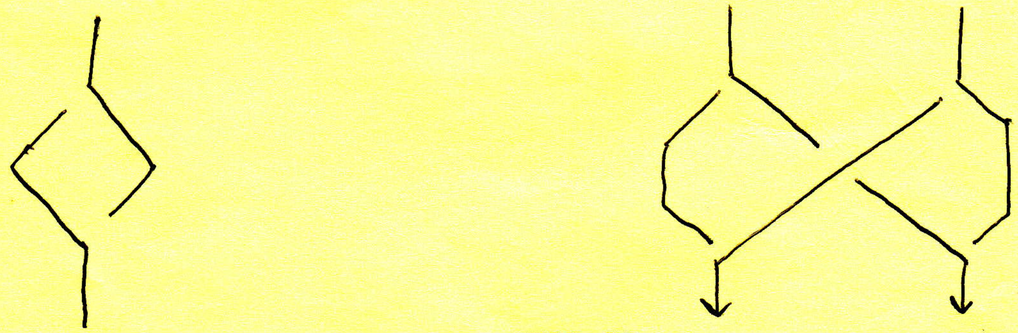


FIG. 23

It's easy to give a nesessary and sufficient condition, when the element of SPINS has a finite degree .

REMARK We may also define the spin-twin and spin-triplet groups and spin-braid-twin-triplet group .

REMARK You may see from ($\star$) that $L_3, L_4, \ldots$ $R_2, R_3, \ldots$ are expressed throught L_1, L_2, R_1 .

REMARK We may construct a link as a closure of a braid . Similarly , we may construct a spin-link from a spin-braid(fig. 24) .

FIG. 24

QUESTION Construct the theory of spin-links (Dehn spin-surgery , Markov's theorem ,Jones polynomial and so on).

The author is grateful to A. Radul and E. Cheporova for translation this text to English . The author is also grateful to I. A. Bass for the help in the design of this article.

{1} R. Fenn, P. Taylor Introduction in doodles Lect. Notes. in Math. ,722, p. 37-43 .