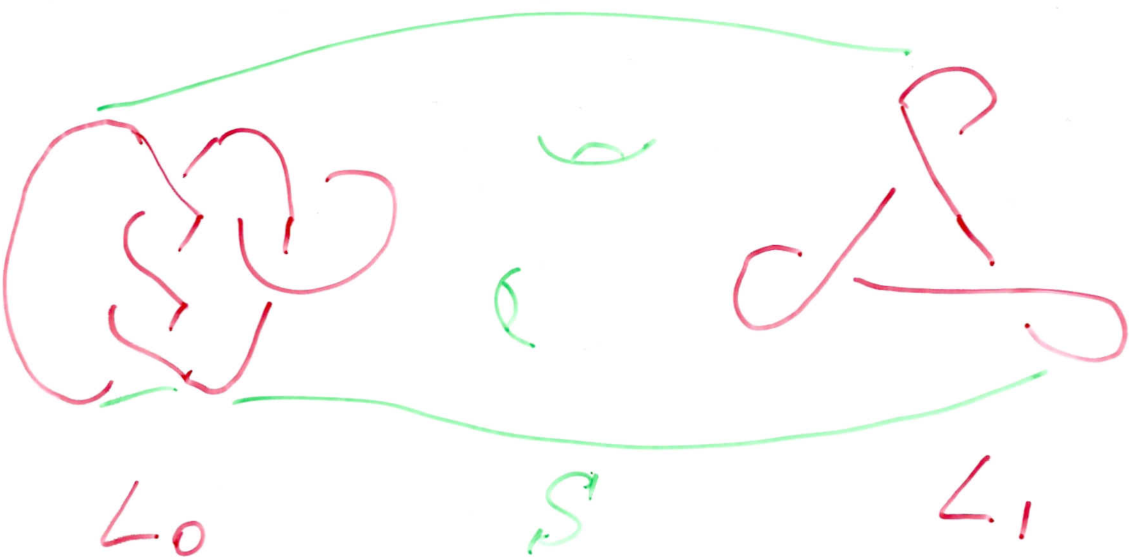
## Theorem (M. Jacobsson, M.K.)

Homology theory $H$ extends to a (projective) functor from the category of link cobordisms to the category of bigraded abelian groups



$$H(L_0) \xrightarrow{H(S)} H(L_1)$$

$L_0, L_1 \subset \mathbb{R}^3$

$S \subset \mathbb{R}^3 \times [0,1]$

$\partial S = L_0 \amalg (-L_1)$

$H(S)$ is well-defined up to overall minus sign.

Sign indeterminacy was taken care of by

David Clark

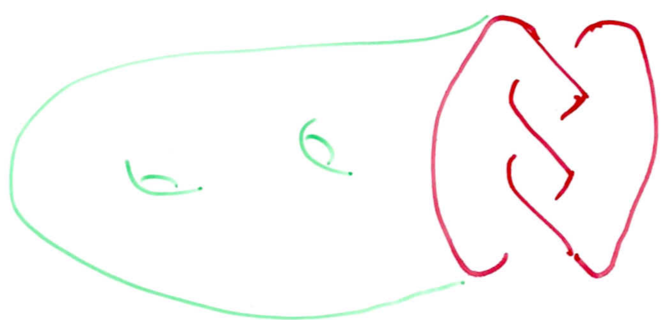
Scott Morrison

Kevin Walker

# Applications

## Jacob Rasmussen

Combinatorial proof of Kronheimer - Mrowka theorem (Milnor conjecture) on slice genus of positive knots.



$M \subset D^4$ $L \subset S^3$

$\partial M = L$

Slice genus $g_s(L)$

Positive 

$c = 3$

$s = 2$

$$\boxed{g_s(L) = \frac{c + 1 - s}{2}}$$

$c$ # of crossings

$s$ # of circles

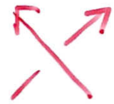
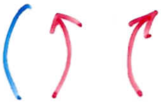
Lenhard Ng

Effective upper bound on Thurston-Bennequin number of Legendrian links

# HOMFLYPT Polynomial

Hoste, Ocneanu, Millett, Freyd,
Lickorish, Yetter, Przytycki, Traczyk

$P(L) \in \mathbb{Z}[a^{\pm 1}, b^{\pm 1}]$

$a\,P($  $) - a^{-1}\,P($  $) = b\,P($  $)$

$a = q^n \,, \; b = q - q^{-1}$

$P(L)$ related to representation theory of $U_q(sl(n))$

Quantum deformation of $U(sl(n))$

$q^n P_n($  $) - q^{-n} P_n($  $) = (q-q^{-1}) P_n($  $)$

$n=0$ Alexander Polynomial

P. Ozsváth, Z. Szabó, J. Rasmussen

Categorification – knot Floer homology

$$H_0(L) = \bigoplus_{i,j\in\mathbb{Z}} H_0^{i,j}(L)$$

$$P_0(L) = \sum (-1)^i q^j \,\mathrm{rk}\, H_0^{i,j}(L)$$

$n=2$ Jones polynomial

$$H_2(L) = \bigoplus_{i,j\in\mathbb{Z}} H_2^{i,j}(L)$$

$$P_2(L) = \sum (-1)^i q^j \,\mathrm{rk}\, H_2^{i,j}(L)$$

Lev Rozansky, M.K.

any $n > 2$

$$H_n(L) = \bigoplus_{i,j\in\mathbb{Z}} H_n^{i,j}(L)$$

$$P_n(L) = \sum_{i,j\in\mathbb{Z}} (-1)^i q^j \operatorname{rk} H_n^{i,j}(L)$$

$$H_n(\bigcirc) = \mathbb{Z}[x]/(x^n) \simeq H^*(\mathbb{CP}^{n-1}, \mathbb{Z})$$

$$P_n(\bigcirc) = [n] \quad \text{quantum } n$$

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \dots + q^{1-n}$$

Functoriality for cobordisms

# Alternative approaches to $H_n$ and related theories

J. Sussan — via highest weight categories for $sl(k)$

V. Mazorchuk, C. Stroppel

S. Cautis, J. Kamnitzer — via derived categories of coherent sheaves

P. Seidel, I. Smith, C. Manolescu — via Fukaya-Floer categories of quiver varieties

M. Mackaay, M. Stosic, P. Vaz — cohomology of partial flag varieties + foams + complex residues functors

natural transformations

Categories

Grothendieck group

vector spaces or free abelian groups

operators

dimension

numbers

# Grothendieck group

$C$ - abelian category

$C = R\text{-mod}$ , $R$ a ring

$G_0(C)$ - abelian group with generators $[M]$ , $M \in \operatorname{Ob} C$ , relations

$$[M_2] = [M_1] + [M_3]$$

for each exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

If $C$ - category of finite length modules over a ring $R$, then $G_0(C)$ is free abelian with basis

$[L]$ $\quad L$ - simple $R$-module

$R = \mathbb{Z}$ $\quad$ basis $[\mathbb{Z}/p]$, $p$ prime

$R = \mathbb{Q}[x]/(x^n)$

$G_0(R\text{-mod}) = \mathbb{Z}$

local ring, has unique simple module $\underline{\mathbb{Q}}$, $x$ acts by $0$