A functor $F : C \longrightarrow C$ is exact if it takes short exact sequences to short exact sequences

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

$$\Downarrow$$

$$0 \longrightarrow FM_1 \longrightarrow FM_2 \longrightarrow FM_3 \longrightarrow 0$$

F induces an endomorphism $[F]$ of $G_0(C)$ taking $[M]$ to $[FM]$

A categorification of the ring of polynomials $\mathbb{Q}[x]$.

Differentiation ∂ and multiplication by x are operators on $\mathbb{Q}[x]$, and

$$\partial x = x\partial + 1$$

Have two integral lattices

$$\mathbb{Z}[x] \text{ and } \mathbb{Z}\left[\frac{x^n}{n!}\right]$$

stable under x and ∂.

Have a bilinear form $(\,,\,)$

$$\mathbb{Z}[x] \times \mathbb{Z}\left[\frac{x^n}{n!}\right] \longrightarrow \mathbb{Z}$$

$$(x^m, x^n) = \delta_{n,m}\, n!$$

$$(x \cdot f, g) = (f, \partial g)$$

After categorification, $(\,,\,)$ will become $\mathrm{Hom}(\,,\,)$ and x, ∂ will become adjoint functors

$$\mathrm{Hom}(XM, N) \simeq \mathrm{Hom}(M, DN)$$

T_i - Newton divided difference operator on polynomials

$f \in \mathbb{C}[y_1, y_2, \dots, y_n]$

$s_i f$ - permute y_i and y_{i+1} in f

$$s_1(y_1^3 y_2) = y_2^3 y_1$$

$$T_i f = \frac{f - s_i f}{y_i - y_{i+1}}$$

$$T_1(y_1^3 y_2) = \frac{y_1^3 y_2 - y_1 y_2^3}{y_1 - y_2} =$$

$$= y_1 y_2 (y_1 + y_2)$$

R_n - nil Coxeter algebra / field $\mathbb{C}$

generators $T_1, T_2, \ldots, T_{n-1}$

relations

$$T_i^2 = 0$$

$$T_i T_j = T_j T_i \qquad |i-j|>1$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$$

$\dim R_n = n!$

Has unique simple module

$L_n = \mathbb{C} v$, $T_i v = 0$

$G_0(R_n) \simeq \mathbb{Z}$, generator $[L_n]$

$$G_0(R_n\text{-mod}) \simeq \mathbb{Z}\frac{x^n}{n!}$$

$$[L_n] \longmapsto \frac{x^n}{n!}$$

R_n has composition series with

$n!$ copies of L_n

$$[R_n] = n!\,[L_n]$$

$$[R_n] \longmapsto x^n$$

Have inclusions $R_n \subset R_{n+1}$

Induction and restriction

functors

$$\begin{tikzcd} R_n\text{-mod} \arrow[r, bend left, "\mathrm{Ind}"] & R_{n+1}\text{-mod} \arrow[l, bend left, "\mathrm{Res}"] \end{tikzcd}$$

$A \subset B$ inclusion of rings

$\mathrm{Ind} : A\text{-mod} \longrightarrow B\text{-mod}$

$\mathrm{Ind}(M) = B \otimes_A M$

$\mathrm{Res} : B\text{-mod} \longrightarrow A\text{-mod}$

$\mathrm{Res}(N) = {}_A N$

Restriction is an exact functor

Induction is exact sometimes.

If B is projective as right A-module, Ind is exact

$$\mathcal{C} = \bigoplus_{n \geqslant 0} R_n\text{-mod}$$

$$G_0(\mathcal{C}) = \bigoplus_{n \geqslant 0} G_0(R_n\text{-mod})$$

$$G_0(\mathcal{C}) \simeq \mathbb{Z}\left[\frac{x^n}{n!}\right]_{n \geqslant 0}$$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "X"] & \mathcal{C} \arrow[l, bend left, "D"] \end{tikzcd}$$

induction

exact functors

restriction

$$X(R_n) = R_{n+1}$$

$$\begin{array}{ccc} [R_n] & \overset{[X]}{\longmapsto} & [R_{n+1}] \\ \shortparallel & & \shortparallel \\ x^n & \longmapsto & x^{n+1} \end{array}$$

$[X]$ is multiplication by x

$$D(R_{n+1}) = R_n^{\oplus n+1}$$

$$x^{n+1} \longmapsto (n+1)x^n$$

$$[R_{n+1}] \xmapsto[{[D]}]{} [R_n^{\oplus n+1}]$$

$[D]$ is differentiation ∂

$$D(\angle_{n+1}) = \angle_n$$

$$\frac{x^{n+1}}{(n+1)!} \longmapsto \frac{x^n}{n!}$$

$$\mathrm{Hom}_C(R_n, \angle_m) = \begin{cases} 0, & n \neq m \\ \mathbb{C}, & n = m \end{cases}$$

$$\left(x^n, \frac{x^m}{m!}\right) = \delta_{n,m}$$

$$\dim \mathrm{Hom}_C(P, M) = ([P], [M])$$

P - projective

$$\mathcal{C} \xrightarrow{X,\, D} \mathcal{C} \qquad \mathcal{C} = \bigoplus_{n \geq 0} R_n\text{-mod}$$

$$\begin{tikzcd}
G_0(\mathcal{C}) \arrow[r, "{[X],\,[D]}"] \arrow[d, phantom, "\cong" rotate=90] & G_0(\mathcal{C}) \arrow[d, phantom, "\cong" rotate=90] \\
\mathbb{Z}\left[\frac{x^n}{n!}\right] \arrow[r, "{x,\, \partial}"] & \mathbb{Z}\left[\frac{x^n}{n!}\right]
\end{tikzcd}$$

<u>Proposition</u> There is an isomorphism of functors

$$DX \simeq XD \oplus \mathrm{Id}$$

$$R_n\text{-mod} \longrightarrow R_n\text{-mod}$$

$$\partial x = x\partial + 1$$

$R_n \otimes R_m \subset R_{n+m}$

Ind, Res functors

sum over all $n, m \geq 0$

$$\mathcal{C} \otimes \mathcal{C} \underset{\text{Res}}{\overset{\text{Ind}}{\rightleftarrows}} \mathcal{C}$$

$$\begin{array}{ccc} G_0(\mathcal{C}) \times G_0(\mathcal{C}) & \underset{[\text{Res}]}{\overset{[\text{Ind}]}{\rightleftarrows}} & G_0(\mathcal{C}) \\ \wr| & & \wr| \\ \mathbb{Z}\left[\frac{x^n}{n!}\right] \times \mathbb{Z}\left[\frac{x^n}{n!}\right] & \rightleftarrows & \mathbb{Z}\left[\frac{x^n}{n!}\right] \end{array}$$

$$x^n \otimes x^m \xmapsto{\text{multiplication}} x^{n+m}$$

comultiplication

$$\sum_{k=0}^{n} \frac{x^k}{k!} \otimes \frac{x^{n-k}}{(n-k)!} \longleftarrow\!\shortmid \frac{x^n}{n!}$$