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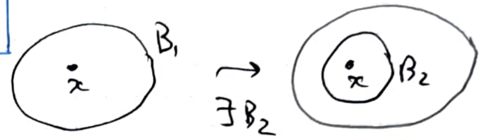
Topology, lecture 2

Last time: topology $\mathcal{T}$ on X
basis of topology

subbasis $\mathcal{S}$ $\xrightarrow{\text{finite intersections}}$ basis $\mathcal{B}$ $\xrightarrow{\text{arbitrary unions}}$ topology $\mathcal{T}$

2 bases lead to the same topology if always can nest one inside the other



~~B_2~~ $\mathcal{T}_2$ finer than $\mathcal{T}_1$

both ways $\Rightarrow$ $\mathcal{T}_2 = \mathcal{T}_1$, same topology

<u>Subbasis</u> $\mathcal{S}$ - any collection of sets that covers X.

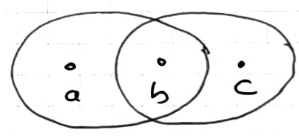
Given $\mathcal{S}$, let $\mathcal{B}_\mathcal{S}$ consist of all finite intersections of elements of $\mathcal{S}$.

<u>Prop</u> $\mathcal{B}_S$ is a basis in X.

<u>Ex</u>: Prove this proposition

$$\mathcal{S} \xmapsto{\text{finite } \cap} \mathcal{B}_S \xrightarrow{\text{unions}} \mathcal{T}_S$$ ← the coarsest topology s.t. each element of $\mathcal{S}$ is open.

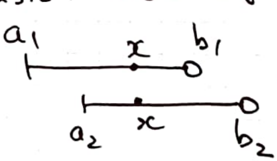
<u>Example</u> $X = \{a,b,c\}$, $\mathcal{S} = \{\{a,b\},\{b,c\}\}$



Get topology $\mathcal{T}_S = \{\emptyset, \{b\}, \{a,b\}, \{b,c\}, \{a,b,c\}\}$

<u>Example</u> Let $\mathcal{B}' = \{[a,b)\}_{a<b}$ $\quad [a,b) = \{x \in \mathbb{R} \mid a \leq x < b\}$

Then $\mathcal{B}'$ is a basis for a topology on $\mathbb{R}$ called the <u>lower limit topology</u> denoted $\mathbb{R}_\ell$. (see Munkres p. 82 lemma 13.4) why a basis? Look at overlaps



$\Rightarrow x \in [\max(a_1,a_2), \min(b_1$

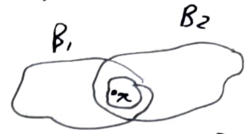
<u>Prop</u> $\mathbb{R}_\ell$ is <u>strictly finer</u> than the standard top. on $\mathbb{R}$.

Munkres introduces $\mathbb{R}_\ell$ and $\mathbb{R}_K$ to use for various examples of top. spaces with strange properties.

The product topology §15

Last time:
Munkres §12, 13

Def The product topology on $X \times Y$ is the topology with basis collection $\mathcal{B}$ of all sets $U \times V$, $U \subset X$ open, $V \subset Y$ open



B_1, B_2, x given
find B_3

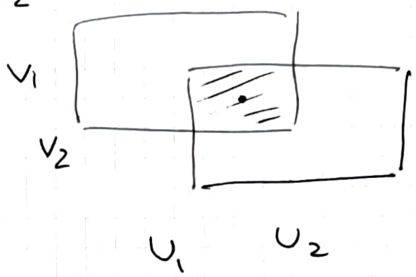
want $x \in B_3 \subset B_1 \cap B_2$

Why topology (why basis?)

(a) covers $X \times Y$

(b) $B_1 = U_1 \times V_1$, $B_2 = U_2 \times V_2$

$x \times y \in B_1 \cap B_2$



$x \times y \in (U_1 \cap U_2) \times (V_1 \cap V_2)$

$X \times Y \times Z$

$\Rightarrow \mathcal{B}$ is a basis, gives rise to product topology on $X \times Y$.

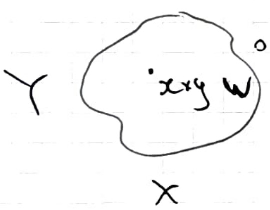
take unions of ~~elts of~~ sets in $\mathcal{B}$ to get all open sets.

Thm (15.1) If $\mathcal{B}$ is a basis for top on X, $\mathcal{C}$ a basis for top. on $Y \Rightarrow$

$\mathcal{D} = \{ B \times C \mid B \in \mathcal{B}, C \in \mathcal{C} \}$ is a basis for product top. on $X \times Y$.

Proof (lemma 13.2) el's of $\mathcal{D}$ are open in $\mathcal{T}_{X \times Y}$

lemma 13.2: X-top. space
$\mathcal{C}$ - some open sets in X
$\forall$ open U, $\forall x$ $\exists C \in \mathcal{C}$
$x \in C \subset U \Rightarrow \mathcal{C}$ is a basis for the top on X.



open set in $X \times Y$

$x \times y \in W$-open $\Rightarrow x \times y \in U \times V \subset W$

pick $x \in B \subset U$, $y \in C \subset W$

$x \times y \subset B \times C \subset U \times V \subset W$

$\Rightarrow$ get standard top on $\mathbb{R}^2$, on $\mathbb{R}^n$.

$$\begin{array}{ccc} X \times Y & \xrightarrow{\pi_2} & Y \\ \downarrow{\scriptstyle \pi_1} & & \\ X & & \end{array}$$

$\pi_1(x,y) = x$
$\pi_2(x,y) = y$
projections

$U \subset X$ open $\quad \pi_1^{-1}(U) = U \times Y$ open

$V \subset Y$ open $\quad \pi_2^{-1}(V) = X \times V$ open

$U \in \mathcal{T}_X$

Thm (15.2) $\mathcal{S} = \{ \pi_1^{-1}(U) \mid U \text{ open in } X \} \cup \{ \pi_2^{-1}(V) \mid V \in \mathcal{T}_Y \}$

is a subbasis for $\mathcal{T}_{X \times Y}$

Proof el's of $\mathcal{S}$ are open and can get any basis elt of $\mathcal{T}_{X \times Y}$ (product basis) as intersection $\pi_1^{-1}(B) \cap \pi_2^{-1}(C)$. $\square$

Subspace topology (§16)

$(X, \mathcal{T}_X)$, $Y \subset X$ subset set $\mathcal{T}_Y = \mathcal{T}_X|_Y$

$\mathcal{T}_Y := \{ Y \cap U \mid U \in \mathcal{T}_X \}$ is a top. on Y. (induced topology or subspace topology)

lemma (16.1) If $\mathcal{B}$ is a basis ~~of~~ for $\mathcal{T}_X$ then $\mathcal{B}_Y = \{ B \cap Y \mid B \in \mathcal{B} \}$ is a basis for the subspace topology on Y

Be careful with the notion of open set when $Y \subset X$

U is <u>open in Y</u> (open rel. to Y) if it's in $\mathcal{T}_Y$ ($\Rightarrow U \subset Y$)

U is <u>open in X</u> if $U \in \mathcal{T}$

U might be open in Y but not in X. For instance, take $Y \subset X$, $Y \notin \mathcal{T}_X$ and $U = Y$.

<u>Ex</u> $\underset{Y}{(0,1)} \subset \underset{X}{\mathbb{R}}$ $\quad [0, 1/2)$ open in Y, not open in X.

lemma (16.2) Let $Y \subset X$. If U is open in Y, Y is open in X $\Rightarrow U$ is open in X.

<u>Proof</u> U open in $Y \Rightarrow U = Y \cap V$, V open in $X \Rightarrow U$ open in X

Thm (16.3) If $A \subset X$, $B \subset Y$ subspaces, then product top. on $A \times B$ is the same as top. ~~on~~ $A \times B$ inherits from $X \times Y$.

basis for subspace top: $(A \times B) \cap (U \times V)$ — U open in X, V open in Y

$$\shortparallel$$

basis for prod. top.: $(U \cap A) \times (V \cap B)$

$$\boxed{\begin{array}{c}\text{first restrict, then form product} \\ \wr\wr \\ \text{form product, then restrict}\end{array}}$$

commutativity of these operations

Order topology

X simply ordered set (total order, complete order)

Order relation $<$ (then $\leq$ means $<$ or $=$)

(1) $\forall\, x, y \in X$ ~~can~~ exactly 1 out of 3 possibilities hold $x=y$, $x<y$, $y<x$

(2) $x<y, y<z \Rightarrow x<z$ transitivity

(weaker notion: partial order)

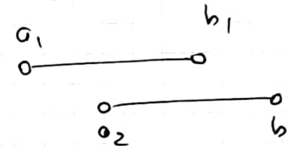
for $a, b \in X$, $a<b$ define $(a,b) = \{x \in X \mid a<x<b\}$ open interval

$[a,b)$, $(a,b]$, $[a,b]$ likewise

Let $\mathcal{B}$ all sets of the form

(1) (a,b), (2) $[a_0, b)$ where a_0 is the smallest (if any) in X

(3) $(a, b_0]$ b_0 the largest (if any) in X



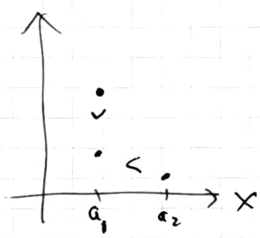
Exercise $\mathcal{B}$ is a basis in X. Get a topology (order topology)

$(a_1, b_1) \cap (a_2, b_2) = (\max(a_1, a_2), \min(b_1, b_2))$

~~Exs~~ 1) $\mathbb{R}$ - familiar topology. order top $(\mathbb{R})$ = standard top $(\mathbb{R})$

X, Y simply ordered $\Rightarrow$ get lexicographic order on $X \times Y$

$a_1 \times b_1 < a_2 \times b_2$ if $a_1 \underset{X}{<} a_2$ or $a_1 = a_2$, $b_1 \underset{Y}{<} b_2$



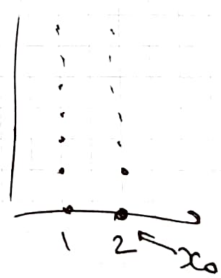
Example 2) Order top on $\mathbb{Z}$ $\Rightarrow$ any set is open, discrete top

3) $X = \{1, 2\} \times \mathbb{Z}_+$ $\quad 1\times 0, 1\times 1, \ldots$

$2\times 0, 2\times 1 \ldots$

$x_0 = 2\times 0$.

any open neighbourhood of x_0 contains all $\{1\times n \mid n > N\}$ some N



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$\mathbb{R}, <$

$Y = [0,1) \cap \{2\}$

$$\text{order } \cancel{\text{top}} \text{ on } \mathbb{R} \xrightarrow{\text{restrict}} \text{order on } Y \xrightarrow[\text{of order}]{\text{take topology}} (Y, \mathcal{J}')$$

$\{2\}$ is <u>not</u> open

$\mathcal{J}' \neq \mathcal{J}''$

$$\text{order on } \mathbb{R} \xrightarrow{\text{take topology}} (\mathbb{R}, \mathcal{J}) \xrightarrow[\text{restrict}]{} (Y, \mathcal{J}'')$$

$\{2\}$ is open
more open sets

X order, $Y \subset X$ subset

$$\begin{tikzcd}
 & \text{order on } Y \arrow[r] & \text{top on } Y \quad \mathcal{J}' \\
\text{order on } X \arrow[ur] \arrow[dr] & & \\
 & \text{topology on } X \arrow[r] & \text{top } \mathcal{J}'' \text{ on } Y
\end{tikzcd}$$

$\mathcal{J}' \subset \mathcal{J}''$

more open sets in $\mathcal{J}''$

We say that $Y \subset X$ is <u>convex</u> in X if $\forall a, b \in Y \quad [a,b] \subset Y$.

Prop (Munkres 16.4) Y - convex $\Rightarrow \mathcal{J}' = \mathcal{J}''$
See there for the proof. $\square$