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Towards functor exponentiation

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A R T I C L E   I N F O

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We consider a possible framework to categorify the exponential map $\exp(-f)$ given the categorification of a generator f of $\mathfrak{sl}_2$ by Lauda. In this setup the Taylor expansions of $\exp(-f)$ and $\exp(f)$ turn into complexes built out of categorified divided powers of f. Hom spaces between tensor powers of categorified f are given by diagrammatics combining nilHecke algebra relations with those for a additional "short strand" generator. The proposed framework is only an approximation to categorification of exponentiation, because the functors categorifying $\exp(f)$ and $\exp(-f)$ are not invertible.

1. Introduction

The exponential function is fundamental in mathematics. In Lie theory, the exponential map connects a Lie algebra and its Lie group. Idempotented version of quantized universal enveloping algebras of simple Lie algebras have been categorified [3,4,8]. This

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paper can be viewed as a small step towards lifting the exponential map to the categorical level.

We focus on the case of $\mathfrak{sl}_2$. Consider the expansion

$$\exp(-f) = \sum_{k\geq 0}(-1)^k \frac{f^k}{k!},$$

in a completion of the universal enveloping algebra of $\mathfrak{sl}_2$, where $f \in \mathfrak{sl}_2$ is a Chevalley generator of the lower-triangular matrix. Categorification of the divided power $f^{(k)} = \frac{f^k}{k!}$ and of its quantized version $\frac{f^k}{[k]!}$ naturally appears in the categorified quantum $\mathfrak{sl}_2$ [7]. The generator f is lifted to a bimodule $\mathcal{F}'$ over a direct sum of the nilHecke algebras $\bigoplus_{n\geq 0} NH_n$. The tensor powers $\mathcal{F}'^{\,k} = \mathcal{F}'^{\,\otimes k}$ of the bimodule admit a direct sum decomposition $\mathcal{F}'^{\,k} \cong \bigoplus_{k!} \mathcal{F}'^{\,(k)}$. It is natural to expect lifting $\exp(-f)$ to a cochain complex whose degree k component is $\mathcal{F}'^{\,(k)}$ for $k \geq 0$. A nontrivial differential is needed to link adjacent components.

The diagrammatic approach is widely used in categorification and plays a significant role in the present paper as well. We provide a modification $\widetilde{NH}$ of $\bigoplus_{n\geq 0} NH_n$ by adding an extra generator to the nilHecke algebras The new generator is described by a short strand which links NH_n and NH_{n+1} together. The induction $\widetilde{NH}$-bimodule still exists, denoted $\mathcal{F}$. Short strand induces a $\widetilde{NH}$-bimodule homomorphism $\widetilde{NH} \to \mathcal{F}$. This morphism and its suitable generalizations $\mathcal{F}^k \to \mathcal{F}^{k+1}$ define a differential on $\bigoplus_{k\geq 0} \mathcal{F}^{(k)}[-k]$. The resulting complex descends to an alternating sum $\sum_{k\geq 0}(-1)^k[\mathcal{F}^{(k)}]$ in the Grothendieck ring of the derived category of $\widetilde{NH}$-bimodules.

Due to the existence of the short strand, the extension group $\mathrm{Ext}^1(\mathcal{F}, \widetilde{NH})$ of bimodules is nontrivial. We lift the expansion $\exp(f) = \sum_{k\geq 0} f^{(k)}$ to a complex $\left(\bigoplus_{k\geq 0} \mathcal{F}^{(k)}, d\right)$, where the differential d consists of certain elements of Ext^1-groups. The absence of the alternating sign in the expansion of $\exp(f)$ comes from the fact that it cancels against the sign coming from the use of Ext^1-groups.

Unfortunately, the two resulting complexes, lifting $\exp(-f)$ and $\exp(f)$, respectively, are not invertible, as explained in Section 4. A more elaborate or just a different construction is needed to more adequately categorify exponentiation.

Problem 1.1. Find a framework for categorification of the exponential map, where an object $\mathcal{F}$ in a monoidal triangulated category $\mathcal{C}$ lifts to two invertible objects $\exp(\mathcal{F})$ and $\exp(-\mathcal{F})$ in some monoidal triangulated category $\mathcal{C}^e$. The objects should descend to $\exp([\mathcal{F}])$ and $\exp(-[\mathcal{F}])$ in the Grothendieck ring of $\mathcal{C}^e$, where $[\mathcal{F}]$ is the class of $\mathcal{F}$ in the Grothendieck ring of $\mathcal{C}$. The Grothendieck rings of $\mathcal{C}$ and $\mathcal{C}^e$ should be suitably related.

The recently developed theory of stability in representations of the symmetric group (see [1], for instance) can be related to a diagrammatical category similar to the one for $\widetilde{NH}$. The nilHecke algebra NH_n should be replaced by the group algebra $\mathbb{C}[S_n]$ of the symmetric group. Adding a short strand with suitable sliding and commutativity relations will enlarge the direct sum $\oplus_{n\geq 0}\mathbb{C}[S_n]$ to a non-unital idempotented algebra $\widetilde{S}$. Alternatively, this idempotented algebra can be viewed as describing a monoidal category with a single generating object. One of the first key results in the stable representation theory of S_n can be restated as the theorem that the category of finitely-generated right $\widetilde{S}$ modules is Noetherian.

The first author used endofunctors in the category of finite-dimensional $\oplus_{n\geq 0}\mathbb{C}[S_n]$ modules to categorify the Heisenberg algebra [2]. The second author studied the category of finite dimensional left $\widetilde{S}$ modules to build a categorical boson–fermion correspondence [9]. Short strands were also used in diagrammatic categorifications of the polynomial ring in [5], and the ring of integers localized at two in [6].

One possible application of categorified exponentiation would be the categorification of integral forms of Lie groups and the exponential map between a Lie algebra and its Lie group. It might also be useful for categorification of Vassiliev invariants, where parameter h appears as the logarithm of q. After categorification q becomes the grading shift, and some sophisticated structure refining the shift functor would be needed to define its logarithm.

2. The algebra

2.1. The definition of $\widetilde{NH}$

Definition 2.1. Define an algebra $\widetilde{NH}$ by generators 1_n for $n \geq 0$, $\partial_{i,n}$ for $1 \leq i \leq n-1$, $x_{i,n}$ for $1 \leq i \leq n$, and $v_{i,n}$ for $1 \leq i \leq n$, subject to the relations consisting of three groups:

(1) Idempotent relations:

$$1_n 1_m = \delta_{n,m} 1_n, \qquad 1_n \partial_{i,n} = \partial_{i,n} 1_n = \partial_{i,n},$$
$$1_n x_{i,n} = x_{i,n} 1_n = x_{i,n}, \qquad 1_{n-1} v_{i,n} = v_{i,n} 1_n = v_{i,n}.$$

(2) NilHecke relations:

$$x_{i,n} x_{j,n} = x_{j,n} x_{i,n},$$
$$x_{i,n}\partial_{j,n} = \partial_{j,n} x_{i,n} \quad \text{if } |i-j|>1, \qquad \partial_{i,n}\partial_{j,n} = \partial_{j,n}\partial_{i,n} \quad \text{if } |i-j|>1,$$
$$\partial_{i,n}\partial_{i,n} = 0, \qquad \partial_{i,n}\partial_{i+1,n}\partial_{i,n} = \partial_{i+1,n}\partial_{i,n}\partial_{i+1,n},$$
$$x_{i,n}\partial_{i,n} - \partial_{i,n} x_{i+1,n} = 1_n, \qquad \partial_{i,n} x_{i,n} - x_{i+1,n}\partial_{i,n} = 1_n.$$

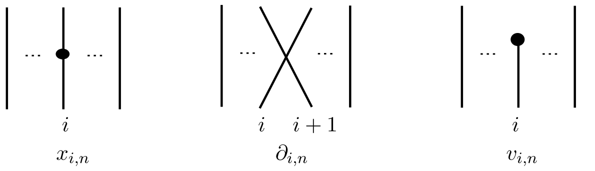

Fig. 1. Generators of $\widetilde{NH}$.

(3) Short strands relations:

$$\begin{aligned} v_{i,n}x_{j,n} &= x_{j,n-1}v_{i,n} \quad \text{if } i > j, & v_{i,n}x_{j,n} &= x_{j-1,n-1}v_{i,n} \quad \text{if } i < j, \\ v_{i,n}\partial_{j,n} &= \partial_{j,n-1}v_{i,n}, \quad \text{if } i > j+1, & v_{i,n}\partial_{j,n} &= \partial_{j-1,n-1}v_{i,n} \quad \text{if } i < j, \\ v_{i,n}v_{j,n+1} &= v_{j,n}v_{i+1,n+1} \quad \text{if } i \geq j. & & \\ v_{i,n}\partial_{i,n} &= v_{i+1,n}\partial_{i,n}. & &\text{(Exchange relation)} \end{aligned}$$

The algebra $\widetilde{NH}$ can be described diagrammatically. The idempotent 1_n is denoted by n vertical strands. In particular, 1_0 is denoted by the empty diagram. The generator $x_{i,n}$ is denoted by n vertical strands with a dot on the i-th strand, and $\partial_{i,n}$ is denoted by n strands with a $(i, i+1)$ crossing. The new generator $v_{i,n}$ is denoted by a diagram with $n-1$ vertical strands and one short strand in the i-th position. The short strand has no endpoint at the top and one endpoint at the bottom, see Fig. 1.

The product ab of two diagrams a and b is a vertical concatenation of a and b, where a is at the top, b is at the bottom. The product is zero unless the numbers of their endpoints match.

The relations of the second group are the defining relations of the nilHecke algebras. The relations of the third group are about short strands. The first three lines are isotopy relations of disjoint diagrams. The last line says that the short strand is exchangeable between i-th and $(i+1)$-th positions when composing with the crossing. We call it the *exchange relation*. In addition to the isotopy relations of disjoint diagrams, other local relations are drawn in Fig. 2.

Let $a \odot b \in \widetilde{NH}$ denote a horizontal concatenation of a and b, where a is on the left, b is on the right. The element $a \odot b$ does not depend on the heights of a and b by the isotopy relations of disjoint diagrams.

The algebra $\widetilde{NH}$ is idempotented, i.e. has a complete system of mutually orthogonal idempotents $\{1_n\}_{n\geq 0}$, so that

$$\widetilde{NH} = \bigoplus_{m,n\geq 0} 1_m\widetilde{NH}1_n.$$

Let $\widetilde{NH}_n^m$ denote its component $1_m\widetilde{NH}1_n$. It is spanned by diagrams with m endpoints at the top and n endpoints at the bottom. The new generator $v_{i,n} \in \widetilde{NH}_n^{n-1}$. Clearly, $\widetilde{NH}_n^m = 0$ if $n < m$. Let NH_n denote the nilHecke algebra with the generators x_i and ∂_i.

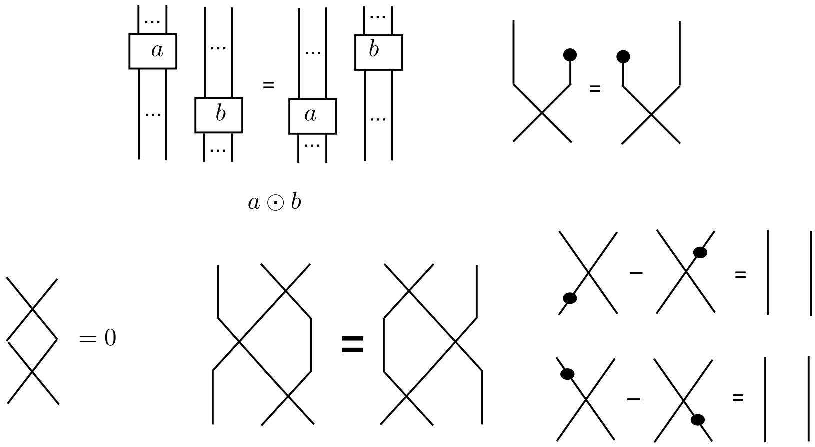

Fig. 2. Local relations of $\widetilde{NH}$.

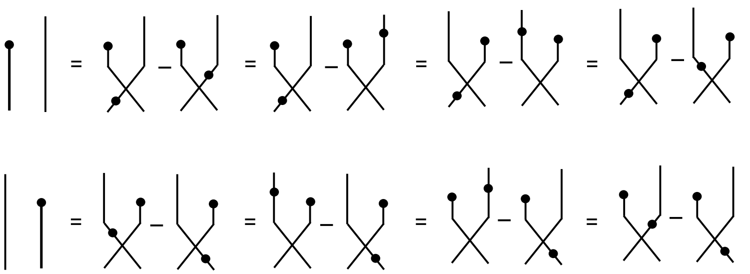

Fig. 3. Two induced relations.

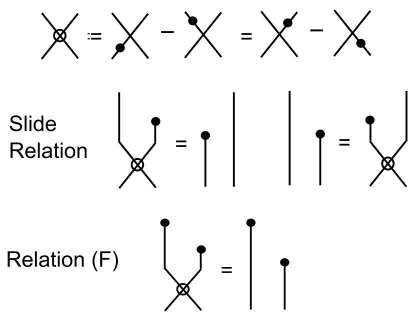

Fig. 4. A new diagram $s_{1,2}$, the slide relation, and the relation (F).

There is a surjective homomorphism $\psi_n : NH_n \to \widetilde{NH}_n^n$ defined by $\psi_n(1) = 1_n, \psi_n(x_i) = x_{i,n}, \psi_n(\partial_i) = \partial_{i,n}$. We will prove that ψ_n is an isomorphism in Section 2.2.

A combination of the exchange relation, the isotopy relation and the nilHecke relations implies the following relations, see Fig. 3.

Motivated by these relations, we introduce a new diagram as a circle crossing, see Fig. 4. It defines an element

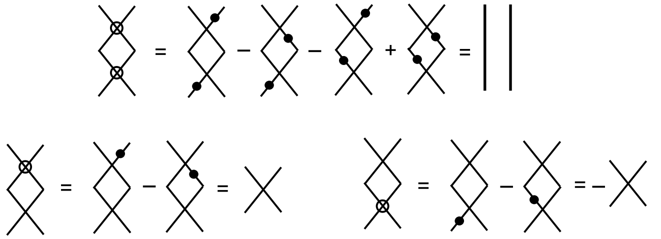

Fig. 5. Local relations about the circle crossing.

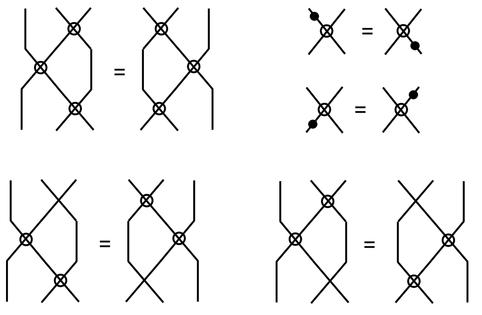

Fig. 6. More local relations for the circle crossings.

$$s_{1,2} := \partial_{1,2}x_{1,2} - x_{1,2}\partial_{1,2} = x_{2,2}\partial_{1,2} - \partial_{1,2}x_{2,2} \in NH_2^2. \tag{1}$$

The relations in Fig. 3 can be rewritten in terms of the new diagram $s_{1,2}$. The short strand can slide through the circle crossing. We call this relation the *slide relation*. As a corollary, the horizontal concatenation of two short strands is fixed by the circle crossing. We call it the *relation (F)*.

We discuss the interaction of the new diagram $s_{1,2}$ with the generators of $\widetilde{NH}$ in the following. Firstly, we check that $s_{1,2}s_{1,2} = 1_2$, and

$$s_{1,2}\partial_{1,2} = \partial_{1,2}, \qquad \partial_{1,2}s_{1,2} = -\partial_{1,2}, \tag{2}$$

using diagrams, see Fig. 5. Secondly, we use the nilHecke relations to deduce more relations, see Fig. 6. For $1 \le i \le n-1$, define

$$s_{i,n} = 1_{i-1} \odot s_{1,2} \odot 1_{n-i-1} \in NH_n^n$$

by adding $i-1$ and $n-i-1$ vertical strands to the left and right of $s_{1,2}$, respectively.

The diagrams used in defining $s_{1,2}$ do not have the short strand. The same diagram defines an element $s_1 := \partial_1 x_1 - x_1\partial_1 \in NH_2$ such that $\psi_2(s_1) = s_{1,2}$. Similarly, define $s_i := \partial_i x_i - x_i\partial_i \in NH_n$ such that $\psi_n(s_i) = s_{i,n}$.

Lemma 2.2.

(1) For $1 \leq i \leq n-1$, the generators $s_{i,n}$ satisfy the defining relations of the symmetric group S_n.
(2) A dot can slide through the circle crossing, and a crossing and two circle crossings satisfy the braid relation, see Fig. 6.

Proof. Under the map $\psi_n : NH_n \to \widetilde{NH}^n_n$, it is enough to check the corresponding relations in NH_n. Consider the action of NH_n on the ring $\mathbf{k}[x_1, \ldots, x_n]$ of polynomials, where ∂_i acts as the divided difference operator, and x_i acts as the multiplication by x_i. For $f \in \mathbf{k}[x_1, \ldots, x_n]$,

$$\begin{aligned} s_i(f) &= \partial_i x_i(f) - x_i \partial_i(f) = \partial_i(x_i f) - x_i \partial_i(f) \\ &= \frac{x_i f - x_{i+1}\widetilde{s}_i(f)}{x_i - x_{i+1}} - x_i \cdot \frac{f - \widetilde{s}_i(f)}{x_i - x_{i+1}} = \widetilde{s}_i(f), \end{aligned}$$

where, $\widetilde{s}_i(f)(\ldots, x_i, x_{i+1}, \ldots) = f(\ldots, x_{i+1}, x_i, \ldots)$. In other words, the operator induced by s_i is the same as the operator $\widetilde{s}_i$ on $\mathbf{k}[x_1, \ldots, x_n]$. The relations hold in NH_n since the action is faithful. $\square$

The slide relation in Fig. 4 says that the short strand can slide through the circle crossing. Thus, we can reduce the generators of $\widetilde{NH}$ in Definition 2.1 to $1_n, \partial_{i,n}, x_{i,n}$, and $v_{n,n}$. Here, $v_{n,n}$ is the diagram with the short strand in the rightmost position.

Notation: Let $v_n = v_{n,n} \in \widetilde{NH}^{n-1}_n$ for simplicity.

Definition 2.3. Define a $\mathbf{k}$-algebra $\widetilde{NH}'$ by generators 1_n for $n \geq 0$, $\partial_{i,n}$ for $1 \leq i \leq n-1$, $x_{i,n}$ for $1 \leq i \leq n$, and v_n for $n \geq 0$, subject to the relations consisting of three groups:

(1) Idempotent relations: the same as in Definition 2.1 (1), except that $v_{i,n}$ is replaced by v_n.
(2) NilHecke relations: the same as in Definition 2.1 (2).
(3) Short strands relations:

$$\begin{aligned} v_n x_{j,n} &= x_{j,n-1} v_n, && \text{if } j < n, \\ v_n \partial_{j,n} &= \partial_{j,n-1} v_n, && \text{if } j < n-1, \\ v_{n-1} v_n &= v_{n-1} v_n s_{n-1,n}, && \text{(Relation (F))} \end{aligned}$$

where $s_{n-1,n} = \partial_{n-1,n} x_{n-1,n} - x_{n-1,n} \partial_{n-1,n} = x_{n,n} \partial_{n-1,n} - \partial_{n-1,n} x_{n,n}$.

Lemma 2.4. *The algebras $\widetilde{NH}$ and $\widetilde{NH}'$ are naturally isomorphic.*

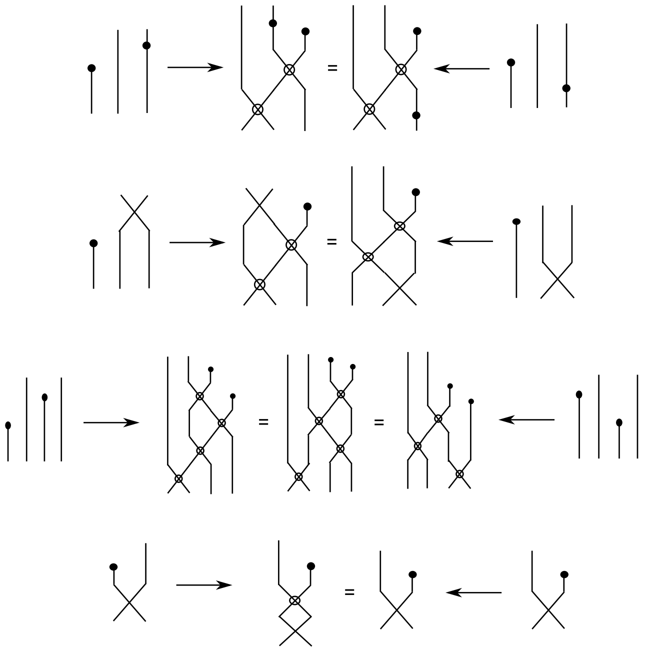

Fig. 7. Checking ϕ preserves the short strand relations, where the diagrams on the left and right sides are in $\widetilde{NH}$, the arrows represent the map ϕ, and the diagrams in the middle between the arrows are in $\widetilde{NH}'$.

Proof. The algebra $\widetilde{NH}'$ has fewer generators and relations than $\widetilde{NH}$, except for the last relation in $\widetilde{NH}'$. This relation is a direct consequence of the relation (F) in Fig. 4. Therefore, there is a homomorphism $\widetilde{NH}' \to \widetilde{NH}$.

We construct a map $\phi : \widetilde{NH} \to \widetilde{NH}'$ in the opposite direction as follows. On generators, define $\phi(a) = a$ for $a = 1_n, \partial_{i,n}, x_{i,n}$, and

$$\phi(v_{n,n}) = v_n, \qquad \phi(v_{i,n}) = v_n s_{n-1,n} \cdots s_{i+1,n} s_{i,n}.$$

The definition of $\phi(v_{i,n})$ is motivated by the slide relation. We need to prove that ϕ respects the defining relations of $\widetilde{NH}$. This is clear for the relations (1), (2) in Definition 2.1. Consider the short strand relations, see Definition 2.1 (3). In Fig. 7, we check some special cases using the relations in Figs. 4, 5 and 6. For $v_{i,n}x_{j,n} = x_{j-1,n-1}v_{i,n}$ when $i < j$, we check that

$$\begin{aligned}\phi(v_{i,n})\phi(x_{j,n}) &= v_n s_{n-1,n} \cdots s_{i+1,n} s_{i,n} x_{j,n} = v_n x_{j-1,n} s_{n-1,n} \cdots s_{i+1,n} s_{i,n} \\ &= x_{j-1,n-1} v_n s_{n-1,n} \cdots s_{i+1,n} s_{i,n} = \phi(x_{j-1,n-1})\phi(v_{i,n}).\end{aligned}$$

The proof for the general cases is similar and left to the reader.

It is obvious that the two maps between $\widetilde{NH}$ and $\widetilde{NH}'$ are inverse to each other. □

We will identify $\widetilde{NH}'$ with $\widetilde{NH}$ under the isomorphism in the lemma. Definition 2.3 has fewer generators and relations, while Definition 2.1 is more symmetric and local diagrammatically. Moreover, it is clear that Definition 2.1 describes Hom spaces between powers of a generating object of a suitable monoidal category. We will use both presentations.

2.2. The basis

We first construct an action of $\widetilde{NH}$ which generalizes the action of nilHecke algebras on the rings of polynomials. Let

$$R = \mathbb{Z}[r_1, r_2, \dots] \tag{3}$$

be a graded ring of polynomials in infinitely many variables, where $\deg(r_i) = 2i$. Let $R_n = R[x_1, \dots, x_n]$ be the graded ring of polynomials over R, where $\deg(x_i) = 2$.

In the following, we define an action α of $\widetilde{NH} = \bigoplus_{m,n\geq 0} \widetilde{NH}_n^m$ on $\widetilde{R} = \bigoplus_{n\geq 0} R_n$, where on each component $\alpha : \widetilde{NH}_n^m \to \mathrm{Hom}_R(R_n, R_m)$. For $m = n$, the action is the same as the nilHecke action on R_n:

$$\alpha(x_{i,n})(f) = x_i f, \qquad \alpha(\partial_{i,n})(f) = \frac{f - s_i(f)}{x_i - x_{i+1}},$$

for $f \in R_n$. For $v_n \in \widetilde{NH}_n^{n-1}$, the operator $\alpha(v_n) : R_n \to R_{n-1}$ is defined on the R-basis $\{x_1^{i_1} \dots x_{n-1}^{i_{n-1}} x_n^{i_n}\}$ of R_n by

$$\alpha(v_n)(x_1^{i_1} \dots x_{n-1}^{i_{n-1}} x_n^{i_n}) = x_1^{i_1} \dots x_{n-1}^{i_{n-1}} r_{i_n} \in R_{n-1}. \tag{4}$$

In other words, $\alpha(v_n)$ fixes the first $n-1$ variables, and maps x_n^k to r_k.

To check that the action is well-defined, we use the presentation of $\widetilde{NH}$ in Definition 2.3. Since $\alpha(v_n)$ acts nontrivially only on the variable x_n, all isotopy relations and nilHecke relations are preserved under α. For the relation (F), both sides map $x_1^{i_1} \dots x_{n-2}^{i_{n-2}} x_{n-1}^{i_{n-1}} x_n^{i_n}$ to $x_1^{i_1} \dots x_{n-2}^{i_{n-2}} r_{i_{n-1}} r_{i_n}$.

To compute the basis of $\widetilde{NH}_n^m$, we first discuss the case $m = 0$. Recall from Lemma 2.2 that the $s_{i,n}$'s generate a symmetric group $S_n \subset NH_n$. Let

$$V_n = \mathrm{Ind}_{\mathbb{Z}[S_n]}^{NH_n} \mathbb{Z}$$

be the right NH_n-module induced from the trivial right $\mathbb{Z}[S_n]$-module $\mathbb{Z}$. Let $NH_n \to V_n$ denote the quotient map which takes $t \in NH_n$ to $\overline{t} \in V_n$. As an abelian group, V_n is generated by elements $\overline{t}$ for $t \in NH_n$, modulo the relation $\overline{wt} = \overline{t}$ for $w \in S_n$.

Let $v_n^0 = v_1 v_2 \cdots v_n \in \widetilde{NH}_n^0$ denote the diagram of n short strands. Define a map of abelian groups:

$$\begin{aligned} \eta_n : V_n &\to \widetilde{NH}_n^0 \\ \bar{t} &\mapsto v_n^0 t. \end{aligned} \tag{5}$$

The map η_n is well-defined since $\eta_n(\overline{wt}) = v_n^0 wt = v_n^0 t = \eta_n(\bar{t})$ for $w \in S_n$ from the relation (F), see Fig. 4. It is surjective by definition. The map η_n is actually a homomorphism of right NH_n-modules. Note that both V_n and $\widetilde{NH}_n^0$ are graded. Let $V_n(m)$ and $\widetilde{NH}_n^0(m)$ be their degree $2m$ components.

We will prove that V_n is a free abelian group and show that η_n is an isomorphism in the following. Recall some results about the action of NH_n on $\mathcal{P}_n = \mathbb{Z}[x_1, \ldots, x_n]$, see [7, Section 3.2] for more detail. Let $\Lambda_n \subset \mathcal{P}_n$ denote the subring of symmetric polynomials. Let

$$\mathcal{H}_n = \{x_1^{j_1} \ldots x_n^{j_n} \mid 0 \le j_k \le n-k\},$$

and $\mathcal{H}_n(m)$ be its subset of elements of degree $2m$. Then $\mathcal{P}_n$ is a finite-rank free module over Λ_n with basis $\mathcal{H}_n$. There is a canonical isomorphism of rings

$$NH_n \cong \mathrm{End}_{\Lambda_n}(\mathcal{P}_n).$$

Let $\partial_w = \partial_{i_1} \cdots \partial_{i_l} \in NH_n$ for a reduced word expression of $w = s_{i_1} \cdots s_{i_l} \in S_n$. The number l is called the length $l(w)$ of w. Define

$$t_{g,w} = g \cdot \partial_w \in NH_n,$$

for $w \in S_n, g \in \mathcal{P}_n$. The collection $\{t_{g,w} \mid w \in S_n, g \text{ a monomial in } \mathcal{P}_n\}$ forms a $\mathbb{Z}$-basis of NH_n, see [7, Proposition 3.5]. Let

$$\mathcal{F}_n(m) = \{x_1^{i_1} \ldots x_n^{i_n} \mid i_1 \ge \cdots \ge i_n \ge 0, \sum_k i_k = m\},$$

$$\mathcal{B}_n(m) = \{t_{f,w} \mid w \in S_n, f \in \mathcal{F}_n(l(w)+m)\}.$$

Here, $\mathcal{F}_n(m)$ is a finite subset of $\mathcal{P}_n$, and $\mathcal{B}_n(m)$ a finite subset of $NH_n(m)$. Moreover, $\mathcal{B}_n(m)$ is empty if $m < -\binom{n}{2}$.

Lemma 2.5. *The abelian group $V_n(m)$ is free with a basis which is in bijection with $\mathcal{B}_n(m)$.*

Proof. Firstly, we claim that $\{\bar{t}_{f,w} \mid t_{f,w} \in \mathcal{B}_n(m)\}$ generates $V_n(m)$ over $\mathbb{Z}$. Let $g = x_{l_1}^{i_1} \ldots x_{l_n}^{i_n}$ such that $i_1 \ge \cdots \ge i_n \ge 0$, and $\sum_k i_k = l+m$. Let $w_g \in S_n$ denote the permutation which maps l_k to k. Then $f = w_g(g) = x_1^{i_1} \ldots x_n^{i_n} \in \mathcal{F}_n(l+m)$. So $w_g \cdot t_{g,w} = t_{f,w} \in NH_n$, and $\bar{t}_{g,w} = \bar{t}_{f,w}$. The claim follows from that $\{t_{g,w}\}$ generates $NH_n(m)$ over $\mathbb{Z}$.

Secondly, we claim that the collection $\{\bar{t}_{f,w} \mid t_{f,w} \in \mathcal{B}_n(m)\}$ is $\mathbb{Z}$-independent in $V_n(m)$. Let

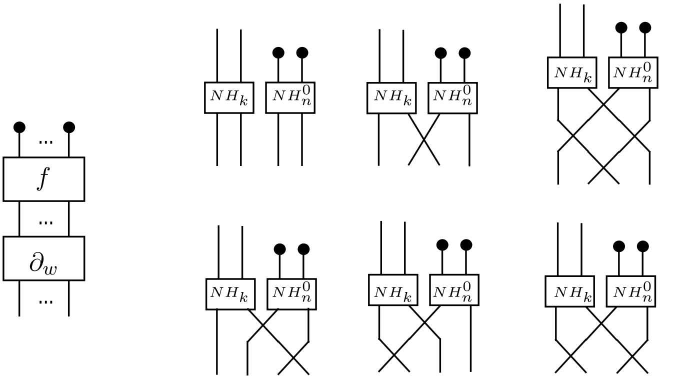

Fig. 8. The picture on the left describes the basis of $\widetilde{NH}_n^0$, where $f \in \mathcal{F}, \partial_w \in NH_n$; the six pictures on the right describe the basis of $\widetilde{NH}_{n+k}^k$ in terms of bases of NH_k and $\widetilde{NH}_n^0$ in the case $n = k = 2$.

$$\beta_n = \alpha \circ \eta_n : V_n \to \widetilde{NH}_n^0 \to \mathrm{Hom}_R(R_n, R).$$

Suppose that $\gamma = \sum k_{f,w}\ \beta_n(\bar{t}_{f,w}) = \sum k_{f,w}\ \alpha(v_n^0 \cdot t_{f,w}) = 0 \in \mathrm{Hom}_R(R_n, R)$ for some integers $k_{f,w}$. We have $0 = \gamma(1) = \sum k_{f,1}\ \alpha(v_n^0 \cdot t_{f,1})(1)$ since $\partial_w(1) = 0$ for $w \neq 1 \in S_n$. The image $\alpha(v_n^0 \cdot t_{f,1})(1) = r_{i_1} \cdots r_{i_n}$ for $f = x_1^{i_1} \cdots x_n^{i_n} \in \mathcal{F}_n$. Since $\{r_{i_1} \cdots r_{i_n} \mid i_1 \geq \cdots \geq i_n \geq 0\}$ is $\mathbb{Z}$-independent in R, we have $k_{f,1} = 0$ for all $f \in \mathcal{F}_n$. For $w_1 \in S_n$ with $l(w_1) = 1$, let $h_{w_1} \in \mathcal{P}_n$ be the Schubert polynomial associated to w_1. Then $\partial_{w_1}(h_{w_1}) = 1$, and $\partial_w(h_{w_1}) = 0$ for $w \neq w_1$ or 1. We have $0 = \gamma(h_{w_1}) = \sum k_{f,w_1}\ \alpha(v_n^0 \cdot t_{f,w_1})(h_{w_1})$. It follows that $k_{f,w_1} = 0$ for all $f \in \mathcal{F}_n$. By applying γ to all Schubert polynomials in $\mathcal{P}_n$, one can inductively show that $k_{f,w} = 0$ for all $f \in \mathcal{F}_n$ and $w \in S_n$. We conclude that $\{\beta_n(\bar{t}_{f,w}) \mid t_{f,w} \in \mathcal{B}_n(m)\}$ is $\mathbb{Z}$-independent. $\square$

The map β_n in the proof above is injective. It implies that η_n is also injective. Combining with the fact that η_n is surjective, we have the following description of $\widetilde{NH}_n^0$.

Proposition 2.6. *The map $\eta_n : V_n \to \widetilde{NH}_n^0$ is an isomorphism of free abelian groups. Moreover, $\widetilde{NH}_n^0$ has a $\mathbb{Z}$-basis $\{v_n^0 t \mid t \in \mathcal{B}_n(m), m \in \mathbb{Z}\}$ which is in bijection with $\mathcal{B}_n = \bigcup_m \mathcal{B}_n(m)$.*

The abelian group $\widetilde{NH}_n^0$ has a $\mathbb{Z}$-basis $\{v_n^0 \cdot f \cdot \partial_w \mid w \in S_n, f \in \mathcal{F}_n\}$, where $\mathcal{F}_n = \bigcup_m \mathcal{F}_n(m)$. Diagrammatically, f only has dots and numbers of dots are non-increasing from left to right, and ∂_w only has crossings. See the picture on the left in Fig. 8.

The basis of $\widetilde{NH}_{n+k}^k$ can be described using bases of NH_k and $\widetilde{NH}_n^0$. Let $S_n \subset NH_{n+k}$ be the symmetric group generated by $s_{i,n+k}$'s for $i = k+1, \ldots, n+k-1$. Let $NH_k \subset NH_{n+k}$ be the inclusion given by adding k vertical strands on the right. Then the left actions of S_n and NH_k on NH_{n+k} commute. Define the induction module

$$V_{n+k}^k = \mathrm{Ind}_{\mathbb{Z}[S_n]}^{NH_{n+k}} \mathbb{Z}.$$

It is a left NH_k, right NH_{n+k} module. We call it an (NH_k, NH_{n+k})-bimodule. Since NH_{n+k} is a free left module over NH_k, V^k_{n+k} is also free over NH_k. There is a canonical surjective map $NH_{n+k} \to V^k_{n+k}$. Define a map

$$\begin{array}{rcl} \eta^k_{n+k} : V^k_{n+k} & \to & \widetilde{NH}^k_{n+k} \\ \overline{t} & \mapsto & v^k_{n+k} t, \end{array} \tag{6}$$

for $t \in NH_{n+k}$, where $v^k_{n+k} \in \widetilde{NH}^k_{n+k}$ has n short strands on the right. It is known that NH_k is a free abelian group and has a basis of the form

$$\mathcal{B}^k_k = \{f \cdot \partial_w \mid w \in S_k,\ f \text{ a monomial in } x_1, \ldots, x_k\}.$$

By a similar argument as in the proof of Proposition 2.6, one can prove the following result.

Proposition 2.7. *The map $\eta^k_{n+k} : V^k_{n+k} \to \widetilde{NH}^k_{n+k}$ is an isomorphism of (NH_k, NH_{n+k})-bimodules. As an abelian group, $\widetilde{NH}^k_{n+k}$ is free with a basis*

$$\{v^k_{n+k} \cdot (t_k \odot b_n) \cdot \partial_w \mid t_k \in \mathcal{B}^k_k, b_n \in \mathcal{B}_n, w \text{ a minimal representative in } (S_k \times S_n)\backslash S_{n+k}\}.$$

See Fig. 8 for an example with $n = k = 2$.

3. The complex lifting exponentials

3.1. The $\widetilde{NH}$-bimodule $\mathcal{F}$

There is an inclusion

$$\begin{array}{rcl} \rho : \widetilde{NH} & \to & \widetilde{NH} \\ a & \mapsto & a \odot 1_1, \end{array} \tag{7}$$

given by adding one vertical strand on the right to any diagram $a \in \widetilde{NH}$. Consider the $\widetilde{NH}$-bimodule corresponding to the induction functor with respect to ρ.

Definition 3.1. Define an abelian group $\mathcal{F} = \bigoplus_{m\geq 0, n\geq 1} \widetilde{NH}^m_n$ with the $\widetilde{NH}$-bimodule structure given by $a \cdot t \cdot b = a\ t\ \rho(b)$, where $a\ t\ \rho(b)$ is the product in $\widetilde{NH}$, for $a, b \in \widetilde{NH}$ and $t \in \mathcal{F}$.

Recall that $\widetilde{NH}^m_n = 0$ if $m > n$. See Fig. 9 for a diagrammatic description of $\mathcal{F}$. The left and right multiplication correspond to stacking diagrams at the top and bottom, respectively. Since the rightmost strand in $\mathcal{F}$ is unchanged under stacking diagrams from the bottom, we call it the *frozen strand* of $\mathcal{F}$, and add a little bar at its lower end.

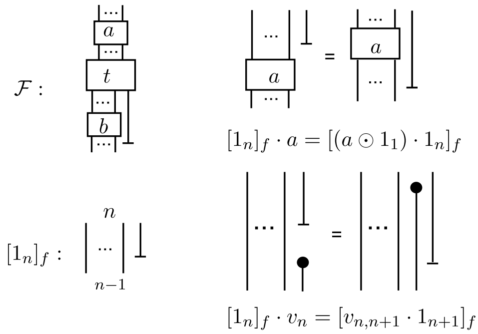

Fig. 9. The bimodule $\mathcal{F}$, its generators $[1_n]_f$, and the right multiplication on $\mathcal{F}$.

Let $P_n = \widetilde{NH} \cdot 1_n$ denote the left projective $\widetilde{NH}$-module associated to the idempotent $1_n \in \widetilde{NH}$. As an abelian group, P_n is spanned by all diagrams with n strands at the bottom. The summand $\bigoplus_{0 \le m \le n} \widetilde{NH}_n^m$ of $\mathcal{F}$ forms a left $\widetilde{NH}$-submodule of $\mathcal{F}$, naturally isomorphic to P_n. So there is a natural isomorphism

$$\mathcal{F} \cong \bigoplus_{n \ge 1} P_n$$

of left $\widetilde{NH}$-modules. In particular, $\mathcal{F}$ is projective as a left $\widetilde{NH}$-module.

For $t \in \widetilde{NH}_n^m$, let $[t]_f$ denote the corresponding element of $\mathcal{F}$, which satisfies

$$1_m \cdot [t]_f = [t]_f = [t]_f \cdot 1_{n-1}.$$

As a left $\widetilde{NH}$-module, $\mathcal{F}$ is generated by $[1_n]_f$ for $n \ge 1$, since $[t]_f = t \cdot [1_n]_f$ for any $t \in \widetilde{NH}_n^m$. The generator $[1_n]_f$ is represented by the diagram of n vertical strands with a little bar at the lower end of the rightmost strand. The right multiplication on the generators is given by

$$[1_n]_f \cdot 1_{n-1} = [1_n]_f, \tag{8}$$

$$[1_n]_f \cdot a = [(a \odot 1_1) \cdot 1_n]_f, \qquad \text{for} \quad a \in NH_{n-1}, \tag{9}$$

$$[1_n]_f \cdot v_n = [1_n \cdot (v_n \odot 1_1)]_f = [v_{n,n+1} \cdot 1_{n+1}]_f, \tag{10}$$

see Fig. 9.

From now on, all tensor products are taken over $\widetilde{NH}$. We will simply write $\otimes$ for $\otimes_{\widetilde{NH}}$. Let $\mathcal{F}^k$ denote the $\widetilde{NH}$-bimodule given by the k-th tensor product $\mathcal{F}^{\otimes k}$ over $\widetilde{NH}$. As an abelian group,

$$\mathcal{F}^k = \bigoplus_{m \ge 0, n \ge k} \widetilde{NH}_n^m,$$

where the $\widetilde{NH}$-bimodule structure is given by $a \cdot f \cdot b = a\ f\ \rho^k(b)$, where $a\ f\ \rho^k(b)$ is the product in $\widetilde{NH}$, for $a, b \in \widetilde{NH}$ and $f \in \mathcal{F}$. Here, $\rho^k = \rho \circ \cdots \circ \rho : \widetilde{NH} \to \widetilde{NH}$ maps b to $b \odot 1_k$ for $b \in \widetilde{NH}$. The map ρ^k is diagrammatically given by adding k vertical strands on the right. Let $[t]_{f^k} \in \mathcal{F}^k$ which corresponds to $t \in \widetilde{NH}_n^m$ for $n \geq k$. The collection $\{[1_n]_{f^k} \mid n \geq k\}$ generates $\mathcal{F}^k$ as a left $\widetilde{NH}$-module.

The nilHecke algebra NH_k naturally acts on $\mathcal{F}^k$ on the right as follows. For any $c \in NH_k$, define a map

$$\begin{array}{rccc} r(c): & \mathcal{F}^k & \to & \mathcal{F}^k \\ & [t]_{f^k} & \mapsto & [t \cdot (1_{n-k} \odot c)]_{f^k} \end{array} \tag{11}$$

for $t \in \widetilde{NH}_n^m$. Diagrammatically, the map $r(c)$ stacks $t \in NH_k$ onto the k frozen strands from the bottom. The action commutes with the left and right multiplication of $\widetilde{NH}$ on $\mathcal{F}^k$. Hence, $r(c)$ is a map of $\widetilde{NH}$-bimodules.

Let $\mathbf{D}(\widetilde{NH}^e)$ denote the derived category of $\widetilde{NH}$-bimodules. It is a monoidal triangulated category whose monoidal bifunctor is given by the derived tensor product over $\widetilde{NH}$. The unit object, denoted by $\mathbf{1}$, is isomorphic to $\widetilde{NH}$ as an $\widetilde{NH}$-bimodule placed in cohomological degree zero. Since $\mathcal{F}$ is projective as a left $\widetilde{NH}$-module, the derived tensor product reduces to the ordinary tensor product between $\mathcal{F}$'s. In particular, we view $\mathcal{F}^k = \mathcal{F}^{\otimes k}$ placed in cohomological degree zero as an object of $\mathbf{D}(\widetilde{NH}^e)$. The subscript $\mathbf{D}(\widetilde{NH}^e)$ will be omitted in $\mathrm{Hom}_{\mathbf{D}(\widetilde{NH}^e)}$ from now on. We compute $\mathrm{End}(\mathbf{1})$ and $\mathrm{End}(\mathcal{F})$ in the following.

Lemma 3.2. *The endomorphism ring* $\mathrm{End}(\mathbf{1})$ *is isomorphic to* $\mathbf{k}$ *with a generator* $id_{\mathbf{1}}$.

Proof. For any $h \in \mathrm{End}(\mathbf{1})$, $h(1_n) \in NH_n$ lives in the center $Z(NH_n)$ which is isomorphic to the ring Λ_n of symmetric functions. Moreover, $h(1_{n-1}) \cdot v_n = h(v_n) = v_n \cdot h(1_n) \in \widetilde{NH}_n^{n-1}$. Suppose $h(1_0) = a \cdot 1_0 \in Z(NH_0)$ for some $a \in \mathbf{k}$. Then $h(1_n) = a \cdot 1_n \in Z(NH_n)$ by induction on n. So $h = a \cdot id_{\mathbf{1}}$. $\square$

There is a natural inclusion $\rho_n : NH_n \to NH_{n+1}$ of algebras which sends a to $a \odot 1_1$. Let $C_{n+1}(n)$ denote the centralizer of NH_n in NH_{n+1} with respect to the inclusion.

Proposition 3.3. *There is an isomorphism* $C_{n+1}(n) \cong Z(NH_n) \otimes \mathbb{Z}[x_{n+1}]$ *of rings.*

Proof. The NH_n-bimodule NH_{n+1} has a decomposition

$${}_{NH_n}(NH_{n+1})_{NH_n} \cong (NH_n \otimes_{\mathbf{k}} \mathbf{k}[x_{n+1}]) \oplus (NH_n \otimes_{NH_{n-1}} NH_n),$$

where $NH_n \otimes_{\mathbf{k}} \mathbf{k}[x_{n+1}]$ is isomorphic to a sum of $\mathbb{N}$ copies of NH_n as NH_n-bimodule, see Fig. 10. The center of the bimodule $NH_n \otimes_{\mathbf{k}} \mathbf{k}[x_{n+1}]$ is isomorphic to $Z(NH_n) \otimes \mathbb{Z}[x_{n+1}]$. It suffices to show that the center of the bimodule $NH_n \otimes_{NH_{n-1}} NH_n$ is trivial.

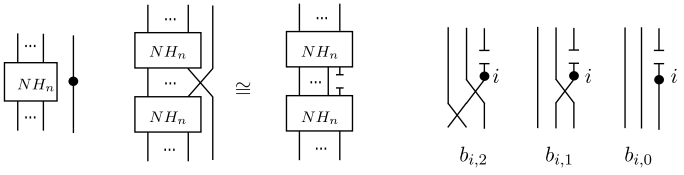

Fig. 10. The NH_n-bimodule NH_{n+1}.

Note that $NH_n \otimes_{NH_{n-1}} NH_n$ is a free left NH_n-module with a basis

$$\{b_{i,k} = 1_n \otimes (x_n^i \partial_{[k]}) \mid i \geq 0, 0 \leq k \leq n-1\},$$

where $\partial_{[k]} = \partial_{n-1} \cdots \partial_{n-k}$ for $k \geq 1$ and $\partial_{[0]} = 1_n$. Let $c = \sum\limits_{i,k} c_{i,k} \cdot b_{i,k} \in NH_n \otimes_{NH_{n-1}} NH_n$, for $c_{i,k} \in NH_n$. Let $k_0(c) = \max\{k \mid c_{i,k} \neq 0 \text{ for some } i\}$, and $i_0(c) = \max\{i \mid c_{i,k_0(c)} \neq 0\}$. Consider $c_1 = x_{n-k_0} \cdot c$, and $c_2 = c \cdot x_{n-k_0}$, for $x_{n-k_0} \in NH_n$. Then $k_0(c_1) = k_0(c_2) = k_0(c)$, but $i_0(c_1) = i_0(c), i_0(c_2) = i_0(c) + 1$. Thus $c_1 \neq c_2$ which implies that the center is trivial. □

Proposition 3.3 together with a similar argument as in the proof of Lemma 3.2 gives the following result.

Corollary 3.4. *There is a ring isomorphism* $\mathbf{k}[x] \to \mathrm{End}(\mathcal{F})$ *which maps* $c \in \mathbf{k}[x]$ *to* $r(c) \in \mathrm{End}(\mathcal{F})$.

3.2. The complex lifting $\exp(-f)$

The goal is to lift the expansion

$$\exp(-f) = \sum_{k \geq 0} (-1)^k \frac{f^k}{k!},$$

to a complex in $\mathbf{D}(\widetilde{NH}^e)$, where 1 and f are lifted to the objects $\mathbf{1}$ and $\mathcal{F}$, respectively. We use certain direct summands of $\mathcal{F}^k$ to lift the divided powers $\frac{f^k}{k!}$. To define the differential, we will use the new generators with short strands.

Recall some basic facts about some idempotents of NH_k as follows. Let $w_0(k) \in S_k$ denote the longest element of S_k, and $\partial_{(k)} = \partial_{w_0(k)} \in NH_k$. There are idempotents

$$e_k = x_1^{k-1} \cdots x_{k-1} \partial_{(k)}$$

in NH_k. Let $e_{i,k} = 1_{i-1} \odot e_2 \odot 1_{k-i-1} \in NH_k$ denote the diagram obtained from e_2 by adding $i-1$ and $k-i-1$ vertical strands on the left and right, respectively, see Fig. 11. It is easy to see that $e_k = e_{i_1,k} \cdots e_{i_l,k}$, where $s_{i_1} \cdots s_{i_l}$ is a reduced expression of $w_0(k) \in S_k$. In other words, e_2 is the building block of e_k.

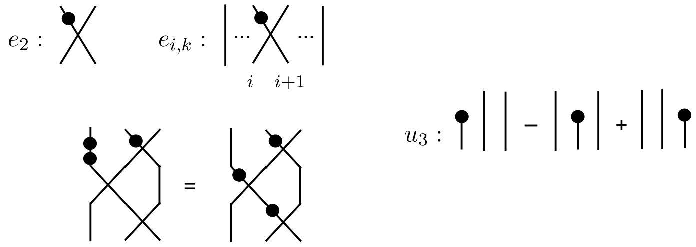

Fig. 11. The diagrams for $e_2, e_{i,k}, e_3 = e_{2,3}e_{1,3}e_{2,3}$, and u_3.

The idempotent e_k induces an idempotent endomorphism $r(e_k)$ of the bimodule $\mathcal{F}^k$.

Definition 3.5. Define the $\widetilde{NH}$-bimodule $\mathcal{F}^{(k)}$ as a direct summand of $\mathcal{F}^k$ corresponding to the idempotent endomorphism $r(e_k)$. Elements of $\mathcal{F}^{(k)}$ are of the form $[t\cdot(1_{n-k}\odot e_k)]_{f^k}$ for $t \in \widetilde{NH}^m_n$.

Define an alternating sum

$$u_k = \sum_{i=1}^{k}(-1)^{i-1}v_{i,k} \in \widetilde{NH}^{k-1}_k, \tag{12}$$

for $k \geq 1$, see Fig. 11 for u_3. Then $u_k \cdot u_{k+1} = 0$ by the isotopy relation of disjoint short strands.

Via the right action of NH_k on $\mathcal{F}^k$ in (11), the element u_k induces a map

$$\begin{aligned} r(u_k):\ \mathcal{F}^{k-1} &\rightarrow \mathcal{F}^k \\ [t]_{f^{k-1}} &\mapsto [t\cdot(1_{n+1-k}\odot u_k)]_{f^k} \end{aligned} \tag{13}$$

for $t \in \widetilde{NH}^m_n$, $n \geq k-1$.

Consider the restriction of $r(u_k)$ to $\mathcal{F}^{(k-1)}$ followed by the projection of $\mathcal{F}^k$ onto $\mathcal{F}^{(k)}$. The resulting map is denoted by $d(u_k)$:

$$\begin{aligned} d(u_k):\quad \mathcal{F}^{(k-1)} &\rightarrow \mathcal{F}^{(k)} \\ [t\cdot(1_{n+1-k}\odot e_{k-1})]_{f^{k-1}} &\mapsto [t\cdot(1_{n+1-k}\odot(e_{k-1}u_k e_k))]_{f^k} \end{aligned} \tag{14}$$

for $t \in \widetilde{NH}^m_n$, $n \geq k-1$.

We compute $d(u_{k+1}) \circ d(u_k)$ in the following.

Lemma 3.6. *There is an equality* $e_{k-1}u_k e_k = e_{k-1}u_k \in \widetilde{NH}^{k-1}_k$.

Proof. We claim that $e_{k-1}u_k e_{i,k} = e_{k-1}u_k$ for all $1 \leq i \leq k-1$. The lemma follows from the claim since e_k can be written as a product of $e_{i,k}$'s.

To prove the claim, we compute

$$e_{k-1}u_k\partial_{i,k} = \sum_{j=1}^{k}(-1)^{j-1}e_{k-1}v_{j,k}\partial_{i,k}$$
$$= \sum_{j\neq i,i+1}(-1)^{j-1}e_{k-1}v_{j,k}\partial_{i,k} + (-1)^{i-1}e_{k-1}v_{i,k}\partial_{i,k} + (-1)^{i}e_{k-1}v_{i+1,k}\partial_{i,k}.$$

Each term of the first summation is zero since $v_{j,k}\partial_{i,k} = \partial_{i',k}v_{j,k}$ for some i' when $j \neq i, i+1$, and $e_{k-1}\partial_{i',k-1} = 0$ for all i'. The remaining two terms cancel each other since $v_{i,k}\partial_{i,k} = v_{i+1,k}\partial_{i,k}$ from the exchange relation in Definition 2.1 (3). So $e_{k-1}u_k\partial_{i,k} = 0$. By the nilHecke relation, $e_2 = 1_2 + \partial_1 x_2$ so that $e_{i,k} = 1_k + \partial_{i,k}x_{i+1,k}$. Thus

$$e_{k-1}u_k e_{i,k} = e_{k-1}u_k(1_k + \partial_{i,k}x_{i+1,k}) = e_{k-1}u_k 1_k = e_{k-1}u_k.$$

The lemma follows. □

Remark 3.7. In general, $e_{k-1}u_k e_k \neq u_k e_k$.

The lemma above implies that

$$(e_{k-1}u_k e_k)\cdot(e_k u_{k+1} e_{k+1}) = e_{k-1}u_k e_k u_{k+1} e_{k+1} = e_{k-1}u_k u_{k+1} e_{k+1} = 0, \tag{15}$$

where the last equality holds because $u_k u_{k+1} = 0$. Hence, $d(u_{k+1}) \circ d(u_k) : \mathcal{F}^{(k-1)} \to \mathcal{F}^{(k+1)}$ is zero by the definition of $d(u_k)$ in (14). We define a complex

$$\exp(-\mathcal{F}) = \left(\bigoplus_{k\geq 0}\mathcal{F}^{(k)}[-k], d = \bigoplus_{k\geq 1} d_k\right), \tag{16}$$

where the components of the differential d are given by $d_k = d(u_k) : \mathcal{F}^{(k-1)} \to \mathcal{F}^{(k)}$.

3.3. *The complex lifting* $\exp(f)$

The goal is to lift the expansion

$$\exp(f) = \sum_{k\geq 0}\frac{f^k}{k!},$$

to an object in $\mathbf{D}(\widetilde{NH}^e)$. We use another direct summand of $\mathcal{F}^k$ induced by an idempotent which is different from e_k. The differential is induced by certain elements in the extension group $\mathrm{Ext}^1(\mathcal{F}^k, \mathcal{F}^{k-1})$.

In the following, we construct an $\widetilde{NH}$-bimodule $\mathcal{G}$ in two steps. We will show that $\mathcal{G}$ is an extension of $\mathcal{F}$ by $\mathbf{1} = \widetilde{NH}$ as $\widetilde{NH}$-bimodules:

$$0 \to \mathbf{1} \to \mathcal{G} \to \mathcal{F} \to 0.$$

Step 1: The left module. We first define

$$\mathcal{G} = \mathbf{1} \oplus \mathcal{F}$$

as a left $\widetilde{NH}$-module. Let $[t]_1 \in \mathbf{1}$, and $[t']_f \in \mathcal{F}$ for $t, t' \in \widetilde{NH}^m_n$. Then

$$t \cdot [1_n]_1 = [t]_1, \qquad t' \cdot [1_n]_f = [t']_f.$$

The left $\widetilde{NH}$-module $\mathcal{G}$ is projective, and generated by $[1_n]_1$ and $[1_{n+1}]_f$, for $n \geq 0$.

Step 2: The right module. The right multiplication is defined on the generators $[1_n]_1$ and $[1_{n+1}]_f$ as follows. The summand $\mathbf{1}$ is a right $\widetilde{NH}$-submodule of $\mathcal{G}$:

$$[1_n]_1 \cdot t = [t]_1, \quad \text{for} \;\; t \in \widetilde{NH}^n_{n'}.$$

The algebra $\widetilde{NH}$ contains $\bigoplus\limits_{n \geq 0} NH_n$ as a subalgebra. The summand $\mathcal{F}$ is a right $\bigoplus\limits_{n \geq 0} NH_n$-submodule of $\mathcal{G}$:

$$[1_{n+1}]_f \cdot t = [t \odot 1_1]_f = (t \odot 1_1) \cdot [1_{n+1}]_f, \quad \text{for} \;\; t \in NH_n.$$

The only nontrivial part of the definition of $\mathcal{G}$ is the right multiplication on $[1_n]_f$ with $v_n \in \widetilde{NH}^{n-1}_n$, for $n \geq 1$:

$$[1_n]_f \cdot v_n = [v_n \odot 1_1]_f + [1_n]_1 = [v_{n,n+1}]_f + [1_n]_1, \tag{17}$$

see Fig. 12. Here we use the presentation of $\widetilde{NH}$ in Definition 2.3. The algebra $\widetilde{NH}$ is generated by NH_{n-1} and v_n for $n \geq 1$. In particular, $v_{i,n} = v_n \cdot g_{i,n}$ for some element $g_{i,n} \in NH_n$. Define $[1_n]_f \cdot v_{i,n} = ([1_n]_f \cdot v_n) \cdot g_{i,n}$.

Define the right multiplication on $[t]_f$ for $t \in \widetilde{NH}^m_n$, $b \in \widetilde{NH}$ as

$$[t]_f \cdot b = t \cdot ([1_n]_f \cdot b).$$

Our construction of $\mathcal{G}$ is complete.

As a left $\widetilde{NH}$-module, $\mathcal{G}$ is projective, and generated by $[1_n]_1$ and $[1_{n+1}]_f$ for $n \geq 0$. Thus, the right multiplication on $\mathcal{G}$ is determined by the right multiplication on the generators $[1_n]_1$ and $[1_{n+1}]_f$. By definition, it commutes with the left multiplication.

Lemma 3.8. *The right multiplication on $\mathcal{G}$ is well-defined.*

Proof. Since the action of NH_n on $\mathcal{G}$ is the ordinary multiplication on $\widetilde{NH}$, we only have to check the relations involving v_n in Definition 2.3 (3).

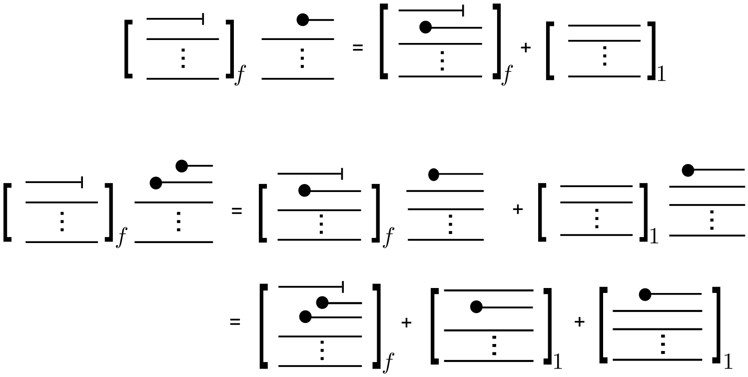

Fig. 12. Diagrams are rotated by $\frac{\pi}{2}$ counterclockwise from the vertical direction to the horizontal direction. The upper part is the definition $[1_n]_f \cdot v_n$; the lower part computes $([1_n]_f \cdot v_n) \cdot v_{n+1}$.

We check the isotopy relation of a short strand with a dot

$$\begin{aligned}([1_n]_f \cdot v_n) \cdot x_{j,n} &= ([v_{n,n+1}]_f + [1_n]_1) \cdot x_{j,n} \\ &= [v_{n,n+1} \cdot (x_{j,n} \odot 1_1)]_f + [x_{j,n}]_1 \\ &= [x_{j,n} \cdot v_{n,n+1}]_f + [x_{j,n}]_1 \\ &= ([1_n]_f \cdot x_{j,n-1}) \cdot v_n.\end{aligned}$$

The proof of the isotopy relation of a short strand with a crossing is similar, and left to the reader.

We check the relation (F) in Definition 2.3 (3): $v_n v_{n+1} = v_n v_{n+1} s_{n,n+1}$

$$\begin{aligned}([1_n]_f \cdot v_n) \cdot v_{n+1} &= ([v_{n,n+1}]_f + [1_n]_1) \cdot v_{n+1} \\ &= [v_{n,n+1} \cdot 1_{n+1}] \cdot v_{n+1} + [v_{n+1}]_1 \\ &= [v_{n,n+1} \cdot v_{n+1,n+2}]_f + [v_{n,n+1} \cdot 1_{n+1}]_1 + [v_{n+1}]_1 \\ &= [v_{n,n+1} \cdot v_{n+1,n+2}]_f + [v_{n,n+1} + v_{n+1}]_1,\end{aligned}$$

see Fig. 12. Recall that $s_{n,n+1} \in NH_{n+1}$ exchanges v_{n+1} and $v_{n,n+1}$. So

$$\begin{aligned}([1_n]_f \cdot v_n \cdot v_{n+1}) \cdot s_{n,n+1} &= [v_{n,n+1} \cdot v_{n+1,n+2}]_f \cdot s_{n,n+1} + [v_{n,n+1} + v_{n+1}]_1 \cdot s_{n,n+1} \\ &= [v_{n,n+1} \cdot v_{n+1,n+2} \cdot (s_{n,n+1} \odot 1_1)]_f + [v_{n,n+1} + v_{n+1}]_1 \\ &= [v_{n,n+1} \cdot v_{n+1,n+2}]_f + [v_{n,n+1} + v_{n+1}]_1.\end{aligned}$$

This proves that the right multiplication on $\mathcal{G}$ is well-defined. □

The $\widetilde{NH}$-bimodule $\mathcal{G}$ fits into a short exact sequence of $\widetilde{NH}$-bimodules:

$$0 \to \mathbf{1} \to \mathcal{G} \to \mathcal{F} \to 0.$$

Lemma 3.9. *The extension $\mathcal{G}$ is not split.*

Proof. Suppose $\mathcal{G}$ is split, i.e. there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{1} & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & \mathbf{1} & \longrightarrow & \mathbf{1}\oplus\mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array}$$

of $\widetilde{NH}$-bimodules. Let $[1_{n-1}]'_1$ and $[1_n]'_f$ for $n \geq 1$ denote the generators of $\mathbf{1} \oplus \mathcal{F}$. The commutative diagram implies that $\psi([1_{n-1}]_1) = [1_{n-1}]'_1$, and $\psi([1_n]_f) = [1_n]'_f + [t]'_1$ for some $t \in \widetilde{NH}^n_{n-1}$. Since $\widetilde{NH}^n_{n-1} = 0$, we have $\psi([1_n]_f) = [1_n]'_f$. As a left $\widetilde{NH}$-module, $\mathcal{G}$ is generated by $[1_{n-1}]_1$ and $[1_n]_f$. So $\psi([t]_f) = \psi(t\cdot[1_n]_f) = t\cdot\psi([1_n]_f) = t\cdot[1_n]'_f = [t]'_f$. Similarly, $\psi([t]_1) = [t]'_1$. On the other hand,

$$\psi([1_n]_f \cdot v_n) = \psi([v_{n,n+1}]_f + [1_n]_1) = [v_{n,n+1}]'_f + [1_n]'_1,$$
$$\psi([1_n]_f) \cdot v_n = [1_n]'_f \cdot v_n = [v_{n,n+1}]'_f.$$

Thus ψ is not a map of right $\widetilde{NH}$-modules. This is a contradiction. □

The extension $\mathcal{G}$ gives rise to a morphism $\widetilde{v}_1 \in \mathrm{Hom}(\mathcal{F}, \mathbf{1}[1])$ in the derived category $\mathbf{D}(\widetilde{NH}^e)$. We write $\mathrm{Hom}^1(\mathcal{F}, \mathbf{1})$ for $\mathrm{Hom}(\mathcal{F}, \mathbf{1}[1])$. Lemma 3.9 implies that $\widetilde{v}_1 \neq 0 \in \mathrm{Hom}^1(\mathcal{F}, \mathbf{1})$.

Define

$$\widetilde{v}_{i,n} = 1_{\mathcal{F}^{i-1}} \otimes \widetilde{v}_1 \otimes 1_{\mathcal{F}^{n-i}} \in \mathrm{Hom}^1(\mathcal{F}^n, \mathcal{F}^{n-1}),$$

for $1 \leq i \leq n$.

Two elements $\widetilde{v}_{1,2}$ and $\widetilde{v}_{2,2}$ correspond to two extensions $\mathcal{G}\otimes\mathcal{F}$ and $\mathcal{F}\otimes\mathcal{G}$ of $\mathcal{F}^2$ by $\mathcal{F}$:

$$\widetilde{v}_{1,2}: \qquad 0 \to \mathcal{F} \to \mathcal{G}\otimes\mathcal{F} \to \mathcal{F}^2 \to 0,$$
$$\widetilde{v}_{2,2}: \qquad 0 \to \mathcal{F} \to \mathcal{F}\otimes\mathcal{G} \to \mathcal{F}^2 \to 0.$$

Recall $s_{1,2} \in NH_2$ in (1) and the map $r(s_{1,2}) : \mathcal{F}^2 \to \mathcal{F}^2$ of $\widetilde{NH}$-bimodules in (11).

Lemma 3.10. *There exists a map $\phi : \mathcal{G}\otimes\mathcal{F} \to \mathcal{F}\otimes\mathcal{G}$ of $\widetilde{NH}$-bimodules such that the following diagram commutes*

$$\begin{array}{cccccccccc} \widetilde{v}_{1,2}: & 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{G}\otimes\mathcal{F} & \longrightarrow & \mathcal{F}^2 & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle r(s_{1,2})} & & \\ \widetilde{v}_{2,2}: & 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}\otimes\mathcal{G} & \longrightarrow & \mathcal{F}^2 & \longrightarrow & 0. \end{array}$$

Fig. 13. The map ϕ defined on the generators $[1_n]^{gf}_f$ and $[1_{n+1}]^{gf}_{f^2}$.

Proof. The left projective $\widetilde{NH}$-module $\mathcal{G} \otimes \mathcal{F} \cong \mathcal{F} \oplus \mathcal{F}^2$ is generated by elements

$$[1_n]^{gf}_f := [1_n]^g_1 \otimes [1_n]_f, \qquad [1_{n+1}]^{gf}_{f^2} := [1_{n+1}]^g_f \otimes [1_n]_f,$$

for $n \geq 1$. Similarly, the left projective $\widetilde{NH}$-module $\mathcal{F} \otimes \mathcal{G} \cong \mathcal{F} \oplus \mathcal{F}^2$ is generated by elements

$$[1_n]^{fg}_f := [1_n]_f \otimes [1_{n-1}]^g_1, \qquad [1_{n+1}]^{fg}_{f^2} := [1_{n+1}]_f \otimes [1_n]^g_f,$$

for $n \geq 1$. Define a map $\phi : \mathcal{G} \otimes \mathcal{F} \to \mathcal{F} \otimes \mathcal{G}$ of left $\widetilde{NH}$-modules on the generators as

$$\phi([1_n]^{gf}_f) = [1_n]^{fg}_f, \qquad \phi([1_{n+1}]^{gf}_{f^2}) = [s_{n,n+1}]^{fg}_{f^2},$$

see Fig. 13. The map ϕ makes the diagram commute.

We need to show that ϕ is a map of right $\widetilde{NH}$-modules. It is true when restricting to the right $\widetilde{NH}$-submodule $\mathcal{F}$ of $\mathcal{G} \otimes \mathcal{F}$. It remains to show that $\phi([1_{n+1}]^{gf}_{f^2} \cdot v_n) = \phi([1_{n+1}]^{gf}_{f^2}) \cdot v_n$. We compute

$$\begin{aligned}
[1_{n+1}]^{gf}_{f^2} \cdot v_n &= [1_{n+1}]^g_f \otimes ([1_n]_f \cdot v_n) \\
&= [1_{n+1}]^g_f \otimes [v_n \odot 1_1]_f \\
&= ([1_{n+1}]^g_f \cdot v_{n,n+1}) \otimes [1_{n+1}]_f \\
&= ([1_{n+1}]^g_f \cdot v_{n+1} \cdot s_{n,n+1}) \otimes [1_{n+1}]_f \\
&= (([v_{n+1} \odot 1_1]^g_f + [1_{n+1}]^g_1) \cdot s_{n,n+1}) \otimes [1_{n+1}]_f \\
&= ([v_{n+1,n+2} \cdot s_{n,n+2}]^g_f + [s_{n,n+1}]^g_1) \otimes [1_{n+1}]_f \\
&= ([v_{n,n+2}]^g_f + [s_{n,n+1}]^g_1) \otimes [1_{n+1}]_f \\
&= [v_{n,n+2}]^{gf}_{f^2} + [s_{n,n+1}]^{gf}_f .
\end{aligned}$$

So $\phi([1_{n+1}]^{gf}_{f^2} \cdot v_n) = [v_{n,n+2} \cdot s_{n,n+1}]^{fg}_{f^2} + [s_{n,n+1}]^{fg}_f$. On the other hand,

$$\begin{aligned}
\phi([1_{n+1}]^{gf}_{f^2}) \cdot v_n &= [s_{n,n+1}]^{fg}_{f^2} \cdot v_n \\
&= [s_{n,n+1}]_f \otimes ([1_n]^g_f \cdot v_n) \\
&= [s_{n,n+1}]_f \otimes ([v_{n,n+1}]^g_f + [1_n]^g_1) \\
&= ([s_{n,n+1}]_f \cdot v_{n,n+1}) \otimes [1_{n+1}]^g_f + [s_{n,n+1}]^{fg}_f \\
&= [s_{n,n+1} \cdot v_{n,n+2}]^{fg}_{f^2} + [s_{n,n+1}]^{fg}_f .
\end{aligned}$$

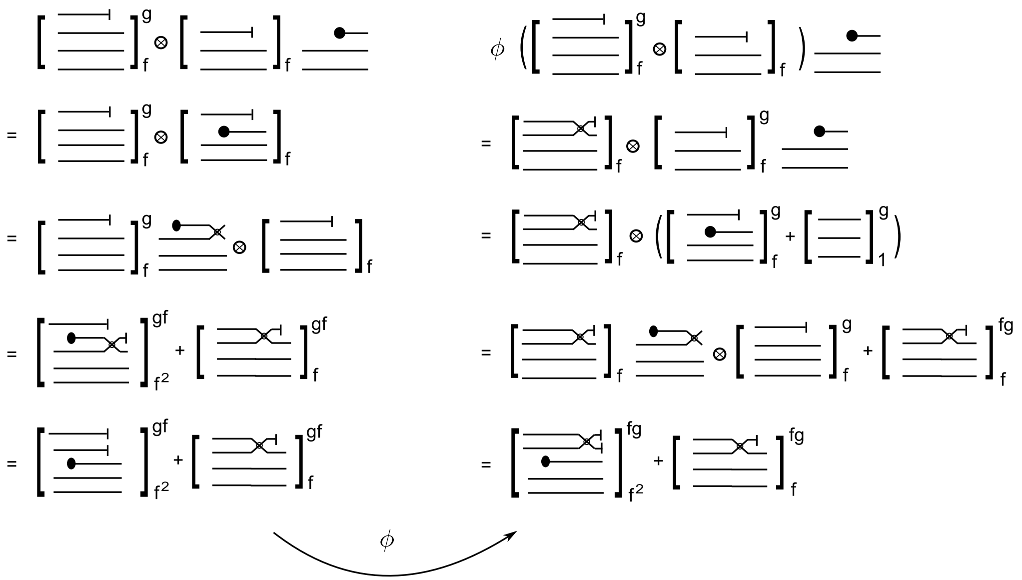

Fig. 14. The pictures compute $\phi([1_{n+1}]^{gf}_{f^2} \cdot v_n)$ and $\phi([1_{n+1}]^{gf}_{f^2}) \cdot v_n$ for $n = 3$ on the left and right, respectively.

Since $v_{n,n+2} \cdot s_{n,n+1} = s_{n,n+1} \cdot v_{n,n+2}$, it follows that $\phi([1_{n+1}]^{gf}_{f^2} \cdot v_n) = \phi([1_{n+1}]^{gf}_{f^2}) \cdot v_n$. A graphic counterpart of this computation is depicted in Fig. 14. □

Recall that $r(s_{1,2}) \in \mathrm{Hom}^0(\mathcal{F}^2, \mathcal{F}^2)$, and $\widetilde{v}_{1,2} \in \mathrm{Hom}^1(\mathcal{F}^2, \mathcal{F})$. So $\widetilde{v}_{1,2} \circ r(s_{1,2}) \in \mathrm{Hom}^1(\mathcal{F}^2, \mathcal{F})$ corresponds to another extension of $\mathcal{F}^2$ by $\mathcal{F}$, denoted by $s(\mathcal{G} \otimes \mathcal{F})$. The lemma above shows that the extensions $s(\mathcal{G} \otimes \mathcal{F})$ and $\mathcal{F} \otimes \mathcal{G}$ are equivalent.

Corollary 3.11. *The equalities* $\widetilde{v}_{2,2} = \widetilde{v}_{1,2} \circ r(s_{1,2})$, *and* $\widetilde{v}_{2,2} \circ r(\partial_{1,2}) = -\widetilde{v}_{1,2} \circ r(\partial_{1,2})$ *hold in* $\mathrm{Hom}^1(\mathcal{F}^2, \mathcal{F}[1])$.

Proof. The first equality directly follows from Lemma 3.10. Precomposing the first equality with $r(\partial_{1,2})$ gives the second one, since $r(s_{1,2}) \circ r(\partial_{1,2}) = r(\partial_{1,2} \cdot s_{1,2}) = r(-\partial_{1,2})$, see Fig. 5. □

In the monoidal category $\mathbf{D}(\widetilde{NH}^e)$, we could diagrammatically represent $\widetilde{v}_1 \in \mathrm{Hom}^1(\mathcal{F}, \mathbf{1})$ as a short strand with one endpoint at the top and one endpoint in the middle decorated by a circle, see Fig. 15. It is of cohomological degree one. The element $\widetilde{v}_{i,n}$ can be obtained from $\widetilde{v}_1$ by adding $i-1$ and $n-i$ vertical strands on the left and right, respectively. The two relations in Corollary 3.11 are depicted in Fig. 15. They are the analogues of the slide relation, and the exchange relation in $\mathrm{Hom}^0(\mathcal{F}, \mathcal{F}^2)$, see Figs. 4 and 2.

The super version of the isotopy relation of disjoint diagrams holds in $\mathbf{D}(\widetilde{NH}^e)$:

$$(a \otimes 1_{N'}) \circ (1_M \otimes b) = (-1)^{|a||b|}(1_N \otimes b) \circ (a \otimes 1_{M'}) \in \mathrm{Hom}^{|a|+|b|}(M \otimes M', N \otimes N'),$$

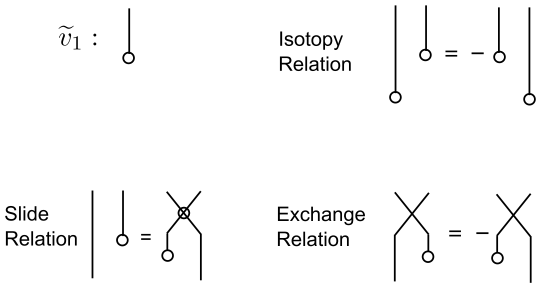

Fig. 15. A graphic presentation of $\widetilde{v}_1 \in \mathrm{Hom}^1(\mathcal{F}, \mathbf{1}[1])$, and the slide, exchange, and isotopy relations in $\mathrm{Hom}^1(\mathcal{F}^2, \mathcal{F})$.

for $a \in \mathrm{Hom}^{|a|}(M, N), b \in \mathrm{Hom}^{|b|}(M', N')$. In particular, $\widetilde{v}_{2,2} \circ \widetilde{v}_1 = -\widetilde{v}_{1,2} \circ \widetilde{v}_1$ from the isotopy of two disjoint short strands of degree one.

As an analogue of u_k defined in (12), define a sum

$$\widetilde{u}_k = \sum_{i=1}^{k} \widetilde{v}_{i,k} \in \mathrm{Hom}^1(\mathcal{F}^k, \mathcal{F}^{k-1}), \tag{18}$$

for $k \geq 1$. Then $\widetilde{u}_k \circ \widetilde{u}_{k+1} = 0$ by the super isotopy relation of disjoint short strands.

Define the idempotent

$$\widetilde{e}_k = \partial_{(k)} x_1^{k-1} \cdots x_{k-1} \in NH_k.$$

It is the flip of e_k with respect to the horizontal axis. There is an induced idempotent endomorphism $r(\widetilde{e}_k) \in \mathrm{Hom}^0(\mathcal{F}^k, \mathcal{F}^k)$. Define $\widetilde{\mathcal{F}}^{(k)}$ to be the direct summand of $\mathcal{F}^k$ corresponding to the idempotent endomorphism $r(\widetilde{e}_k)$.

The morphism $\widetilde{u}_k \in \mathrm{Hom}^1(\mathcal{F}^k, \mathcal{F}^{k-1})$ induces a morphism in $\mathrm{Hom}^1(\widetilde{\mathcal{F}}^{(k)}, \widetilde{\mathcal{F}}^{(k-1)})$ as the restriction of $\widetilde{u}_k$ to $\widetilde{\mathcal{F}}^{(k)}$ followed by a projection onto $\widetilde{\mathcal{F}}^{(k-1)}$. Let $d(\widetilde{u}_k)$ denote the resulting morphism.

Lemma 3.12. *There is an equality* $r(\widetilde{e}_{k-1}) \circ \widetilde{u}_k \circ r(\widetilde{e}_k) = r(\widetilde{e}_{k-1}) \circ \widetilde{u}_k \in \mathrm{Hom}^1(\mathcal{F}^k, \mathcal{F}^{k-1})$.

Proof. The proof is similar to that of Lemma 3.6 except that there is a minus sign in the exchange relation $\widetilde{v}_{2,2} \circ r(\partial_{1,2}) = -\widetilde{v}_{1,2} \circ r(\partial_{1,2})$, as in Corollary 3.11. □

The lemma above implies that

$$\begin{aligned}
&(r(\widetilde{e}_{k-1}) \circ \widetilde{u}_k \circ r(\widetilde{e}_k)) \circ (r(\widetilde{e}_k) \circ \widetilde{u}_{k+1} \circ r(\widetilde{e}_{k+1})) \\
=& r(\widetilde{e}_{k-1}) \circ \widetilde{u}_k \circ r(\widetilde{e}_k) \circ \widetilde{u}_{k+1} \circ r(\widetilde{e}_{k+1}) \\
=& r(\widetilde{e}_{k-1}) \circ \widetilde{u}_k \circ \widetilde{u}_{k+1} \circ r(\widetilde{e}_{k+1}) = 0,
\end{aligned}$$

where the last equality holds because $\widetilde{u}_k \circ \widetilde{u}_{k+1} = 0$.

Hence, $d(\widetilde{u}_k) \circ d(\widetilde{u}_{k+1}) \in \mathrm{Hom}^2(\widetilde{\mathcal{F}}^{(k+1)}, \widetilde{\mathcal{F}}^{(k-1)})$ is zero. We define a complex

$$\exp(\mathcal{F}) = \left(\bigoplus_{k \geq 0} \widetilde{\mathcal{F}}^{(k)}, d = \bigoplus_{k \geq 1} d_k \right), \tag{19}$$

where the components of the differential d are given by $d_k = d(\widetilde{u}_k) \in \mathrm{Hom}^1(\widetilde{\mathcal{F}}^{(k)}, \widetilde{\mathcal{F}}^{(k-1)})$.

4. Discussions

Non-invertibility of $\exp(\mathcal{F})$ and $\exp(-\mathcal{F})$.

Suppose that $\exp(\mathcal{F}), \exp(-\mathcal{F})$ are invertible objects. They should induce invertible endofunctors via categorical actions. We will consider certain actions of $\mathbf{D}(\widetilde{NH^e})$ or its variants, and show that the induced functors cannot be invertible.

The derived tensor product over $\widetilde{NH}$ induces an action of $\mathbf{D}(\widetilde{NH^e})$ on the derived category $\mathbf{D}(\widetilde{NH})$ of left $\widetilde{NH}$-modules. Let $P_{(k)} = \widetilde{NH} \cdot e_k$ denote the left projective $\widetilde{NH}$-module. There are isomorphisms

$$\mathcal{F}^{(k)}(P_0) \cong \widetilde{\mathcal{F}}^{(k)}(P_0) \cong P_{(k)} \in \mathbf{D}(\widetilde{NH}).$$

By definition, $\exp(\mathcal{F})$ is an iterated extension of $\widetilde{\mathcal{F}}^{(k)}$'s. So $\exp(\mathcal{F})(P_0)$ is isomorphic to an iterated extension of $\widetilde{\mathcal{F}}^{(k)}(P_0)$'s. But there is no nontrivial extension between $\widetilde{\mathcal{F}}^{(k)}(P_0)$'s since $\widetilde{\mathcal{F}}^{(k)}(P_0) \cong P_{(k)}$ are projective for all k. Thus,

$$\exp(\mathcal{F})(P_0) \cong \bigoplus_{k=0}^{\infty} P_{(k)}.$$

The induced map: $\mathrm{End}(P_0) \to \mathrm{End}(\exp(\mathcal{F})(P_0))$ is very far from being surjective. Hence, $\exp(\mathcal{F})$ cannot be an invertible endofunctor of $\mathbf{D}(\widetilde{NH})$, nor an invertible object in $\mathbf{D}(\widetilde{NH^e})$.

In the case of $\exp(-\mathcal{F})$, we consider a cyclotomic quotient $\widetilde{NH}(1)$ of $\widetilde{NH}$, where the two-sided ideal under quotient is generated by $x_{1,n}$ for $n > 0$. Idempotents 1_n are in the ideal for $n > 1$, and $\widetilde{NH}(1)$ has a $\mathbb{Z}$-basis $\{1_0, 1_1, v_1\}$. There are two projective $\widetilde{NH}(1)$-modules P_0 and P_1. Let $\mathcal{F}(1)$ denote the corresponding quotient of $\mathcal{F}$ as an $\widetilde{NH}(1)$-bimodule. It induces an endofunctor of $\mathbf{D}(\widetilde{NH}(1))$. Then $\mathcal{F}(1)(P_0) \cong P_1, \mathcal{F}(1)(P_1) = 0$, and $\mathcal{F}(1)^2 = 0$. The object $\exp(-\mathcal{F}(1))$ reduces to a complex $(\mathbf{1} \to \mathcal{F})$ of two terms. Then

$$\exp(-\mathcal{F}(1))(P_0) \cong (P_0 \xrightarrow{\cdot v_1} P_1), \qquad \exp(-\mathcal{F}(1))(P_1) \cong P_1.$$

So $\mathrm{Hom}(\exp(-\mathcal{F}(1))(P_0), \exp(-\mathcal{F}(1))(P_1)) = 0$, while $\mathrm{Hom}(P_0, P_1) \neq 0$. Hence, the endofunctor $\exp(-\mathcal{F}(1))$ is not invertible.

Lifting $\sum_{n\geq 0}(-1)^n x^n$ and $\sum_{n\geq 0} x^n$.

If we remove the nilHecke generators $x_{i,n}, \partial_{i,n}$ from $\widetilde{NH}$, we obtain a simpler diagrammatic algebra R generated by vertical strands 1_n and short strands $v_{i,n}$ only. They satisfy the same relations as in Definition 2.1. The construction of the induction bimodule $\mathcal{F}$, the morphisms $v \in \mathrm{Hom}(\mathbf{1}, \mathcal{F}), \widetilde{v} \in \mathrm{Hom}^1(\mathcal{F}, \mathbf{1})$ still works in the case of R. The difference is that we do not have the idempotents $e_k, \widetilde{e}_k$, and the corresponding direct summands $\mathcal{F}^{(k)}, \widetilde{\mathcal{F}}^{(k)}$ of $\mathcal{F}^k$. Like $\exp(\mathcal{F}), \exp(-\mathcal{F})$ using $\mathcal{F}^{(k)}, \widetilde{\mathcal{F}}^{(k)}$, one can define similar objects in the derived category of R-bimodules where $\mathcal{F}^{(k)}, \widetilde{\mathcal{F}}^{(k)}$ are replaced by $\mathcal{F}^k$. The two objects lift $\sum_{n\geq 0}(-1)^n x^n$ and $\sum_{n\geq 0} x^n$, respectively, but these liftings are not invertible functors either.

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