Rasmussen's combinatorial proof of the Milnor conjecture

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Have a knot $K \subset S^3$. Slice (or 4-ball) genus $g_4(K)$ is the min. genus of a smooth orientable surface $M \subset D^4$, $\partial M = K$.

$g_4(K) = 0 \iff K$ is slice.

$\sigma(K)$ signature

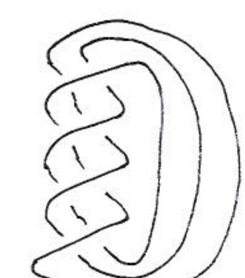
$$\text{(Seifert) genus } g(K) \geq g_4(K) \geq \frac{|\sigma(K)|}{2}$$

$$\text{unknotting number } u(K) \geq g_4(K)$$

$\sigma(K) = \mathrm{sign}(A + A^T)$ even

A - Seifert matrix.

Torus knot $T_{n,m}$ $\qquad T_{3,4}$



Milnor conjecture: $\boxed{g_4(T_{n,m}) = \frac{(n-1)(m-1)}{2}}$

easy to see $g_4(T_{n,m}) \leq u(T_{n,m}) \leq \frac{(n-1)(m-1)}{2}$, $\quad g_4(T_{n,m}) \leq g(T_{n,m}) \leq \frac{(n-1)(m-1)}{2}$

Hard: prove inequality in the opposite direction.

asymptotically $\sigma(T_{n,m}) \approx \frac{-nm}{2} \Rightarrow g_4(T_{n,m}) > \frac{nm}{4}$ large n, m

not good enough

2D TQFT $\qquad$ R - commutative ring

$$\text{2d cobordisms} \xrightarrow[\text{functor}]{\text{tensor}} R\text{-modules}$$

$\emptyset \longmapsto R$

$\bigcirc \longmapsto A \qquad R\text{-module} \qquad A \simeq R^{\oplus n}$

$\bigcirc\,\bigcirc \longmapsto A \otimes_R A$

$A^{\otimes 2} \xrightarrow{m} A$

unit

$R \to A$

counit (trace)

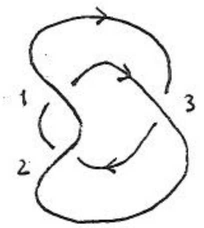
$A \to R$

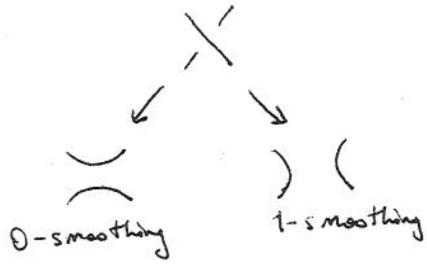
m commutative, associative

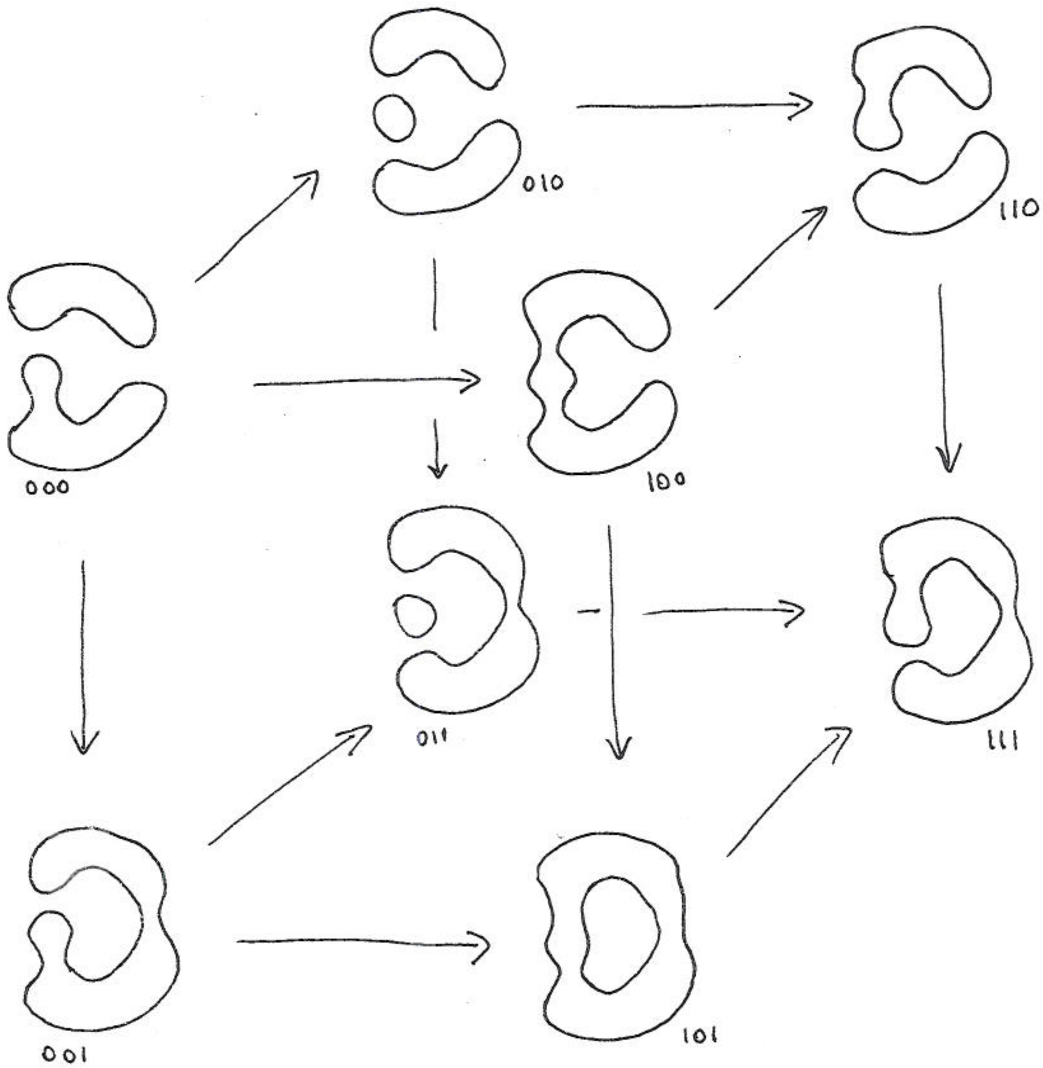
A Frobenius R-algebra

$x(D) = \#$ 2

$y(D) = \#$ 1







$R = \mathbb{Q}[t] = H^*(\mathbb{HP}^\infty, \mathbb{Q}$

$A = \mathbb{Q}[X] = H^*(\mathbb{CP}^\infty, \mathbb{Q}$

$X^2 = t$

$A^{\otimes 2} = A \otimes_R A$, etc.

$A^{\otimes 2} \xrightarrow{m} A$

$A \xrightarrow{\Delta} A^{\otimes 2}$

$\Delta(1) = 1 \otimes X + X \otimes 1$

$\Delta(X) = X \otimes X + t\, 1 \otimes 1$

Commutative cube

$$\begin{tikzcd}
 & A^{\otimes 3} \arrow[rr, "m"] \arrow[dd, "\Delta"] & & A^{\otimes 2} \arrow[dd, "m"] \\
A^{\otimes 2} \arrow[ur, "\Delta"] \arrow[rr, "m"] \arrow[dd, "m"] & & A \arrow[ur, "\Delta"] \arrow[dd, "\Delta"] & \\
 & A^{\otimes 2} \arrow[rr, "m"] & & A \\
A \arrow[ur, "\Delta"] \arrow[rr, "\Delta"] & & A^{\otimes 2} \arrow[ur, "m"] &
\end{tikzcd}$$

Total complex

add grading shifts

$$0 \to \underset{-x(D)}{A^{\otimes 2}} \xrightarrow{d} \begin{matrix} A^{\otimes 3} \\ \oplus \\ A \\ \oplus \\ A \end{matrix} \xrightarrow{d} \begin{matrix} A^{\otimes 2} \\ \oplus \\ A^{\otimes 2} \\ \oplus \\ A^{\otimes 2} \end{matrix} \xrightarrow{d} \underset{y(D)}{A} \to 0$$

$H(L)$ - link homology, functorial

$H(L) = \bigoplus H^{i,j}(L)$

$H(L_1) \xrightarrow{H(S)} H(L_2)$

$H^{i,j} \longrightarrow H^{i,j-2\chi(S)}$

$H(L)$ is an $R = \mathbb{Q}[t]$ module.

fix component of L $\Rightarrow$ $H(L)$ is an $A = \mathbb{Q}[x]$-module.

ε knot K

$H(K) \supset Tor(K)$ t-torsion $H(K)/Tor(K) = H'(K)$

Prop (E. S. Lee) $H'(K) \simeq A\{m\}$ $\square$

Proof Specialize $t=1$. The resulting theory is almost trivial. $\square$.

$m = -s(K)$ Rasmussen invariant.

Prop (J. Rasmussen). If S is a connected knot cobordism  K_1 K_2

$$H'(K_1) \xrightarrow{H'(S)} H'(K_2)\{-2g\}$$ is nontrivial, grading-preserving.

$\Rightarrow$ $s(K)$ is concordance invariant

S

K

$$A \longrightarrow H'(K)\{-2g\}$$
$$H'(K) \longrightarrow A\{-2g\}.$$

$\Rightarrow |s(K)| \leq 2g$

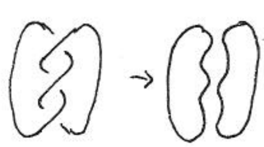
$\Rightarrow$ $g_4(K) \geq \dfrac{|s(K)|}{2}$

K - alternating $\Rightarrow -s(K) = \delta(K)$ E. S. Lee

K - positive

$s(K) = n+1-c$ $\Rightarrow g_4(K) \geq \dfrac{n+1-c}{2}$

easy to see $g(K) \leq \dfrac{n+1-c}{2}$



n # of crossings c # of Seifert circles

$$\boxed{g(K) = g_4(K) = \frac{n+1-c}{2}} \quad (*)$$

K - torus $\Rightarrow$ Milnor conjecture if $\Delta(K)=1 \Rightarrow K$ is top. slice (M. Freedman) A. Casson

K $(-3,5,7)$-pretzel or untwisted double of trefoil

($*$) implies $s(K) \neq 0 \Rightarrow K$ not smoothly slice

Previous proofs relied on Donaldson's thm.

$\Rightarrow$ fake $\mathbb{R}^4$'s exist (Gompf).