Notion of top. space generalizes familiar properties of $\mathbb{R}^n$ & its subsets (subspaces)
give up the properties of the distance function in favor of something much more flexible.

Have a notion of an open subset U of $\mathbb{R}$, $\mathbb{R}^n$, etc.

$\forall$ point $x \in U$ all the points "suff. close to x" are in U.

Abstract this notion.

Def A topology on a set X is a collection $\mathcal{T}$ only of subsets of X s.t.

(1) $\emptyset, X \in \mathcal{T}$

(2) Union of elements of any subcoll of $\mathcal{T}$ is in $\mathcal{T}$

(3) Intersection of elements of any finite subcoll of $\mathcal{T}$ is in $\mathcal{T}$

Pair $(X, \mathcal{T})$ is called a top. space. Sets in $\mathcal{T}$ are called open ~~subsets~~ sets of X.

Asymmetry between $\cap$ (finite) and $\cup$ (any)

$U \subset \mathbb{R}^n$ open if $\forall x \in U\ \exists \varepsilon > 0$ $\forall y$ s.t. $|y - x| < \varepsilon$, $y \in U$.

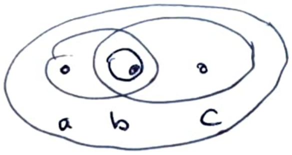
Ex! Convince yourself that ~~the set~~ $\mathbb{R}^n$ with the usual notion of an open subset is a top. space.

There are many examples. For instance, here are some topologies on a

(a) 3-el't set $\{a, b, c\}$

§12 See ex 1. in textbook p.76 for complete list, up to perm of $\{a, b, c\}$

$\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}$



$\mathcal{T} = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$

(b) $\forall$ ~~set~~ X, $\mathcal{T} = P(X)$ all subsets. discrete topology

$\forall X$ $\mathcal{T} = \{\emptyset, X\}$ indiscrete topology.

Ex 2 ~~If $(X, \mathcal{T}_0)$~~, Describe a natural topology on $X \amalg Y$ (disjoint union) given topologies (X, T) and (Y, T'). What are the open sets in this topology?

$X, Y \subset Z$

$Z \times \{0,1\}$

$X \times \{0\} \cup Y \times \{1\}$

$$T'' = \{U \cup V \mid U \in \mathcal{T}, V \in \mathcal{T}'\}$$ Ex this is a topology

(c) finite complement topology. X - set, T_f = all subsets $U \subset X$ s.t. $X \setminus U$ is finite or all of X ($U = \emptyset$).

<u>This is a top. on X.</u> Open set is either empty or "very large".

Union of any el't of T_f (non empty) with anything is in T_f.

intersection $\bigcap_{i=1}^{n} U_i$ $\qquad X \setminus \bigcap_{i=1}^{n} U_i = \bigcup (X \setminus U_i)$ finite set.

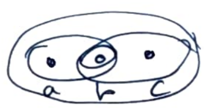
$\bigcap (U_i \cap Y) = (\bigcap U_i) \cap Y$

$\bigcup (U_i \cap Y) = (\bigcup U_i) \cap Y$

(d) Induced topology (Subspace): if $Y \subset X$, $(X, T) \rightsquigarrow X, T|_Y \rightsquigarrow (Y, T')$ (T_Y or)

$U \in T'$ iff $\exists V \in T$ s.t. $\underline{U = V \cap Y}$ $\qquad \mathcal{T}|_Y = \{U \cap Y \mid U \in \mathcal{T}\}$

Example $X =$



$Y = \{a, c\}$ $\quad$ ~~T' on Y.~~ $T|_Y$. ~~On~~ $T|_Y$ is discrete.

~~Label top on Fig 12.1.~~ from left to right $X_1, X_2, X_3 \ldots$ $1, 2, 3,$ $T_1, \ldots$

Ex 3 Which of the topologies ~~on fin~~ are

(a) ~~Inde~~ Find discrete and indiscrete top.

(b) which of them ~~and~~ admit a subset Y with 2 el't s.t.

(a) $T|_Y$ is discrete? (b) $T|_Y$ is indiscrete?

(c) finer/coarser.

(d) Why a top? Closed under fin intersections, arb. unions

$$(U_1 \cap Y) \cap \ldots \cap (U_n \cap Y) = (U_1 \cap \ldots \cap U_n) \cap Y.$$

$$\bigcup_{\alpha \in J} (U_\alpha \cap Y) = \left(\bigcup_{\alpha \in J} U_\alpha\right) \cap Y.$$

distributivity

$$\left(\bigcup_i U_i\right) \cap \left(\bigcup_j V_j\right) = \bigcup_{i,j} (U_i \cap V_j)$$

$$\left(\bigcup_i U_i\right) \cap V = \bigcup_i (U_i \cap V)$$

$$\left(\bigcap_i U_i\right) \cup V = \bigcap_i (U_i \cup V)$$

$\mathcal{T}, \mathcal{T}'$ are top on X,

If $\mathcal{T}' \supset \mathcal{T}$ say $\mathcal{T}'$ is finer than $\mathcal{T}$ (in part $\mathcal{T}$ is finer than $\mathcal{T}$)
and if $\mathcal{T}' \supset \mathcal{T}$ properly, $\mathcal{T}'$ is strictly finer than $\mathcal{T}$.

$\mathcal{T}$ is coarser than $\mathcal{T}'$; strictly coarser

$$\boxed{\underset{\text{coarser}}{\mathcal{T}} \subset \underset{\text{finer}}{\mathcal{T}'}}$$

$\mathcal{T}$ comparable with $\mathcal{T}'$ if either $\mathcal{T}' \supset \mathcal{T}$ or $\mathcal{T} \subset \mathcal{T}'$.

Ex ~~Indis~~ X_d - discrete top is finer than any top on X.

X_{ind} indiscrete top is ~~is~~ coarser than any top on X.

~~Exam~~ Problem ~~Bf.~~ Find as long as you can chain of top $\mathcal{T}_1 \subset \mathcal{T}_2 \dots \mathcal{T}_n$ on a 3-elt set ~~of~~ $\{a, b, c\}$. The starting top should be ind, $\mathcal{T}_n$ = discrete. Squeeze as much as you can in between - ~~Fi~~ ~~Addi~~ Even better if you can find an efficient way to encode your chain

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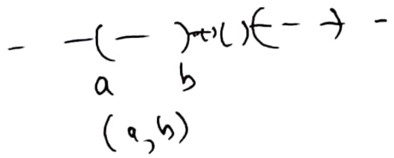
Discuss $\mathbb{R}$ & open sets in $\mathbb{R}$



Prop $\mathcal{T}'$ finer than $\mathcal{T}$, $\mathcal{T}$ finer than $\mathcal{T}' \Rightarrow \mathcal{T} = \mathcal{T}'$.

<u>Basis for topology</u> X set, Basis $\mathcal{B}$ is a coll. of subsets, called <u>basis</u> elements, s.t

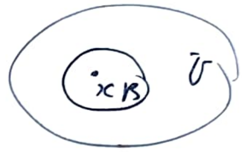
(1) $\forall x \in X \ \exists B \in \mathcal{B}$, $x \in B$. ($\mathcal{B}$ "covers" X) $\mathcal{B}$ contains many small sets that contain x.

(2) if $x \in B_1 \cap B_2$ $B_i \in \mathcal{B} \Rightarrow \exists B_3 \in \mathcal{B}$ $x \in B_3 \subset B_1 \cap B_2$

Generate topology $\mathcal{T}$ from $\mathcal{B}$.. by taking arb. unions of el's of $\mathcal{B}$

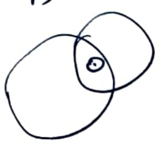
U is a union of elements of $\mathcal{B}$ <u>subsets in $\mathcal{B}$.</u>

(*) $\underbrace{U \subset \mathcal{T}}_{\text{Define}}$ open if $\underbrace{\forall x \in U \ \exists B \in \mathcal{B} \text{ s.t } x \in B \subset U.}_{\text{prop } (*)}$



<u>Ex</u> Collection of circular regions (interiors of circles) in the plane

(a) $\mathcal{B}$.



(b) $\mathcal{B}'$ all rect regions, with sides parallel to x, y axes.

(c) All 1-point subsets - basis for discrete top

$\emptyset$ may or may not be in $\mathcal{B}$. Union of 0 el's of $\mathcal{B}$

<u>Prop</u> This is a top on X.

(a) Union of sets with prop (*) has prop (*)

(b) Inf. $U_1 \cap .. \cap U_n$ Ind. $(U_1 \cap .. \cap U_{n-1}) \cap U_n$

First for two $U_1 \cap .. \cap U_{n-1} \in \mathcal{T}$, $U_n \in \mathcal{T}$.

$\downarrow$

$U_1 \cap U_2$

$x \in B_3 \subset B_1 \cap B_2 \subset U_1 \cap U_2$.

$\underset{\text{basis}}{\mathcal{B}} \xrightarrow{\text{take arb. unions of el}} \underset{\text{top}}{\mathcal{T}}$

take arb. unions of el's of $\mathcal{B}$.

plus empty set

Example: $\mathbb{R}^2$, usual top

basis $\mathcal{B}_1$

basis $\mathcal{B}_2$

$\forall x \in B_1$ is in some $V \in \mathcal{B}_2$

Subbasis $\qquad$ Basis $\qquad$ Topology

$$\mathcal{S} \xrightarrow[\text{intersections}]{\text{take all finite}} \mathcal{B} \xrightarrow[\text{arbitrary unions}]{\text{add}} \mathcal{T}$$

covers X $\qquad$ covers X

~~take all~~

<u>why basis?</u> Convenient way to describe topology.

How do test whether $\mathcal{B}$ is a basis for $\mathcal{T}$?

<u>Prop</u> (test) If $\mathcal{T}$ top. on X, $\mathcal{B} \subset \mathcal{T}$ (subcollection, collection of subsets) is a basis iff $\forall U \in \mathcal{T}, \forall x \in U \ \exists B \in \mathcal{B}$ s.t. $x \in B \subset U$.

<u>Example</u> basis for $\mathbb{R}$ usual top. all open intervals (a,b) $a<b$.

what are open sets? ~~( ) ∪ ( )~~ $\qquad (a,+\infty)$, $(-\infty,a)$, $\emptyset, \mathbb{R}$

Countable unions of disjoint intervals

countable vs. uncountable

More dramatic diff in complexity of description for $\mathbb{R}^2$ with usual top.

$X \qquad U \subset X$ open $\Leftrightarrow X \setminus U$ closed

Complement of open is closed

$\emptyset, X$ - closed

$\forall$ intersection of closed is closed

fin union of closed is closed

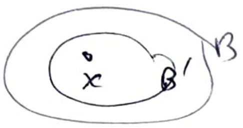
$1/2$

Prop (Comparing bases). Let $\mathcal{B}, \mathcal{B}'$ be bases for $\mathcal{T}, \mathcal{T}'$

TFAE:

(1) $\mathcal{T}'$ is finer than $\mathcal{T}$

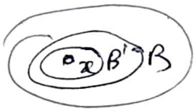
(2) $\forall x \in X \ \forall B \in \mathcal{B}, x \in B \quad \exists B' \in \mathcal{B}'$ s.t. $x \in B' \subset B$.



Can squeeze B' between B and x

Proof $(2) \Rightarrow (1)$ Let $U \in \mathcal{T}$ want to show $U \in \mathcal{T}'$

$x \in U$

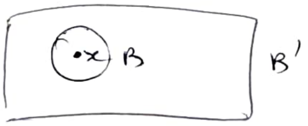


U find $B \Rightarrow$ find $B' \Rightarrow U \in \mathcal{T}'$

$(1) \Rightarrow (2)$. given x, $B \in \mathcal{B}$, $x \in B$ $\quad B \in \mathcal{T} \Rightarrow B \in \mathcal{T}' \Rightarrow \exists B'$

$\mathcal{B}$ - all open rect regions

a_1, a_2
b_1, b_2 $\quad a_1 < x < a_2$
$b_1 < y < b_2$



$\mathcal{B}'$ - all open circular regions.

same top

Are there interesting bases for finite complement topology?

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Example: Topologies on $\mathbb{R}$

1) standard $\mathcal{B} = \{(a,b)\}_{a<b}$ $\quad (a,b) = \{x \mid a<x<b\}$

2) $\mathcal{B}' = \{[a,b)\}$ $\quad [a,b) = \{x \mid a \le x < b\}$

lower limit topology on $\mathbb{R}$, denoted $\mathbb{R}_\ell$

3). $K = \{\frac{1}{n}\}_{n \geq 1}$ $\quad \mathcal{B}'' = \mathcal{B} \cup$ sets of form $(a,b) \setminus K$

K-topology, $\mathbb{R}_K$

why are they bases? finite intersections. check that $\mathcal{B}''$ is a basis for a top.

Explain why $\mathbb{R}_K$

Prop $\mathcal{T}'$ gen by $\mathcal{B}'$ is strictly finer than $\mathbb{R}$.

$\mathcal{T}''$ ~~gen~~ gen by $\mathcal{B}''$

Proof $\forall B \in \mathcal{B}, \forall x \in B$ $\quad \overset{a}{(}\underset{x}{|}\overset{b}{)}$ take $[x,b) \in \mathcal{B}' \Rightarrow \mathcal{B}'$ finer than $\mathcal{B}$

$[x,d)$ in $\mathcal{B}'$ can't find $B \in \mathcal{B}$ $\quad x \in B \subset [x,d)$.

likewise for $\mathbb{R}_K$ $\quad (\quad \overset{0}{|} \dots \circ \circ)$

a) Explain why the 3 ~~top.~~ are bases

Ex b) Show that $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are not comparable.

Subspace topology : examples'. $\mathbb{Z} \subset \mathbb{R}$

$\mathbb{Z}$ a) $\mathcal{T}_\mathbb{Z}$ is discrete

b) $K = \{\frac{1}{n}\}_{n \geq 1}$.

$\mathcal{T}_K$ is discrete

c). $K_0 = K \cup \{0\}$

$\mathcal{T}_{K_0}$ is not discrete.

* A <u>subbasis</u> S for a top. on X is a collection of subsets that covers X.

<u>Prop</u> All finite ints of el't sets in a subbasis constitute a basis of X.

$x \in X$. $x \in$ el't of $S \Rightarrow$ el't of B.

$B_1 = S_1 \cap .. \cap S_m$ $\quad B_2 = S_1' \cap .. \cap S_n'$ $\quad B_1 \cap B_2 = (S_1 \cap .. \cap S_m) \cap (S_1' \cap .. S_n')$

can take

Can we get any basis of T this way? No, only some

$$\text{subbasis } S \longrightarrow \text{basis } B \longrightarrow \text{topology } T$$

<u>Hw</u> Ex. 6, 8ab.

list all possible top on