Topology, fall 2015

Homework 11, due Wednesday December 9 before class.

Read §70 – 73 (in §71 you can read up to and including Theorem 71.3, the rest of §71 is optional reading).

I. Exercise 3ab on page 433.

II. Give an example where the Seifert-van Kampen theorem fails if one does not assume that \( U \cap V \) is path-connected.

III. (a) What is the relation between punctured \( \mathbb{R}P^2 \) (that is, \( \mathbb{R}P^2 \) with a point deleted) and the Möbius band?
   (b) Remove two points from \( \mathbb{R}P^2 \) to get a space \( X = \mathbb{R}P^2 \setminus \{p, q\} \).
   Determine the fundamental group of \( X \). (Hint: if you solve (a) first, you’ll be able make a picture of \( X \). Then look for a homotopy retract of \( X \) with the fundamental group easy to determine).

IV. Find fundamental groups of the following spaces:
   (a) Wedge \( \mathbb{R}P^2 \vee \mathbb{R}P^2 \) of two projective planes.
   (b) Wedge \( \mathbb{R}P^2 \vee S^1 \vee S^2 \).
   (c) Once punctured Klein bottle \( KB \setminus \{p\} \).
   (d) Twice punctured Klein bottle \( KB \setminus \{p, q\} \).
   (e) Once punctured oriented surface \( M \) of genus 2.
   (f) Two-sphere with a disc \( B^2 \) attached along the equator.
   (g) Take two two-dimensional tori \( T^2_1 \) and \( T^2_2 \) and identify a longitudinal circle of the first tori with a meridianal circle of the second tori. Determine \( \pi_1 \) of the resulting space.

V. Take a solid torus \( B^2 \times S^1 \), a Hausdorff space \( Y \), and a continuous map \( \psi : T^2 \to Y \) from the boundary surface \( T^2 = S^1 \times S^1 \) of the solid torus. Glue the solid torus and \( Y \) via this map (by identifying point \( s \in T^2 \) with its image in \( Y \) for all \( s \)) and denote the resulting space \( X \). Explain the relation between the fundamental groups of \( Y \) and \( X \) by analogy with the Theorem 72.1 (also discussed in class).